Exact Number of Flips Required to Sort a Burnt Stack of Pancakes

Gerold Jäger*, **Nacim Oijid***,

Department of Mathematics and Mathematical Statistics, Umeå University, Umeå, Sweden
gerold.jager@umu.se, nacim.oijid@umu.se

Abstract

In this work, we consider the burnt pancake problem, which is a well-studied problem going back to a work of Gates and Papadimitriou from 1979. The problem is to sort a stack of n one-sided burnt pancakes of different sizes, by a sequence of flips of the top pancakes, such that at the end of the flipping sequence the pancakes have increasing size and the burnt sides of all pancakes are face-down. The pancakes are denoted by $1, 2, \ldots, n$, and a number is multiplied by -1, if the corresponding pancake has burnt side face-up.

Let $T(n)$ be the minimum number of flips to sort a special stack of n pancakes, namely $\overline{I_n} := [-1, -2, ..., -n]$. The instance $\overline{I_n}$ has strong relevance because of its easy structure and as it has been shown to be a worst-case instance for several small n. Heydari and Sudborough gave in 1997 the currently best upper bound of $T(n)$, namely $\lfloor (3n + 3)/2 \rfloor$ for $n \equiv 3 \bmod 4$, which later has been shown to be exact by a work of Cibulka from 2011. Except these two works, no progress regarding lower and upper bounds has been made until now. In our work, we present that $\lfloor (3n + 3)/2 \rfloor$ is also an upper bound of $T(n)$ for $n \equiv 1 \bmod 4$, which again matches the lower bound of Cibulka in 2011 and thus is exact. Furthermore, we show that our construction approach for $n \equiv 1 \bmod 4$ and the one of Heydari and Sudborough for $n \equiv 3 \bmod 4$ cannot be applied for even n. However, as there might be different construction approaches, the case of even n remains an open problem, where two possible values for $T(n)$ are possible, namely $(3/2)n + 1$ or $(3/2)n + 2$. Finally, we found two values, namely $n = 24$, $n = 26$, where the lower bound is attained.

Code — https://github.com/noijid/burnt-pancake

1 Introduction

The problem of this work can be motivated as follows.

For the buffet, the waiter of a restaurant gets a large stack of pancakes from the overworked cook. As usual, one side is burnt, and as the level of batter decreases, the pancakes became smaller and smaller. Hence, the waiter ends up with a stack of one-sided burnt pancakes sorted by size, with the largest at the bottom and burnt side to be face-up. However, the waiter cannot serve them this way. He needs to turn all the burnt sides down, without changing the order. Having only a spatula, he can only perform flips to the top of the stack. How can he perform this transformation in a minimum number of flips?

The *pancake sorting problem* was first mentioned by Dweighter (a pseudonym for J.E. Goodman) in (Dweighter 1975) and its study was formally introduced by (Gates and Papadimitriou 1979) under two variants: unburnt and burnt. For a stack of n pancakes of different sizes, the unburnt variant of this problem is to find the minimum number of flips required to sort it, where each flip reverses the order of the top k pancakes for some $1 \leq k \leq n$. The burnt variant considers the extra condition that each pancake has a burnt side and the stack needs both to be sorted and all the pancakes have their burnt side to be face-down. The (un)burnt n-pancake graph is defined as the graph containing one vertex per possible stack of n pancakes, and two vertices are connected if it is possible to transform one stack to the other one in a single flip.

Despite the simplicity of the problem, several applications have been found, in binary tree simulation (Akers and Krishnamurthy 1986), genome arrangements (Fertin et al. 2009) and parallel processing (Heydari and Sudborough 1997). From a theoretical point of view, the study of this problem is interesting both regarding structure and algorithmics. From a structural point of view, the pancake stack and its flipping sequences is a simple way to represent large objects. The pancake graph is of theoretical interest as it is a Cayley graph, and its structural properties have been an important area of research in structural graph theory (Kanevsky and Feng 1995; Qiu, Meijer, and Akl 1991; Wang and Cheng 2024). On the algorithmic side, the unburnt pancake sorting problem has been shown to be NP-hard (Bulteau, Fertin, and Rusu 2015), whereas the complexity state of the burnt pancake sorting problem is open. Furthermore, efficient algorithms have been developed to handle some restrictions or approximations of the problem (Seo, Kim, and Lee 2015).

The investigations of the two variants of the problem has led to the study of two functions, $f(n)$ and $g(n)$ that count the minimum number of flips to sort any pancake stack in the unburnt and burnt variant, respectively.

When they introduced the problem formally, (Gates and Papadimitriou 1979) proved that $\frac{17}{16}n \leq f(n) \leq \frac{5n+5}{3}$. Whereas Chitturi *et al.* improved the upper bound to

*These authors contributed equally.

$\frac{18}{11}n$ (Chitturi et al. 2009), (Heydari and Sudborough 1997) improved the lower bound to $\frac{15}{14}n$ and (Peczarski 2025) further by the additive constant $\frac{9}{14}$. (Heydari and Sudborough 1997) also studied the worst-case instance regarding the lower bound and showed an upper bound of $\frac{9}{8}n + 2$ for it. Recently, Amano (Amano 2024) improved this upper bound to $\frac{15}{14}n + 2$.

Considering the burnt pancake problem, (Gates and Papadimitriou 1979) showed that $\frac{3}{2}n - 1 \leq g(n) \leq 2n + 3$. Let $T(n)$ be the minimum number of flips to sort the stack of pancakes $\overline{I_n}$, which is the pancake stack in which all the pancakes are in increasing order with regard to their size, but with their burnt side face-up instead of down. Naturally, it implies that $T(n) \leq g(n)$. (Cohen and Blum 1995) proved that, for $n \geq 10$, it holds that $\frac{3}{2}n \leq T(n) \leq g(n) \leq 2n - 2$. They also conjectured that $\overline{I_n}$ is always a worst-case burnt pancake stack, i.e., one requiring the largest number of flip to be sorted. They proved that this conjecture holds for $n \leq 10$. The upper bound on $T(n)$ was improved by (Heydari and Sudborough 1997) who proved that $T(n) \leq \frac{3n+3}{2}$ if $n \equiv 3 \bmod 4$, being very close to the lower bound. This upper bound appeared to be optimal when (Cibulka 2011) proved that it is also the general lower bound. Hence, only the values for $n \equiv 0, 1, 2 \bmod 4$ remained to be determined. (Bouzy 2016) listed all values $T(n)$ until $n \leq 27$ and upper bounds for $n = 28, 29, 30$, where as we mention later, we have shown that the values for $n = 24$ or $n = 26$ are wrong, but all others are correct.

Finally, computing the values of $T(n)$ is challenging on its own, and we refer the reader to the sequence A078942 of the On-Line Encyclopedia of Integer Sequences (OEIS Foundation Inc. 2025) to have a more general overview of the known values.

In the same article as mentioned above, (Cibulka 2011) disproved the conjecture that $\overline{I_n}$ requires the largest number of flips with a counterexample for $n = 15$. However, until now, $n = 15$ is the only known value for which $\overline{I_n}$ is not a stack requiring the largest number of flips, and all the worst-case instances until $n = 22$ have been tested by (Bouzy 2016).

In this work, we show the same upper bound for $T(n)$ as in (Heydari and Sudborough 1997), namely $\frac{3n+3}{2}$, but for the case $n \equiv 1 \bmod 4$. Together with the case $n \equiv 3 \bmod 4$ from (Heydari and Sudborough 1997), the whole case of odd n is solved. Using results of (Cibulka 2011), this also restricts the case of even n to two possible values, namely $\frac{3}{2}n+1$ or $\frac{3}{2}n+2$. For $n = 24$ and $n = 26$ we found flipping sequences which attain this lower bound, namely with 37 and 40 flips, respectively. Note that this disproves the values given for $n = 24$ and $n = 26$ in (Bouzy 2016) and note that these two values are the only known ones attaining this lower bound except the value $n = 2$.

Our main result is obtained by an efficient algorithmic search, which explores only few states of the pancake graph. This kind of efficient search could find other applications in problems whose encoding is much smaller than the number of states to consider, as it is the case in reconfiguration and enumeration problems.

This work is organized as follows. First, we present notations and basic results which are needed in our main result and its proof. Then we present our main result, a search algorithm leading to this result and a sketch of its proof. We also prove that the construction of our main result does not work for even n. We close this work with some conclusions and suggestions for future work.

We provide an additional online supplemental which includes the following:

- Three pdf files with full proofs of the correctness of the flipping sequences,
- three pdf files consisting of example instances for each of the investigated cases $1, 5, 9 \bmod 12$, namely $n = 61, 53, 57$, in which each step of our constructed flipping sequences is listed,
- a source code in Python which produces optimal flipping sequences for odd n and optimal (or possibly optimal up to an additive factor 1) for even n explicitly,
- a source code in Python which is able to check correctness of the flipping sequences for $n \equiv 1 \bmod 4$, i.e., of our main result

  (note that this correctness test can show only correctness of a given flipping sequence, and note that this correctness test works also for all other n, namely the case $n \equiv 3 \bmod 4$ which has been solved by (Heydari and Sudborough 1997) and the case of even n).

2 Notations and Basic Results

2.1 Notations

We will use the notations of (Cibulka 2011), that we recall in the following.

We denote by I_n the sorted stack of pancakes, and by $\overline{I_n}$ the burnt pancake stack of size n in which all pancakes are in increasing order with regard to their size, but burnt face-up. i.e., we have $\overline{I_n} = [-1, -2, \ldots, -n]$.

If $S = [x_1, x_2, \ldots, x_s]$, where $1 \leq s \leq n$, is a stack of pancakes, we denote by $-S$ its flip, and by $\overline{S}$ its opposite i.e., $-S = [-x_s, -x_{s-1}, \ldots, -x_1]$ and $\overline{S} = [-x_1, -x_2, \ldots, -x_s]$. We denote by $T(n)$ the minimum number of flips required to sort $\overline{I_n}$.

In some parts, it will be helpful to add the (virtual) pancake $\pm(n + 1)$ which cannot be moved.

An adjacency is a pair of consecutive pancakes that are already in the right relative order: either both face-up and in increasing order $[s, s + 1]$, or both face-down and in decreasing order $[-(s + 1), -s]$, for some $1 \leq s \leq n$. In other words, no flip is needed between them. An *anti-adjacency* is a pair of consecutive pancake in the same relative order with respect to $\overline{I_n}$: either both face-up and in decreasing order $[s + 1, s]$, or both face-down and in increasing order $[-s, -(s + 1)]$, for some $1 \leq s \leq n$. In other words, no flip has been performed between them from the original stack $\overline{I_n}$. In both adjacency and anti-adjacency, we also include the virtual pancakes $\pm(n + 1)$ to handle boundary cases.

A set of consecutive pancakes is *a block* of size $s \geq 2$ if it is of the form $[x, x + 1, x + 2, \ldots, x + s - 1]$, where x can be either positive or negative, and it is *a clan* of size $s \geq 2$

if it is of the form $[x, x-1, \dots, x-s+1]$. Note that clans and blocks are necessarily disjoint, and that S is a clan if and only if $\overline{S}$ is a block. A pancake that is neither in a block nor in a clan is said to be *free*. Note that $\overline{I_n}$ is a clan of size n and I_n is a block of size n.

Example: Consider the stack $[-5, -4, -1, -2, 3, 6, 7, 8]$. In this stack, 3 is a free pancake, $-5, -4$ and $6, 7, 8, 9$ are blocks (where the pancake 9 is a virtual pancake added at the end of the stack to show that the pancake "8" is well placed) and $-1, -2$ is a clan.

If H_1 and H_2 are two stacks, we denote by $H_1 \overset{k}{\rightsquigarrow} H_2$ if H_2 is obtained by one flip (k) of k pancakes in H_1.

A flip $H_1 \overset{k}{\rightsquigarrow} H_2$ is said to be an *improve* if the number of adjacencies in H_2 is strictly larger than it is in H_1. A flip that is not an improve is said to be a *waste*.

2.2 Basic Results

The following results are used to reach our main result.

We start with a result which corresponds to our main result, but holds for the case $n \equiv 3 \bmod 4$ (instead of $n \equiv 1 \bmod 4$ in our result).

Theorem 1. *(Heydari and Sudborough 1997, Observation 3) Let $n \geq 23$ be an integer with $n \equiv 3 \bmod 4$. Then it holds that $T(n) = \frac{3}{2}n + \frac{3}{2}$.*

It can easily be seen that each flipping sequence leading to I_n needs to have the flip of the entire stack at least twice. The following theorem states that there is even a flipping sequence which begins with the flip of the entire stack.

Theorem 2. *(Cohen and Blum 1995, Theorem 6) There exists a shortest sequence of flips for sorting $\overline{I_n}$, that begins with the flip of the entire stack.*

The following theorem states that the function $T(n)$ increases not more than by 2 if n increases by 1. Having results for $T(n)$ for the case of odd n, we will use this theorem to obtain also an upper bound for the case of even n.

Theorem 3. *(Cohen and Blum 1995, Theorem 7) Let $n \geq 1$ be an integer. Then it holds that $T(n+1) \leq T(n) + 2$.*

Note that the proof of Theorem 3 is constructive, i.e., given a flipping sequence for $\overline{I_n}$ with $T(n)$ flips, where $n \geq 1$ is an integer, you can construct a flipping sequence for $\overline{I_{n+1}}$ with $T(n) + 2$ flips.

3 Main Result and Method

Our main result is as follows.

Theorem 4. *Let $n \geq 29$ be an integer with $n \equiv 1 \bmod 4$. Then it holds that $T(n) = \frac{3}{2}n + \frac{3}{2}$.*

We prove this result in the additional online supplemental. However, we have independently tested its correctness by a computer program, also provided in the additional online supplemental, until $n = 1501$.

In this main result, exact values for the specific instance $\overline{I_n}$ of the burnt pancake sorting problem are computed, namely for $n \equiv 1 \bmod 4$. On the one hand, the proof of this theorem is based on a precise description of the flipping sequences together with the states of the stack in each step. However, these flipping sequences required an advanced algorithm to be obtained. We will present this algorithm in the next section.

We conducted our computational experiments on a Linux-based high-performance computing cluster. Each compute node was equipped either with two Intel® Xeon® Gold 6132 processors running at 2.6 GHz or two AMD EPYC 9754 (Zen 4) processors running at 2.25 GHz. For each run, we used 16 physical CPU cores and 64 GB of RAM. Our experiments consisted of 20 independent runs executed in parallel.

3.1 Algorithmic Search of the Sequences

This search can be divided into two steps. The first one is to reach optimal flipping sequences for burnt stack of pancakes of sizes 29, 33 and 37, and the second one to generalize these sequences to sizes $5 \bmod 12, 9 \bmod 12$ and $1 \bmod 12$.

The main idea behind the use of "$\bmod 12$" is that the flipping sequence for $n = 3$ is $[3, 2, 3, 2, 3, 2]$ and the one for $n = 15$ is $[15, 10, 4, 6, 14, 6, 4, 10, 15, 10, 4, 6, 14, 6, 4, 10, 15, 10, 4, 6, 14, 6, 4, 10]$ (see (Heydari and Sudborough 1997)), which is obtained from the previous one by either adding 12 (the number of new pancakes) to the value or by inserting $[10, 4, 6]$ or $[6, 4, 10]$. This is the first value for which we obtained this simplicity in the computation from a value to the next one (modulo 12), which made it easy to implement and testing if this phenomenon can appear in other flipping sequences. It turns out that this is the case.

While analyzing the flipping sequences of (Heydari and Sudborough 1997), for $n = 3 \bmod 4$, we realized that the main arguments making this flipping sequence sort $\overline{I_n}$ are based on the assumption that the number of pancakes is odd. Thus, it was natural to try to extend their result from $3 \bmod 4$ to $1 \bmod 4$. However, the known values on $T(n)$ showed that, if we want to prove that $T(n) = \frac{3}{2}n + \frac{3}{2}$ for $n \equiv 1 \bmod 4$, it would be necessary to start at least at $n = 21$, since $T(17) = 28 = \frac{3}{2}n + \frac{3}{2} + 1$ (see (Cohen and Blum 1995; Bouzy 2016)).

Base Case The first step was to compute the base cases that can be generalized to larger values. In order to have a generalizable flipping sequence, following a similar idea to the flipping sequence of (Heydari and Sudborough 1997), we aimed to find a flipping sequence that first starts with all the wastes, and after that only performs improves.

We first restricted ourselves to the flipping sequences that start with a flip of the entire stack. Using Theorem 2, we know that such a flipping sequence exists. The next wastes cannot be performed in any order, but we have the following lemma that implies some structure on any flipping sequence that starts with all the wastes.

Lemma 1. *Let P be a stack of pancakes containing a clan C as a substack of size $|C| \geq 3$. Then P cannot be sorted without waste.*

Proof. Let x be an integer and
$C = [x, x-1, x-2] \,\big/\, [-(x-2), -(x-1), -x]$
be a clan of size 3 in P. If the first flip modifying C is a waste, then there is nothing to do.

Otherwise, the resulting stack is either $\left[-(x-1), -x, \ldots, x-3, x-2\right]$ or $\left[x-1, x-2, \ldots, -(x+1), -x\right]$. In both cases, the next flip cannot be an improve since to be an improve the pancake that follows $-(x-1)$ ($(x-1)$ resp.) after the next flip must be x ($-(x-2)$ resp.), which currently has the wrong side up in the stack. $\square$

In total, we aim at having $\frac{3}{2}n+\frac{3}{2}-n = \frac{n}{2}+\frac{3}{2}$ wastes, and because of Lemma 1, we have to perform them before reaching the state where all clans have size at most 2. Since one waste was already used to flip the entire stack, and since we need at least $\frac{n-1}{2}$ wastes to split the stack into clans of size 2, this forces all the wastes but one to be used to split clans. Following the idea of (Heydari and Sudborough 1997), we again restricted ourselves to the flipping sequences that have a flip of the entire stack as last waste. If such a flipping sequence exists, this last waste ensures a very precise structure of the rest of the flipping sequence as it will be described later.

After all these hypotheses, we know that we can generate candidate flipping sequences as follows, for a permutation σ of size $\frac{n}{2} - \frac{3}{2}$.

- We flip the entire stack.
- For $1 \leq i \leq \frac{n}{2} - \frac{3}{2}$, we flip between the clan $[2\sigma(i) + 1, 2\sigma(i)]$ and the clan $[2\sigma(i) + 3, 2\sigma(i) + 2]$
- We flip the entire stack.

Note that this algorithm can easily be parallelized since each permutation has to be handled separately.

We implemented this algorithm in Python and run the corresponding computer program for $n = 21$ and $n = 25$, but we did not obtain the desired flipping sequences. Hence, our results started at $n = 29$. For $n = 29$, this leads us to $13! \simeq 6.2 \cdot 10^9$ flipping sequences to try. For each of them, since we know that no more wastes are allowed, we can easily check in quadratic time if it is possible to sort the resulting stack of pancakes using only improves: each time a flip is an improve, if we denote by i the first pancake of the stack, we know that the flip has to occur before pancake $(-i) + 1$. This computer program provided us 20 flipping sequences of size 45, which were obtained in 31 hours using 16 cores. Among them, we chose one arbitrarily and we managed to generalize it to any values $n \equiv 5 \bmod 12$ as we will describe in the next subsection.

For $n = 33$ and $n = 37$, this would lead us to $15! \simeq 1.3{\cdot}10^{12}$ and $17! \simeq 3.6{\cdot}10^{14}$ permutations to test. Compared to the values for 29 the computation time would not end in reasonable time. To solve this problem, we randomized the generation of the permutations, hoping that the number of correct flipping sequences will also increase sufficiently so that we can find one with this random search. It happens that we reached both flipping sequences for 33 and 37 in less than two days of computation.

Generalized Sequences Generalizing the flipping sequences from $n = 29$, 33 and 37 to any $n \equiv 1 \bmod 4$ was mostly done inductively with the help of another computer program in Python. For sake of simplicity, we explain here the process that generalizes the flipping sequence from $n = 29$ to all larger values $n \equiv 5 \bmod 12$. We have applied the same process from $n = 33$ to all larger values $n \equiv 9 \bmod 12$ and from $n = 37$ to all larger values $n \equiv 1 \bmod 12$.

The main idea is to consider the wastes of the flipping sequences for $n = 15$ which are $[15, 10, 4, 6, 14, 6, 4, 10, 15]$ and to observe that a good solution to go from 29 to 41 is to introduce the subsequence of three wastes $[10, 4, 6]$ somewhere at the beginning, and the subsequence of three wastes $[6, 4, 10]$ somewhere at the end of the waste sequence for $n = 29$. The intuition is that the flipping sequence for $n = 15$ is obtained by adding these wastes to the sorting sequence for $n = 3$, hence we can naturally think that these wastes placed at the correct place in the flipping sequence manage to handle the 12 new pancakes using exactly 6 wastes and then we have 12 additional improves.

Denote by W the set of wastes of a flipping sequence. The idea is that inserting these two subsequences into W will split it into three (possibly empty) sets W_1, W_4 and W_6. Then, $W_2 = [10, 4, 6]$ is inserted between W_1 and W_4, and $W_5 = [6, 4, 10]$ is inserted between W_4 and W_6 (W_3 will be defined later as we will need one more observation to introduce it, see the end of this paragraph). The computation has shown that to apply this method a second time, i.e., from $n = 41$ to $n = 53$, works naturally for W_5, which is always inserted in the same place going from any integer k to the integer $k + 12$. I.e., W_5 for $n = 53$ is obtained by inserting a new $[6, 4, 10]$ subsequence before W_6 and adding 12 to the flip (10) of the previous $[6, 4, 10]$ subsequence in the flipping sequence of 41, i.e., adding $[6, 4, 22, 6, 4, 10]$ before W_6 in the flipping sequence of $n = 29$. However, the next insertions of W_2 will happen in the previous $[10, 4, 6]$ subsequence by adding also 12 to all their values. I.e., the part $[10, 4, 6]$ of the flipping sequence for $n = 41$ becomes $[10 + 12, 4 + 12, 10, 4, 6, 6 + 12] = [22, 16, 10, 4, 6, 18]$ in the flipping sequence for 53. Hence, to simplify the notation, we divide W_2 into two parts, and we now set $W_2 = [10, 4]$ and $W_3 = [6]$, such that the flipping sequence is easier to describe.

However, even by knowing which subsequence of flips to insert, finding their positions in the flipping sequence is not so easy. Hence, we used the computer program to do so. This aims to find exactly how many flips should be in W_1 before inserting W_2 and W_3, and how many flips should be in W_4 before inserting W_5. All other values of the sequence should not be modified, but, depending on whether we count from the beginning or the end of the stack, some flips may require a "+12" in their value. Hence, we tested all the possibilities to obtain the general form of W_1 and W_4. We generated all the possible flipping sequences that satisfy all these properties, and tested whether it was possible to complete them with only improves. Note that this algorithm is much faster than the program to find a flipping sequence for $n = 29$ mentioned in the last section, as it only requires to test $(\frac{n-1}{2})^2$ positions to insert the subsequences $[10, 4, 6]$ and $[6, 4, 10]$ (which leads to the positions of W_2, W_3 and W_5 in the flipping sequence). Moreover, while going from $n = 29$ to 41, each value of W_1, W_4 and W_6 is preserved or increased by 12. Thus, only $2^{\frac{n-1}{2}-1}$ flips have to be tested for these values. Once one flipping sequence was found this way, we tried by hand to generalize it by repeating the same process to reach the general formula. All this process is possible, since testing whether a flipping sequence is correct can be done in quadratic time.

3.2 Flipping Sequences

In the following we provide optimal flipping sequences to sort the stack $\overline{I_n}$ for any $n \geq 29$ with $n \equiv 1 \bmod 4$. Similarly to the flipping sequences of (Heydari and Sudborough 1997), our flipping sequences depend on the value of $n \bmod 12$. Due to the length of the proof and as they are mostly based on a technical analysis of the stack structure after each flip, we only provide the flipping sequences and an overview of how they are built. We refer the reader to the additional online supplemental to have the full versions of the proofs, with examples of optimal flipping sequences applied to stacks of 61, 53 and 57 pancakes, and with source codes providing the flipping sequences and checking their correctness.

The Case $n \equiv 1 \bmod 12$

Let n be an integer such that $n \equiv 1 \bmod 12$ and $n \geq 37$ and let $s = \frac{n-37}{12}$. We define the flipping sequence $S_1 = W^1 + A^1 + B^1$ as follows:

- $W^1 = W^1_1 + W^1_2 + W^1_3 + W^1_4 + W^1_5 + W^1_6$:
  - $W^1_1 = \{n, n-9, n-21, n-13, n-15\}$,
  - $W^1_2 = \{n-39-6i\}_{0\leq i\leq 2s-1}$,
  - $W^1_3 = \{6+12i\}_{0\leq i\leq s-1}$,
  - $W^1_4 = \{n-19, 10, n-1, n-13, n-15, 14, n-9, n-19, n-15, 12, n-3, 2, 14, n-17\}$,
  - $W^1_5 = \{6, 4, n-39-12i\}_{0\leq i\leq s-1}$,
  - $W^1_6 = \{n\}$.
- $A^1 = A^1_1 + A^1_2 + A^1_3 + A^1_4 + A^1_5$:
  - $A^1_1 = \{6, 32, 14, 20, 10, 2, 28, 4, 30, 8\}$,
  - $A^1_2 = \{44+12i, 4, 6\}_{0\leq i\leq s-1}$,
  - $A^1_3 = \{n-1\}$,
  - $A^1_4 = \{n-9-6i, n-11-12i, n-5-6i\}_{0\leq i\leq s-1}$,
  - $A^1_5 = \{\frac{n-1}{2}, \frac{n-1}{2}+4, \frac{n-1}{2}-2, 10, \frac{n-1}{2}-4, \frac{n-1}{2}+2, \frac{n-1}{2}+8, n\}$.
- $B^1 = B^1_1 + B^1_2 + B^1_3$:
  - $B^1_1 = \{\frac{2n+1}{3}+3\}$,
  - $B^1_2 = \{\frac{2n+1}{3}-3-8i, \frac{2n+1}{3}-1-8i, \frac{2n+1}{3}+7+4i, 6+12i, 4+12i, \frac{2n+1}{3}+3+4i\}_{0\leq i\leq s-1}$,
  - $B^1_3 = \{10, 24, 2, n-3, n-15, n-19, n-17, n-11, n-27, n-9, n-1, n-21, n-9, 4, 6, 18, 14\}$.

Lemma 2. *Let $n \geq 37$ be an integer with $n \equiv 1 \bmod 12$. The sequence S_1 sorts $\overline{I_n}$ and has size $\frac{3}{2}n + \frac{3}{2}$.*

The Case $n \equiv 5 \bmod 12$

Let n be an integer such that $n \equiv 5 \bmod 12$ and $n \geq 29$. Let $s = \frac{n-29}{12}$. We define the flipping Sequence $S_5 = W^5 + A^5 + B^5$ as follows:

- $W^5 = W^5_1 + W^5_2 + W^5_3 + W^5_4 + W^5_5 + W^5_6$:
  - $W^5_1 = \{n, n-9, 4, 10, n-3, n-15, n-25\}$,
  - $W^5_2 = \{n-31-6i\}_{0\leq i\leq 2s-1}$,
  - $W^5_3 = \{6+12i\}_{0\leq i\leq s-1}$,
  - $W^5_4 = \{n-7, n-1, n-11, 6, 4, n-5, 10, n-15\}$,
  - $W^5_5 = \{6, 4, n-31-12i\}_{0\leq i\leq s-1}$,
  - $W^5_6 = \{n\}$.
- $A^5 = A^5_1 + A^5_2 + A^5_3 + A^5_4 + A^5_5$:
  - $A^5_1 = \{18, 20, 12, 4, 6, 24, 16, 26, 16, 4, 18\}$,
  - $A^5_2 = \{36+12i, 4, 6\}_{0\leq i\leq s-1}$,
  - $A^5_3 = \{n-1\}$,
  - $A^5_4 = \{n-9-6i, n-11-12i, n-5-6i\}_{0\leq i\leq s-1}$,
  - $A^5_5 = \{\frac{n-1}{2}-6, \frac{n-1}{2}+6, n\}$.
- $B^5 = B^5_1 + B^5_2 + B^5_3$:
  - $B^5_1 = \{\frac{2n-1}{3}-3\}$,
  - $B^5_2 = \{\frac{2n-1}{3}-9-8i, \frac{2n-1}{3}-7-8i, \frac{2n-1}{3}+1+4i, 6+12i, 4+12i, \frac{2n-1}{3}-3+4i\}_{0\leq i\leq s-1}$,
  - $B^5_3 = \{12, n-5, 4, n-13, n-15, n-21, n-25, n-11, n-9, n-1, 6, 4, 14\}$.

Lemma 3. *Let $n \geq 29$ be an integer with $n \equiv 5 \bmod 12$. The sequence S_5 sorts $\overline{I_n}$ and has size $\frac{3}{2}n + \frac{3}{2}$.*

The Case $n \equiv 9 \bmod 12$

Let n be an integer such that $n \equiv 9 \bmod 12$ and $n \geq 33$. Let $s = \frac{n-33}{12}$. We define the flipping Sequence $S_9 = W^9 + A^9 + B^9$ as follows:

- $W^9 = W^9_1 + W^9_2 + W^9_3 + W^9_4 + W^9_5 + W^9_6$:
  - $W^9_1 = \{n, 14, 4, 10, n-7, n-29\}$,
  - $W^9_2 = \{n-35-6i\}_{0\leq i\leq 2s-1}$,
  - $W^9_3 = \{6+12i\}_{0\leq i\leq s-1}$,
  - $W^9_4 = \{n-11, n-1, n-9, 4, n-13, n-11, 8, 10, n-5, 10, n-19\}$,
  - $W^9_5 = \{6, 4, n-35-12i\}_{0\leq i\leq s-1}$,
  - $W^9_6 = \{n\}$.
- $A^9 = A^9_1 + A^9_2 + A^9_3 + A^9_4 + A^9_5$:
  - $A^9_1 = \{24, 16, 14, 4\}$,
  - $A^9_2 = \{40+12i, 4, 6\}_{0\leq i\leq s-1}$,
  - $A^9_3 = \{n-1\}$,
  - $A^9_4 = \{n-9-6i, n-11-12i, n-5-6i\}_{0\leq i\leq s-1}$,
  - $A^9_5 = \{\frac{n-1}{2}+10, 12, 10, 4, 14, 22, 6, 24, \frac{n-1}{2}+14, \frac{n-1}{2}-6, \frac{n-1}{2}, n\}$.
- $B^9 = B^9_1 + B^9_2 + B^9_3$:
  - $B^9_1 = \{\frac{2n}{3}\}$,
  - $B^9_2 = \{\frac{2n}{3}-6-8i, \frac{2n}{3}-4-8i, \frac{2n}{3}+4+4i, 6+12i, 4+12i, \frac{2n}{3}+4i\}_{0\leq i\leq s-1}$,
  - $B^9_3 = \{18, n-5, n-19, n-13, 8, n-19, n-29, n-9, 14, n-1, n-17, n-13, 10, 4, n-9\}$.

Lemma 4. *Let $n \geq 33$ be an integer with $n \equiv 9 \bmod 12$. The sequence S_9 sorts $\overline{I_n}$ and has size $\frac{3}{2}n + \frac{3}{2}$.*

3.3 Description of the Flipping Sequences

Even though the flipping sequences are different depending on the value of $n \bmod 12$, they are divided in three parts, W, A and B, where each of them is divided in several steps.

Part W contains $\frac{n+3}{2}$ flips. It aims first to split the stack into $\frac{n-1}{2}$ clans of size 2, and the free pancake 1. Then it performs a full flip of the stack to put the free pancake -1 to the top of the stack. To achieve this result, Step W_1 and Step W_4 are the ones that create the clans for a *small* stack, i.e., of size $29, 33$ or 37, respectively. They are generalized to handle the first and last pancakes of the stack, while for large n, most of the pancakes are handled by the other steps. When the size of the stack increases, Step W_2 creates new clans of size 6, one set of clans starting from the larger pancakes (of value almost n) and one set of clans from the smaller ones (of constant value), until reaching pancakes of value $\frac{n}{2}+c$ for some constant c. Step W_3 and Step W_5 aim to separate these clans of size 6 into clans of size 2. First, Step W_3 handles the clans of one set to transform them into a clan of size 2 and one of size 4. Then, Step W_5 handles the remaining clans of size 4 and 6, and separates them into regularly separated clans of size 2. Intuitively, the clans are split like they are in the flipping sequence for $n = 15$. Finally, Step W_6 consists of fully flipping the stack to put the free pancake -1 to the top. This makes it possible to only have improves after this flip.

Then, part A consists of $\frac{n+1}{2}$ improves to transform all the clans into blocks of size 2. In particular, we start with the free pancake -1 to the top of the stack. Then, for $1 \leq i \leq \frac{n+1}{2}$, the i-th flip of part A starts with the pancake $-(2i-1)$ on the top of the stack and is flipped between the clan $(2i+1, 2i)$, transforming it into the block $(2i-1, 2i)$ and putting the free pancake $-(2i+1)$ to the top. The division of part A in several steps is due to the structure of the stack achieved during Part W.

Finally, part B performs $\frac{n-1}{2}$ improves to merge these blocks together to obtain I_n. The flip of Step B_1 puts back this pancake of value $\frac{n}{2}+c$ to the top of the stack. Then, Step B_2 merges all the blocks except the beginning and the end of the stack (which both correspond to a small stack of pancakes), ending up with a large block containing all the pancakes, except some of the smaller ones and some of the larger ones. Step B_3 finally merges the remaining blocks that correspond to a small base stack, i.e., corresponding to a stack of size $29, 33$ or 37, with the big block created during Step B_2.

3.4 Consequences

Corollary 1. *Let $n \geq 19$ be an odd integer. Then it holds that $T(n) = \frac{3}{2}n + \frac{3}{2}$.*

Proof. Let $n \geq 19$ be an odd integer. The lower bound is already known from (Cibulka 2011), i.e., $T(n) \geq \frac{3}{2}n + \frac{3}{2}$.

For the upper bound, if $n = 19, 21, 23, 25, 27$, the result is provided by (Bouzy 2016), and we obtained it also independently by a Python program available in our online supplemental. If $n \geq 29$, depending on the value of $n \bmod 12$, the result is provided by Theorem 4 or by Theorem 1 from (Heydari and Sudborough 1997). $\square$

Corollary 2. *Let $n \geq 14$ be an even integer. We have:*

$$T(n) \in \left\{ \frac{3}{2}n+1, \frac{3}{2}n+2 \right\}.$$

Proof. Let $n \geq 14$ be an even integer. For $n = 14, 16, 18$, the result is provided by (Bouzy 2016), and we obtained it also independently by a Python program available in our online supplemental. If $n \geq 20$, by Theorem 3, we know that $T(n+1) - 2 \leq T(n) \leq T(n-1) + 2$. Since n is even, both $n-1$ and $n+1$ are odd. Thus, by Corollary 1, we have $T(n+1) = \frac{3}{2}(n+1) + \frac{3}{2} = \frac{3}{2}n + 3$ and $T(n-1) = \frac{3}{2}(n-1) + \frac{3}{2} = \frac{3}{2}n$, which leads us to:

$$\frac{3}{2}n + 1 \leq T(n) \leq \frac{3}{2}n + 2. \quad \square$$

4 Limit of the Method

In this section, we prove that the method which we and (Heydari and Sudborough 1997) used to obtain flipping sequences for odd n cannot be used to obtain a flipping sequence for even n and to reach the best known lower bound, i.e., $\frac{3}{2}n + 1$ flips. However, even though these sequences do not exist for small values, by a computer program in C++, based on BnB, we found the following flipping sequences that sort $\overline{I_{24}}$ and $\overline{I_{26}}$ and attain the lower bound, i.e., they have 37 and 40 flips, respectively. Note that this disproves two values of Table 1 of (Bouzy 2016).

The flipping sequence for $\overline{I_{24}}$: 24, 10, 12, 20, 5, 15, 5, 8, 19, 17, 5, 10, 14, 7, 22, 17, 6, 13, 10, 21, 23, 18, 11, 20, 6, 18, 24, 9, 7, 19, 5, 16, 23, 9, 15, 11, 18.

The flipping sequence for $\overline{I_{26}}$: 26, 12, 8, 25, 10, 14, 26, 21, 18, 13, 8, 24, 9, 12, 8, 22, 4, 7, 23, 19, 25, 11, 23, 16, 6, 11, 13, 9, 6, 22, 10, 14, 12, 18, 15, 24, 7, 14, 18, 21.

For proving the mentioned limitation, we first need the following lemma that is not explicitly stated in this form in (Cibulka 2011) but which is shown in the proof of his Lemma 4 as follows.

Lemma 5. *Let n be an even positive integer. If a flipping sequence of size $\frac{3}{2}n+1$ sorts $\overline{I_n}$, all its flips either create an adjacency or break an anti-adjacency.*

To make the paper self-contained, we provide a short proof of this lemma even though most of the arguments are from (Cibulka 2011).

Proof. Let C be a stack of pancakes and let C' be a stack obtained from a single flip from C.

A block (clan resp.) is called a *surface block (clan resp.)* if the topmost pancake of the stack is part of it, otherwise it is called *deep*.

For a stack C of pancakes, we define the value $v(C)$ as follows: [1]

$$v(C) \stackrel{\text{def}}{=} a(C) - a^-(C) + l(C) - l^-(C) - \frac{1}{3}\big(b(C) - b^-(C)\big) + \frac{1}{3}\big(o(C) - o^-(C)\big) + \frac{1}{3}\big(ll(C) - ll^-(C)\big),$$

[1]Note that in comparison to (Cibulka 2011) for a stack of pancakes C, we have interchanged the notations of $\bar{C}$ and $-C$.

where

$$a(C) \overset{\text{def}}{=} \text{number of adjacencies}$$

$$l(C) \overset{\text{def}}{=} \begin{cases} 1 & \text{if the lowest pancake is } n \\ 0 & \text{otherwise} \end{cases}$$

$$b(C) \overset{\text{def}}{=} \text{number of deep blocks}$$

$$o(C) \overset{\text{def}}{=} \begin{cases} 1 & \text{if the pancake on top of the stack} \\ & \text{is the free } -1 \\ & \text{or if 1 is in a block (necessarily with 2)} \\ 0 & \text{otherwise} \end{cases}$$

$$ll(C) \overset{\text{def}}{=} \begin{cases} 1 & \text{if the lowest pancake is } n \\ & \text{ and the second lowest is } n-1 \\ 0 & \text{otherwise} \end{cases}$$

$$a^-(C) \overset{\text{def}}{=} a(\overline{C}) = \text{number of anti-adjacencies in } C$$

$$l^-(C) \overset{\text{def}}{=} l(\overline{C})$$

$$b^-(C) \overset{\text{def}}{=} b(\overline{C}) = \text{number of deep clans in } C$$

$$o^-(C) \overset{\text{def}}{=} o(\overline{C})$$

$$ll^-(C) \overset{\text{def}}{=} ll(\overline{C}).$$

Cibulka (Cibulka 2011) proved that any flip changes the value of $v(C)$ by at most $4/3$, and that $v(I_n) = -v(\overline{I_n}) = n + \frac{2}{3}$. Hence, as $\frac{4}{3}(\frac{3}{2}n + 1) = 2n + \frac{4}{3} = v(I_n) - v(\overline{I_n})$, a flipping sequence of size $\frac{3}{2}n+1$ sorts $\overline{I_n}$ if and only if all the flips of the sequence increase the value of v by $\frac{4}{3}$. We also define the following value to handle the different changes after a flip:

$$\Delta a \overset{\text{def}}{=} a(C') - a(C) \qquad \Delta a^- \overset{\text{def}}{=} -\big(a^-(C') - a^-(C)\big)$$
$$\Delta l \overset{\text{def}}{=} l(C') - l(C) \qquad \Delta l^- \overset{\text{def}}{=} -\big(l^-(C') - l^-(C)\big)$$
$$\Delta b \overset{\text{def}}{=} -\frac{1}{3}\big(b(C') - b(C)\big) \qquad \Delta b^- \overset{\text{def}}{=} \frac{1}{3}\big(b^-(C') - b^-(C)\big)$$
$$\Delta o \overset{\text{def}}{=} \frac{1}{3}\big(o(C') - o(C)\big) \qquad \Delta o^- \overset{\text{def}}{=} -\frac{1}{3}\big(o^-(C') - o^-(C)\big)$$
$$\Delta ll \overset{\text{def}}{=} \frac{1}{3}\big(ll(C') - ll(C)\big) \qquad \Delta ll^- \overset{\text{def}}{=} -\frac{1}{3}\big(ll^-(C') - ll^-(C)\big).$$

Cibulka proved that $\Delta a, \Delta a^-, \Delta l, \Delta l^- \in \{-1, 0, 1\}$ and $\Delta b, \Delta b^-, \Delta o, \Delta o^-, \Delta ll, \Delta ll^- \in \{-1/3, 0, 1/3\}$.

By definition of a, a^-, l and l^-, if one of these four values increases, the flip either creates an adjacency or removes an anti-adjacency. Cibulka also proved the following implications:

$$\Delta l = \Delta l^- = 0, \Delta ll > 0 \implies \text{it creates an adjacency}$$
$$\Delta l = \Delta l^- = 0, \Delta ll^- > 0 \implies \text{it breaks an anti-adjacency}$$
$$\Delta o > 0, \Delta o^- > 0 \implies \Delta b = \Delta b^- = 0$$

Hence, if a flip does neither break an anti-adjacency nor create an adjacency, it satisfies that $\Delta a = \Delta a^- = \Delta l = \Delta l^- = 0$ and at least three terms among $\Delta ll, \Delta ll^-, \Delta o, \Delta o^-, \Delta b, \Delta b^-$ are non-positive. Thus, this flips verifies that $v(C') - v(C) \leq 1$, which cannot happen in a sorting sequence of size $\frac{3}{2}n + 1$. □

We now can prove that no flipping sequence with the structure we used can sort $-I_n$ in $\frac{3}{2}n + 1$ flips when n is even. This structure means starting with the flip (n), then containing all the wastes, another flip (n) and finally only improves. Note that in the odd case the second flip (n) is not an improve, but instead the third one is. (Note that in the even case, the third one may not exist.) Suppose by contradiction that such a flipping sequence exists.

Claim 1. *The second flip (n) has to be an improve.*

Proof of the claim. By Lemma 5, if a flipping sequence of size $\frac{3}{2}n + 1$ exists, all its flips have either to create an adjacency, or break an anti-adjacency. Hence, since the first flip (n) already breaks the anti-adjacency between $-n$ and the end of the stack (a virtual $-(n+1)$-pancake), when the second flip (n) is performed, the last pancake of the stack cannot be the pancake $-n$. Thus, it cannot break an anti-adjacency, and therefore it has to create an adjacency with the virtual pancake $+(n+1)$, which is only possible if it puts the pancake n to the end of the stack. ◇

Claim 2. *The anti-adjacency $(n, n-1)$ has to be broken by a waste.*

Proof of the claim. Since the second flip (n) puts the pancake n to the end of the stack, the flip before has to put the pancake $-n$ to its top. Suppose that this flip is a flip (k), for some integer $1 \leq k \leq n-1$. Before this flip (k), the pancake n has to be positively in position k. Hence, either the clan $(n, n-1)$ has already been broken, and there is nothing to do, or the $(k+1)$-th pancake of the stack is the pancake $n-1$, and this flip (k) breaks this clan, which concludes the proof. In both cases, as this happens before the second flip (n), by hypothesis, the flip that breaks this clan is a waste. ◇

Claim 3. *The anti-adjacency $(2, 1)$ has to be broken by a waste.*

Proof of the claim. Suppose by contradiction that it does not. As after the first flip (n), the two last pancakes of the stack are $[2, 1]$ in that order, all the other flips before the second flip (n) have to be flips (k) with $1 \leq k \leq n-2$. Hence, the two pancakes 1 and 2 cannot be moved before the second flip (n), which will put $[-1, -2]$ on the top of the stack. However, by hypothesis, all the flips after the second flip (n) have to be improves, which is not possible as the only pancake that can go after an improve when -1 is the top pancake of the stack is the pancake 2 which has the wrong side up in the stack. ◇

Since we supposed that a $(\frac{3}{2}n + 1)$-flipping sequence exists, note that n flips have to be improves, and we already determined three of the wastes: the first flip (n), the ones that breaks the anti-adjacency $(n, n-1)$ and $(2, 1)$. Hence $\frac{n}{2} - 2$ wastes remain to be determined.

Claim 4. *Let $1 \leq k \leq \frac{n}{2}$ be an integer. There is one waste that breaks the clan $(2k, 2k-1)$.*

Proof of the claim. We already proved through Claim 2 and Claim 3 that for $k = 1$ and $k = \frac{n}{2}$, there must be a waste breaking the anti-adjacency between the pancakes $2k$ and $2k-1$. Consider now the clan $[n-1, n-2, \ldots, 2]$ of size $n-2$. By Lemma 1, it has to be split into clans of size at most 2 before the second flip (n), otherwise, it would not be possible to finish the sorting with only wastes. As only $\frac{n}{2}-2$ wastes remain to perform this splitting of the clans into clans of size 2, the only solution is to split exactly between the pancakes $2k$ and $2k-1$ for each $2 \leq k \leq \frac{n}{2}-1$. $\diamond$

We can now prove that the method presented in this paper cannot provide a $\left(\frac{3}{2}n+1\right)$-flipping sequence that sorts $\overline{I_n}$ for even n.

Theorem 5. *Let n be an even positive integer. If $T(n) = \frac{3}{2}n+1$, an optimal flipping sequence of $-I_n$ cannot start with a flip (n) (a waste) followed by all further wastes, then a flip (n) (an improve) followed by all further improves.*

Proof. Suppose by contradiction that such a flipping sequence exists. The proof is a case distinction according to the second flip.

- If the second flip is a flip $(n-1)$, the resulting pancake stack is $[-2, -3, \ldots, -n, 1]$. By Claim 4, all the following wastes cannot be flips $(n-1)$, otherwise they would not break a clan. Hence, it is not possible to have the pancake $-n$ at the top of the stack after all the wastes, contradicting Claim 1.
- Otherwise, the second flip is a flip $(2k-1)$ for some $1 \leq k \leq \frac{n}{2}-1$, creating the substack $[-n, n-2k+1, n-2k]$. By Claim 4, this substack cannot be broken by a waste (but it can be fully flipped). Hence, after the last waste, the stack has to start with $[-n, n-2k+1, n-2k]$, to ensure that the flip (n) is an improve, and that this substack has not been split. After the flip (n), the stack becomes $[-1, X, -(n-2k), -(n-2k+1), n]$ for some stack X, and has to be sorted with only improves. Furthermore, by Claim 4, for all $1 \leq i \leq \frac{n}{2}-1$ the clan $(2i+1, 2i)$ has not been split.
  Moreover, since all the remaining flips have to be improves, we can prove by induction that after the j-th improve after the second flip (n), the first pancake of the stack is $-(2j-1)$ and all the clans $(2i+1, 2i)$ for $j \leq i \leq \frac{n}{2}-1$ have not been split.
  Indeed, by hypothesis the first improve is the flip (n) which puts the pancake -1 to the top of the stack and no clan $(2i+1, 2i)$ for $1 \leq i \leq \frac{n}{2}-1$ has been split.
  Then suppose that after the j-th improve, the pancake at the top of the stack is $-(2j-1)$ and that all the clans $(2i+1, 2i)$ for $j \leq i \leq \frac{n}{2}-1$ have not been split. The $(j+1)$-th improve has to set the pancake $-(2j-1)$ next to the pancake $2j$. Hence, since the clan $(2j+1, 2j)$ has not been split, and since the pancake $2j$ has to be burnt face-down for this flip to be an improve, necessarily, this clan is in that order in the stack and thus the j-th improve is a flip between these two pancakes, putting the pancake $-(2j+1)$ to the top of the stack and the only clan than has been split is the clan $(2j+1, 2j)$, which concludes the induction.
  By construction, for $1 \leq j \leq \frac{n-2k}{2}$, the j-th improve after the second flip (n) cannot modify the last three pancakes of the stack. Hence, the $\left(\frac{n-2k}{2}\right)$-th improve after the second flip (n) puts the pancake $-(n-2k-1)$ to the top of the stack. This is a contradiction, as the next flip has to be an improve and thus put it adjacent to the pancake $n-2k$, which currently has the wrong side up in the stack. $\square$

5 Conclusion and Future Work

Burnt pancake sorting is a classical problem going back to the work of (Gates and Papadimitriou 1979). The original problem is to find lower and upper bounds for the number of flips which are required to sort an arbitrary stack of burnt pancakes of size n. As $\overline{I_n}$ has been shown to belong to the worst-case instances for all $1 \leq n \leq 22$, except for $n = 15$ (Cohen and Blum 1995; Bouzy 2016), and as it has a very easy and natural structure, it is very interesting to compute upper and lower bounds for the number of flips which this instance needs to be sorted, called $T(n)$.

(Heydari and Sudborough 1997) gave an upper bound for $T(n)$ for $n \equiv 3 \bmod 4$, and (Cibulka 2011) a global lower bound, which matches the upper bound of (Heydari and Sudborough 1997) for $n \equiv 3 \bmod 4$. (Bouzy 2016) computed the values $T(2), T(3), \ldots, T(27)$. [2] Apart from that, we are not aware of any other general result about $T(n)$ since then. In this work, we have presented an upper bound for $n \equiv 1 \bmod 4$, which again matches the lower bound of (Cibulka 2011) .

Although the problem appears quite simple at first glance, the size of the burnt pancake graph and the complexity of finding optimal strategies make efficient computation highly challenging. Consequently, the tools developed in this work to tackle this problem are of independent interest, beyond the (burnt) pancake sorting problem itself.

For future work we suggest the following:

- The proofs of this work contains three sub-cases, namely $n \equiv 1 \bmod 12$, $n \equiv 5 \bmod 12$ and $n \equiv 9 \bmod 12$, which have similar, long and technical proofs. To shorten them it would be interesting to find unified flipping sequences and corresponding proofs for $n \equiv 1 \bmod 4$ or even for all odd n.
- The only open case is $T(n)$ for even n, where only two values are possible, namely $\frac{3}{2}n+1$ and $\frac{3}{2}n+2$. In this context, we have already shown that there are at least two further values (except the case $n = 2$) where the lower bound is attained, namely $n = 24$ and $n = 26$.
- Another problem would be to investigate whether $n = 15$ is the only positive integer, where $\overline{I_n}$ is not a worst-case instance. Note that whenever $\overline{I_n}$ is a worst-case instance for some integer n, this work ensures that the diameter of the pancake graph is either $\frac{3}{2}n+1$ or $\frac{3}{2}n+2$.

[2] We have recomputed the values of $T(24)$ and $T(26)$ to be 37 and 40, instead of being 38 and 41, respectively, as mentioned in (Bouzy 2016). On the other hand, we have confirmed the correctness of all other values up to 27, for n odd by our main result and for n even by a BnB program in C++.

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