A Theoretical Comparison of the Bounds of MM, NBS, and GBFHS

Vidal Alcázar, 1 Mike Barley, 2 Pat Riddle 2
1Riken AIP, Tokyo, Japan
2University of Auckland, Auckland, New Zealand
vidal.alcazar@riken.jp, {barley, pat}@cs.auckland.ac.nz

Abstract
Recent work in bidirectional front-to-end heuristic search (Bi-HS) has led to the development of three different algorithms with very different behavior: MM, NBS and GBFHS. This paper presents ongoing research about their lower and upper bounds and the underlying reasons for them.

Introduction
Recently, new lines of research in Bi-HS have been explored with success. First, Holte et al. (2017) proposed MM 1, which uses a priority function so no nodes are expanded beyond the midpoint. MM was generalized to fMM (Shaham et al. 2017) so it can meet at different splits. Soon after came NBS (Chen et al. 2017). NBS expands at most twice the minimum number of necessarily-expanded nodes up to the last f layer, making it near-optimal. Finally, GBFHS (Barley et al. 2018) keeps explicit f and g limits, the latter for each side and updated by a split function. GBFHS is well-behaved (guaranteed not to expand more nodes for the same split if the heuristic improves up to the last f layer), and ensures optimality upon first collision in unit-cost domains.

Still, a clear picture of the relationship between them is missing - i.e. fMM and GBFHS never expand nodes beyond the meeting point, but GBFHS has other beneficial properties and a lower upper bound. Also, NBS is near-optimal, but it is unclear whether other Bi-HS algorithms may be too. We study this and, after identifying the specific conditions of these desirable properties, we propose an easy to implement and to understand algorithm that is near-optimal.

Comparing Must- and Never-Expand Nodes
In unidirectional search, any node n such that f(n) < C* must be expanded. However, in Bi-HS one side may rise C (the current lower bound on C*) up to C* while nodes with f(n) < C* remain unexpanded. If the heuristic is consistent and the graph is undirected, nevertheless, a set of sufficient conditions depending on C*, ε and γ (the cost of the edge with the highest cost) can be found, and they are the same for both fMM and GBFHS. Lemma 1 compares MM and GBFHS with a meet-in-the-middle split function (MSF).

Lemma 1. The set of must-expand nodes Sme of MM and GBFHS with a MSF is, ∀n ∈ Sme, (1) and (2) hold true:
1. f(n) < C* − 2γ
2. g(n) < C* − ε − γ

This is consistent with the experiments from Barley et al. (2018), in which the lower bound becomes the same as the strength of the heuristic degrades. On the other hand, in the experiments the upper bound remains consistently higher for MM. The sets of never-expand nodes are as follows:

Definition 1. The set of never-expand nodes Sne of MM is, ∀n ∈ Sne, (1) or (2) holds true:
1. f_f(n) > C* ∧ f_b(n) > C*
2. g_f(n) > C* − ε ∧ g_b(n) > C* − ε

Definition 2. The set of never-expand nodes S_n of GBFHS with a MSF is, ∀n ∈ S_n, (1) or (2) holds true:
1. f_f(n) > C* ∧ f_b(n) > C*
2. g_f(n) ≥ γLim_f ∧ g_b(n) ≥ γLim_b

gLim_f = \left\lfloor \frac{C* − ε + 1}{2} \right\rfloor \text{ and } gLim_b = \left\lfloor \frac{C* − ε + 1}{2} \right\rfloor

vice versa. Hence, when \left\lfloor \frac{C* − ε + 1}{2} \right\rfloor = \left\lfloor \frac{C* − ε + 1}{2} \right\rfloor, a whole g layer may be expanded by MM but not by GBFHS. This highlights the existence of two different g layers within the last f layer in Bi-HS, which has an important impact on the performance of the algorithm. GBFHS also commits to a whole g layer, which allows stopping upon first collision in unit-cost domains, affecting the average case too.

The meeting point of NBS is not known beforehand, but under the assumption that NBS, fMM and GBFHS meet at the same point, we can establish a comparison about S_n. NBS does not impose sufficient g conditions for S_n, so its S_n will be equal or smaller than MM’s and GBFHS’s, with a worse worst-case. In fact, neither fMM nor GBFHS expands nodes beyond the meeting point, but NBS may.

Lastly, GBFHS is well-behaved because it uses g limits tied to C, which avoids committing to heuristic plateaus without raising C. This does not mean that GBFHS is superior to fMM in all cases: expanding by g means that a solution in the last f layer may be found later, so GBFHS may have to expand nodes that fMM does not.
Pruning Power of NBS

Other Bi-HS algorithms can expand nodes alternatively on both sides as NBS does. Any algorithm whose \( f_{\text{Min}} \) is monotonically increasing enforces a lower bound based on \( f \). Thus, the advantage of NBS comes from delaying the expansion of pairs of nodes such that \( g_f(u) + g_v(v) + \epsilon > C \).

**Theorem 1.** Assume all expanded nodes were expanded with optimal \( g \). Let \( n \) be a forward node and \( gCM_{inb} = \min_{s \in \text{Open}_b} g(s) \) such that \( f_b(s) \leq C \). If \( g_f(n) + gCM_{inb} + \epsilon > C \) then there cannot be a solution path of cost \( C \) through \( n \). The backwards case is analogous.

Using Theorem 1 we propose Near-optimal Bidirectional Baseline (NBB, Algorithm 1). \( \text{Expand}(d) \) expands a node in direction \( d \) with minimum \( g \) among \( n \) such that \( f_b(n) \leq C \), updating \( \text{bestSolution} \). \( C \) is updated using Theorem 1 (lines 6 and 7 of Algorithm 2). \( \text{nextFMin}(C) \) returns the minimum \( f \) value bigger than \( C \) in the open lists. \( \text{RunTieBreaker}() \) is an optional procedure invoked after increasing \( C \) that aims to quickly find a collision along paths of \( C \). This allows raising \( C \) quickly while having the possibility of implementing a good tie-break for the last layer.

### Algorithm 1 Near-optimal Bidirectional Baseline

1: \( b_{\text{estSolution}} \leftarrow \infty; C \leftarrow 0; f_w \leftarrow \text{true} \)
2: while \( \text{Open}_f \neq \emptyset \land \text{Open}_b \neq \emptyset \) do
3: if \( \text{UpdateC()} \) then
4: \( \text{RunTieBreaker}() \)
5: end if
6: if \( \text{bestSolution} \leq C \) then
7: \( \text{return bestSolution} \)
8: end if
9: \( \text{Expand}(f_w); f_w \leftarrow \neg f_w \)
10: end while
11: \( \text{return bestSolution} \)

### Algorithm 2 UpdateC

1: \( \text{updatedC} \leftarrow \text{false} \)
2: if \( f_{\text{Min}} > C \) then
3: \( C \leftarrow f_{\text{Min}} \)
4: \( \text{updatedC} \leftarrow \text{true} \)
5: end if
6: while \( gMin_f + gMin_b + \epsilon > C \) do
7: \( C \leftarrow \min(gMin_f + gMin_b + \epsilon, \text{nextFMin}(C)) \)
8: \( \text{updatedC} \leftarrow \text{true} \)
9: end while
10: \( \text{return updatedC} \)

Barring \( \text{RunTieBreaker}() \), NBB is near-optimal. Its behavior is analogous to NBS’s (Shperberg et al. 2019), so it could be seen as a version of NBS. However, NBB was developed independently from NBS with different pseudocode and does not require priority queues for nodes, so we use \( g-f \) buckets (Burns et al. 2012) instead.

Table 1 compares NBS and NBB on: 16 Pancakes (GAP heuristic, the first \& pancakes are ignored to get a weaker heuristic - 100 random problems); Korf’s 15-Puzzle instances with Manhattan distance; and the Dragon Age: Origin (DAO) maps and the ten hardest mazes from Sturtevant’s grid-based pathfinding benchmarks with the octile heuristic.

<table>
<thead>
<tr>
<th>Domain</th>
<th>NBS</th>
<th>NBS_b</th>
<th>NBB</th>
<th>NBB_b</th>
<th>0-C^*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pancake (0)</td>
<td>260</td>
<td>68</td>
<td>733</td>
<td>58</td>
<td>4%</td>
</tr>
<tr>
<td>Pancake (1)</td>
<td>1716</td>
<td>1716</td>
<td>1437</td>
<td>1262</td>
<td>82%</td>
</tr>
<tr>
<td>Pancake (2)</td>
<td>685808</td>
<td>685808</td>
<td>442382</td>
<td>434923</td>
<td>94%</td>
</tr>
<tr>
<td>Pancake (3)</td>
<td>6550k</td>
<td>6550k</td>
<td>4725k</td>
<td>4717k</td>
<td>98%</td>
</tr>
<tr>
<td>15-Puzzle</td>
<td>12748k</td>
<td>12709k</td>
<td>12067k</td>
<td>11739k</td>
<td>98%</td>
</tr>
<tr>
<td>DAO</td>
<td>11858</td>
<td>11713</td>
<td>12092</td>
<td>11661</td>
<td>55%</td>
</tr>
<tr>
<td>Mazes</td>
<td>6558</td>
<td>6556</td>
<td>6557</td>
<td>6556</td>
<td>75%</td>
</tr>
</tbody>
</table>

Table 1: Comparison of NBS and NBB. Best result in bold.

References


