Proceedings of the Ninth International
Symposium on Combinatorial Search (SoCS 2016)

Calculating the Number of Order-6 Magic Squares with Modular Lifting (Extended Abstract)

Skylar R. Croy, Jeremy A. Hansen, Daniel J. McQuillan
Norwich University
158 Harmon Dr.
Northfield, VT 05663 USA

Introduction

There is no known general formula to calculate the total number of magic squares of a given order. However, using various methods, the number of magic squares has been calculated for orders smaller than 6. In this work we describe an approach to bounding the computational complexity of enumerating order-6 squares and provide a related method to approximate the number of such squares.

Though many similar definitions of magic squares appear in the literature, we restrict ourselves here to the "traditional" definition. A magic square of order n is an $n \times n$ array containing an arrangement of each of the numbers 1 through n^2 such that the numbers in every column, row, and diagonal sum to the same number. This number M is called the *magic constant*, which can be expressed as:

$$M = \frac{n(n^2+1)}{2} \tag{1}$$

When counting magic squares, it is common to disregard trivial transformations of magic squares that also produce magic squares. For example, a magic square may both undergo rotation and reflection while retaining the magic square property. For every magic square, there are seven transformations that include all combinations of rotation and reflection. The below shows the original square (in bold), then reflections across two axes, three rotations, then the remaining unique reflections of the rotations.

$$\begin{pmatrix} \mathbf{A} & \cdots & \mathbf{B} \\ \vdots & \ddots & \vdots \\ \mathbf{C} & \cdots & \mathbf{D} \end{pmatrix} \begin{pmatrix} A & \cdots & C \\ \vdots & \ddots & \vdots \\ B & \cdots & D \end{pmatrix} \begin{pmatrix} D & \cdots & B \\ \vdots & \ddots & \vdots \\ C & \cdots & A \end{pmatrix}$$

$$\begin{pmatrix} C & \cdots & A \\ \vdots & \ddots & \vdots \\ D & \cdots & B \end{pmatrix} \begin{pmatrix} D & \cdots & C \\ \vdots & \ddots & \vdots \\ B & \cdots & A \end{pmatrix} \begin{pmatrix} B & \cdots & D \\ \vdots & \ddots & \vdots \\ A & \cdots & C \end{pmatrix}$$

$$\begin{pmatrix} C & \cdots & D \\ \vdots & \ddots & \vdots \\ A & \cdots & B \end{pmatrix} \begin{pmatrix} B & \cdots & A \\ \vdots & \ddots & \vdots \\ D & \cdots & C \end{pmatrix}$$

Modulo k Magic Squares

A *modulo k magic square* of order n is an $n \times n$ array containing an arrangement of each of the numbers 1 modulo k through n^2 modulo k such that each column, row, and diagonal sum to the magic constant M modulo k, or simply M_k. This magic constant modulo k can be expressed as:

$$M_k = \frac{n(n^2+1)}{2} \bmod k \tag{2}$$

To distinguish between modulo k squares and those described in the previous section, we refer to those from the previous section as "full" magic squares.

Enumerating Order-6 Modulo 2 Magic Squares

Using Formula 2 above, the magic constant modulo 2 for order-6 magic squares is $M_2 = 1$. This suggests that in each row, column, and diagonal, there must be an odd number of elements with the value 1. Because there are an equal number of even and odd numbers, there must be an equal number of elements with the values 0 and 1. Using these shortcuts (and not counting reflections and rotations), we enumerated 278,266 modulo 2 order-6 magic squares.

"Lifting" Modulo k Squares to Modulo $2k$ Squares

The algorithm for enumerating all order-6 magic squares involves starting with modulo 2 magic squares and converting them successively into modulo 4, modulo 8, modulo 16, and modulo 32 magic squares. The modulo 32 magic squares are then converted into full magic squares. Briefly, the algorithm for any n follows the following rough sequence:

1. Enumerate all modulo 2 magic squares.
2. Perform "modular lifting": for each modulo 2 magic square, enumerate all possible "descendant" modulo 4 magic squares. These are squares that satisfy the modulo 2 magic constant *and* satisfy the modulo 4 magic constant.
3. As in the previous step, for each modulo k magic square, enumerate all possible "descendant" modulo $2k$ magic squares. Repeat this step until k is the largest power of 2 less than n^2. (For $n = 6$, we repeat until we have enumerated all modulo 32 magic squares.)

16	32	8	24	4	27
36	20	12	28	2	13
34	18	10	26	22	1
6	9	25	23	19	29
5	17	21	7	31	30
14	15	35	3	33	11

Table 1: The first order-6 magic square discovered by the modular lifting algorithm.

4. For each "final" modulo k magic square, enumerate all full magic squares that satisfy the modulo k magic constant and the magic constant.

The first full magic square found by application of the modular lifting algorithm may be found in Table 1.

To lift a modulo k magic square to a modulo $2k$ magic square, each element in the square has either zero or k added to it. Neither addition would change that the square satisfied the modulo k magic constant (nor any previous magic constant). Because each of the n^2 elements will have one of two values added, lifting requires $O(2^{n^2})$ operations. In an order-6 square, this requires $O(2^{36})$ operations.

Because the number of elements in an order-6 square is divisible by 4, we expect that there are equal numbers of zeroes, ones, twos and threes. In order to construct such a square from the modulo 2 squares, there will be an equal number of elements that have zero added as have two added. This means that we can lower the bound on the number of possible modulo 4 descendants to $\binom{n^2}{\frac{n^2}{2}}$. In the case of order-6 squares, this makes the new upper bound roughly 2^{33}. We can continue to improve this bound by considering all the ways that individual rows, columns, and diagonals change during lifting, though space considerations prevent a deeper explanation.

Estimating the number of 6×6 Magic Squares

A weak upper bound on the number of magic squares of of $\frac{(n^2)!}{8(2n+1)!}$ was established in (Ward 1980). For order-6 magic squares, this provides an upper bound of approximately 7.467×10^{30}. Using a Monte Carlo simulation, the number of order-6 magic squares was estimated to be $(1.7745 \pm 0.0016) \times 10^{19}$ (Pinn and Wieczerkowski 1998). Unfortunately, a statistical methods like Monte Carlo simulations do not provide a concrete number of magic squares of a given order, or even bound that number. Further, it does not provide a method for enumerating the squares.

While we have not enumerated all order-6 squares, our initial results are promising. To get a sense of how many descendants each lifting might produce, we lifted the first 85 modulo 2 magic squares to modulo 4. The number of descendants ranged from 574,000 to 589,152. A handful of iterations of lifting at each level was conducted, and Table 2 shows the minimum, maximum, and mean number of descendants for each lifting level.

We do not rule out the possibility that these small samples serve as poor representatives for the remaining squares, though the closeness of the minima and maxima and how close our approximation below is to the Monte Carlo results suggest that the means will not shift much as additional squares are produced.

Lift from	Lift to	Mean	Min.	Max.
mod 2	mod 4	577,081	574,000	589,152
mod 4	mod 8	61,590	61,431	61,639
mod 8	mod 16	3,027	2,958	3,143
mod 16	mod 32	105	78	168

Table 2: The approximate mean, minimum, and maximum number of descendants at each lifting level, rounded to the nearest integer.

Using the approximate means in Table 2, we can calculate an approximation of the number of full order-6 magic squares if we assume that true magic squares are distributed uniformly across all modulo 32 magic squares and modulo $2k$ magic squares are distributed uniformly across all modulo k magic squares. With these assumptions, the approximate number is simply the number of modulo 2 squares (which we know precisely) multiplied by each of the mean number of modulo 4 squares, modulo 8 squares, modulo 16 squares, and modulo 32 squares. This total is then multiplied by the frequency of true magic squares per modulo 32 square, approximately 0.27%.

Our approximate number of full order-6 magic squares is then 8.6×10^{18}. Even with the small number of "lifts", we note that this figure is strikingly close to that produced by the Monte Carlo simulation of (Pinn and Wieczerkowski 1998). We expect that our approximation will become more accurate as more modulo k magic squares are produced.

Counting with Distributed Computing

We are in the process of launching a distributed computing project using BOINC (Anderson 2004) to enlist volunteers' idle computing resources to calculate the number of order-6 magic squares and continue to improve our approximation. While the software currently lacks a number of shortcuts identified since it was written, and has not been fully optimized to take advantage of modern hardware, we are confident that with these improvements and a robust volunteer effort, it could produce a precise count of order-6 magic squares in a reasonable amount of time, on the order of years.

References

Anderson, D. P. 2004. Boinc: A system for public-resource computing and storage. In *Grid Computing, 2004. Proceedings. Fifth IEEE/ACM International Workshop on*, 4–10. IEEE.

Pinn, K., and Wieczerkowski, C. 1998. Number of magic squares from parallel tempering monte carlo. *International Journal of Modern Physics C* 9(04):541–546.

Ward, J. E. 1980. Vector spaces of magic squares. *Mathematics Magazine* 53(2):108–111.