The Directed Closure Process in Hybrid Social-Information Networks, with an Analysis of Link Formation on Twitter

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Abstract

It has often been taken as a working assumption that directed links in information networks are frequently formed by "short-cutting" a two-step path between the source and the destination — a kind of implicit "link copying" analogous to the process of triadic closure in social networks. Despite the role of this assumption in theoretical models such as preferential attachment, it has received very little direct empirical investigation. Here we develop a formalization and methodology for studying this type of directed closure process, and we provide evidence for its important role in the formation of links on Twitter. We then analyze a sequence of models designed to capture the structural phenomena related to directed closure that we observe in the Twitter data.

Introduction

Information networks, which connect Web pages or other units of information, and social networks, which connect people, are related notions, but they exhibit fundamental differences. Two of the principal differences are based on directionality and heterogeneity. First, information networks are generally directed structures, with links created by one author to point to another; social networks, on the other hand, tend to be represented in most basic settings as undirected structures, expressing relationships that are approximately mutual. Second, information networks tend to contain a few nodes with extremely large numbers of incoming edges — documents or pages that are "famous" and hence widely referenced —- while social networks exhibit disparities in connectivity only to a smaller extent, since even the most gregarious people have some practical limit on the number of genuine social ties they can form.

The link structure of the Web, and of well-defined subsets of the Web such as the blogosphere and Wikipedia, are clear examples of information networks; social-networking sites such as Facebook have provided us with very large representations of social networks that are derived from social struc-
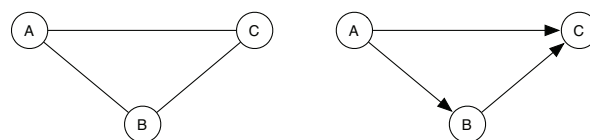
(a) *Undirected feed-forward triangle* (b) *Directed "feed-forward" triangle*

Figure 1: (a) Triadic closure in an undirected graph produces a triangle when an edge connects two nodes who already have a common neighbor. (b) Analogously, in a directed information network, directed closure occurs when a node A links to a node C to which it already has a two-step path (through a node B). This creates a directed triangle (a "feed-forward" structure on three nodes).

ture in the off-line world. An interesting recent development has been the growth of social media sites that increasingly interpolate between the properties of information networks and social networks. The micro-blogging site Twitter is a compelling example of such an interpolation. A user on Twitter is able to create links to other users whose content he or she is interested in; this is referred to as *following* these users, and the set of all such follower relations defines a network. The structure of this network reflects properties both of a social network, since it exposes underlying friendship relations among people, and also of an information network, since it is directed and also contains huge concentrations of links to specific "celebrities" and automated generators of news content that reflect fundamentally informational relations.

Link Formation in Information Networks. In a social network, triadic closure is one of the fundamental processes of link formation: there is an increased chance that a friendship will form between two people if they already have a friend in common (Rapoport 1953; Granovetter 1973). (For example, we could imagine the A-C friendship in Figure 1(a) as forming after the existence of the A-B and B-C edges, and accelerated by the existence of these two edges.) Recent empirical analysis has quantified this effect on large social network datasets (Kossinets and Watts 2006). Is there
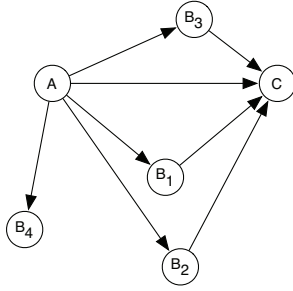
Figure 2: In this example, the edge from A to C exhibits closure if there is already a two-step path from A to C (i.e., through B_1, B_2, B_3) when the A-C edge arrives.

an analogous process in information networks?

A natural hypothesis for such a process is the following: if a node A in an information network links to B, and B links to C, then one should arguably expect an increased likelihood that A will link to C — since the author of A has an increased ability to become aware of C via the two-step path through B. (See Figure 1(b).) We will refer to this as the *directed closure process*. In addition to its intuitive appeal, this process contains an implicit hypothesis about how links are formed in information networks — through the "copying" of a link from something you already point to — and such copying mechanisms form a crucial part of the motivation for the fundamental notion of preferential attachment (Albert and Barabási 2002; Kumar et al. 2000; Newman 2003). Despite the importance of the notion, however, there has been remarkably little empirical analysis of the extent to which this type of directed closure is truly at work in real information networks, and of the effects it may have on network structure.

The Present Work: The Directed Closure Process. In this paper, we analyze the directed closure process using data from Twitter: we provide some of the first evidence on large information networks that directed closure is taking place at a rate significantly above what would be expected by chance; we identify a surprising level of heterogeneity in how strongly it operates across different parts of the network; and we analyze models that capture these effects.

An important difference between triadic closure in social networks and directed closure in information networks is the following observation, which in a sense serves as the starting point for our analysis: while the extent of triadic closure can be assessed from a single snapshot of an undirected graph, the evaluation of directed closure inherently requires some form of temporal sequence information. Indeed, when we see an undirected triangle such as the one in Figure 1(a), we know that whichever edge formed last will complete a two-step path consisting of the earlier two edges, and hence will satisfy the definition of triadic closure. On the other hand, the structure in Figure 1(b) satisfies the definition of directed closure only if the A-C edge formed after the other two.

This means that the amount of directed closure in a di-

rected graph depends not just on the graph's structure, but also on the order in which edges arrive. Because of this we are able to develop a natural *randomization test* to evaluate whether directed closure is taking place in a given network at a rate above chance. Specifically, we say that an edge in a directed graph *exhibits closure* if, at the time it forms, it completes a directed two-step path between its endpoints. For example, in Figure 2, the A-C edge exhibits closure if and only if it arrives after the pair of edges in one of the three possible two-step A-C paths through B_1, B_2, or B_3. For a given network, we can thus ask: how many edges exhibit closure, and how many would have exhibited closure (in expectation) if the edges had arrived in a random order? The point is that in any arrival order of the edges, some number of the edges will close directed triangles; but if directed closure is a significant effect, then we may expect to see a larger number of such triangle-closings compared to what we'd see under a random arrival order.

To investigate this empirically, we choose a random sample of *micro-celebrities* on Twitter, which we define to be users with between 10,000 and 50,000 followers. (We will abbreviate the term as μ-*celebrity*.) For each such μ-celebrity C, we determine the number of edges to C that exhibit closure, and compare it to the expected number of edges to C that would exhibit closure in a random ordering — we will refer to this latter number as the *random-ordering baseline*. Given that we are studying the followers of users with high numbers of in-links, one would conjecture that there are two competing forces at work. In one direction is the intuitively natural tendency of directed closure to create short-cuts in the presence of two-step paths. In the other direction, however, is the plausible tendency for people to link first to celebrities, before they link to more obscure users; that is, it is not clear that closure processes are necessary in order for people to discover and link to very prominent users. This latter effect would tend to cause triangles as in Figure 1(b) to appear with the A-C and B-C edges first, reducing the extent of directed closure in the real data.

We find in the Twitter data that the number of edges to a μ-celebrity that exhibit closure is higher than the random-ordering baseline, indicating that even in linking to celebrities, there is an above-chance tendency to do this by closing an existing two-step path. This finding suggests a range of further interesting questions — specifically, whether the high rate of directed closure is due to overt copying of follower lists (as in the intuitive basis for the definition), or due to more subtle, implicit mechanisms that produce copying behavior at a macroscopic level. To address this question, as we discuss below, we consider the extent to which directed closure can arise even in models that do not explicitly build in copying as a mechanism.

The Present Work: Directed Closure and Network Structure. Given the prevalence of directed closure in the Twitter network, one might suppose that it operates according to a relatively uniform underlying mechanism. But what we find, surprisingly, is significant heterogeneity in the amount of directed closure. We define the *closure ratio* of

a μ-celebrity C to be the fraction of C's incoming edges that exhibit closure. If we track the closure ratio of C as edges to C are added in their temporal order, we find that the ratio stabilizes to an approximately constant value fairly early. However, the value to which the closure ratio stabilizes varies considerably from one μ-celebrity to another, and is not closely related to the number of followers. Thus, the closure ratio appears to be an intrinsic and diverse property of users with large numbers of followers: some such users receive a clear majority of their incoming links via the closing of a directed triangle, while others receive a much smaller proportion of their links this way.

The cause of this is at some level a mystery, but to get a better understanding we look at the predictions of some basic network formation models. We present a heuristic calculation based on the preferential attachment model, suggesting that a user's closure ratio should be related to the sum of the in-degrees of the user's followers, and we find on the Twitter data that the closure ratio indeed follows this quantity more closely than simpler quantities such as the user's own number of followers. However, preferential attachment is not able to explain either the diversity of different closure ratios, or the fact that they can be large on nodes of small in-degree; to understand these effects better, we analyze more complex models that do not incorporate copying as an overt or explicit mechanism in link formation, including preferential attachment with fitness (Bianconi and Barabási 2001) and a version of preferential attachment with embedded community structure which is related to a model of Menczer (2002).

We also note that the closure ratio of a user is distinct from — and exhibits qualitatively different properties than — the *clustering coefficient* (Watts and Strogatz 1998). The clustering coefficient is the fraction of pairs in a node's neighborhood that are directly linked, and in the neighborhood of a high-degree node it is almost always a small quantity, for the fundamental reason that most of a high-degree node's neighbors don't have enough incident edges to produce a significant clustering coefficient (Vazquez 2003). The closure ratio, on the other hand, is a quantity that can be quite large even for the neighborhoods of nodes with extremely large degrees.

Twitter Data and Micro-Celebrities

We collected a random sample of μ-celebrities on Twitter, each with between 10,000 and 50,000 followers. For each of these μ-celebrities C, we determine the subset of edges to C that exhibit closure.

It is an interesting fact that determining this subset does not require exact time-stamps or full network structure. Rather, it is enough to have a chronologically ordered list $L_{in}(C)$ of the followers of C, and for each user $A \in L_{in}(C)$, a chronologically ordered list $L_{out}(A)$ of the users that A follows.[1] From these lists, we can conclude that an edge from A to C exhibits closure if and only if there exists

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[1]Such ordered lists were available via the Twitter API at the time we performed these analyses (Kalucki 2009).
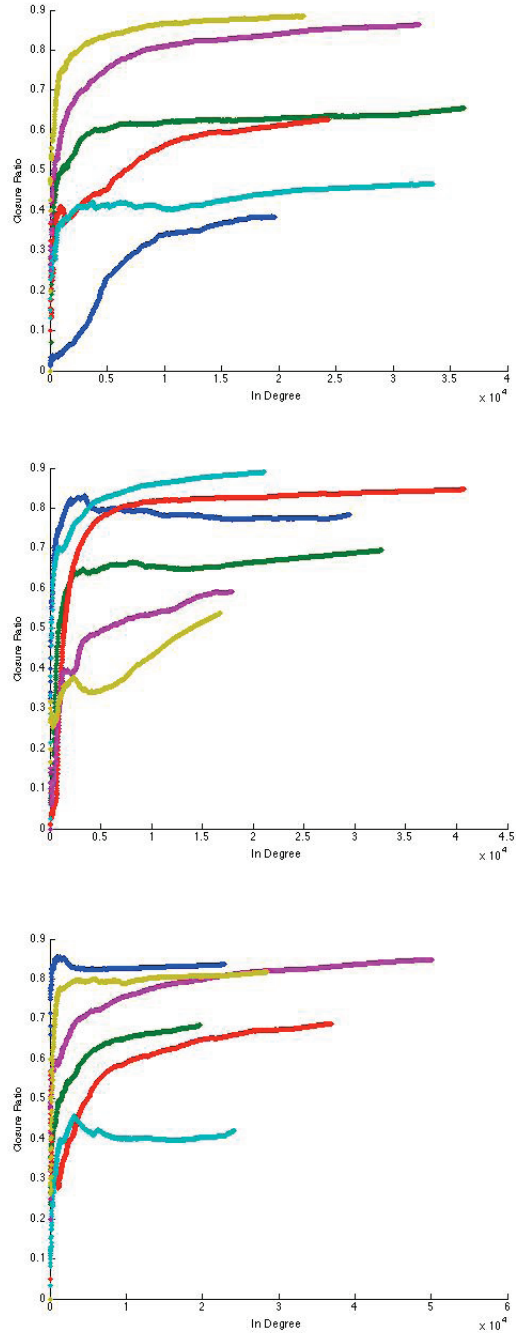

Figure 3: Closure ratio as a function of the arrival order of incoming edges for 18 Twitter μ-celebrities. The following are the professions of the μ-celebrities in each figure (from top to bottom curve). Top figure: Journalist, Venture Capital Blogger, Actor, Actor, DJ, Skateboarder. Middle figure: Comedian, Film Producer, Social Media Blogger, Musician, Actor, Journalist. Bottom figure: Comedian, TV Presenter, Actor, Musician, Filmmaker, Actor.

a B such that B precedes A in $L_{in}(C)$ and B precedes C in $L_{out}(A)$.

In Figure 3, we show the running fraction of edges that exhibit closure as the followers of a μ-celebrity C arrive in chronological order. As noted in the introduction, in most cases this fraction reaches a relatively stable value quite quickly, and this stable value varies a lot from one μ-celebrity to another. Our models in the subsequent sections will help us investigate this phenomenon.

Evidence for Directed Closure

We now use the randomization test described in the introduction to identify evidence for the directed closure process at work. We take the subgraph induced on the nodes in $\{C\} \cup L_{in}(C)$, and we insert the edges in an order selected uniformly at random from among all permutations of the edges.

Specifically, we say that a user A is *k-linked* to a user C if A follows C, and A also follows k followers of C. (For example, in Figure 2, A is 3-linked to C.) Let $S_k(C)$ denote the set of all users who are k-linked to C, and let f_k denote the fraction of users in $S_k(C)$ whose edge to C exhibits closure.

Now, for each k with $|S_k| > 10$, we approximate the expected value of f_k under the assumption that the order in which the edges are created is chosen uniformly at random. To do this, we run a simulation in which we generate a network consisting simply of a node A pointing to a node C and to k other nodes which also point to C; we randomly choose $|S_k|$ different orderings of the edges of this network (one corresponding to each of the $|S_k|$ followers who are k-linked to the real μ-celebrity); and we then determine the fraction of these random orderings in which the A-C edge exhibit closure. We approximate the expected value of f_k over randomly ordered edges by the average closure ratio among 100 runs of this simulation, and we define error bars using the minimum and the maximum fraction among the 100 simulations.

We find the same trend for all the μ-celebrities in our sample, as shown in Figure 4: there is some K such that for all $k < K$ the actual value of f_k is higher than the maximum fraction from the 100 simulations. This means that at least for small values of k the fraction of edges exhibiting closure is much higher than expected by chance. This suggests the existence of an underlying mechanism — copying of links or something producing similar observed behavior — that makes it more likely than chance to see edges that appear to be copied. For large values of k, the expected value of f_k assuming random ordering of edges becomes very large, and it is hard for the values observed in the data to lie above the error bars; we find that for large k, the actual value of f_k is very close to the average fraction among the 100 simulations and is inside the error bars.

Preferential attachment

We would like to use probabilistic models of network formation to investigate the following two fundamental properties
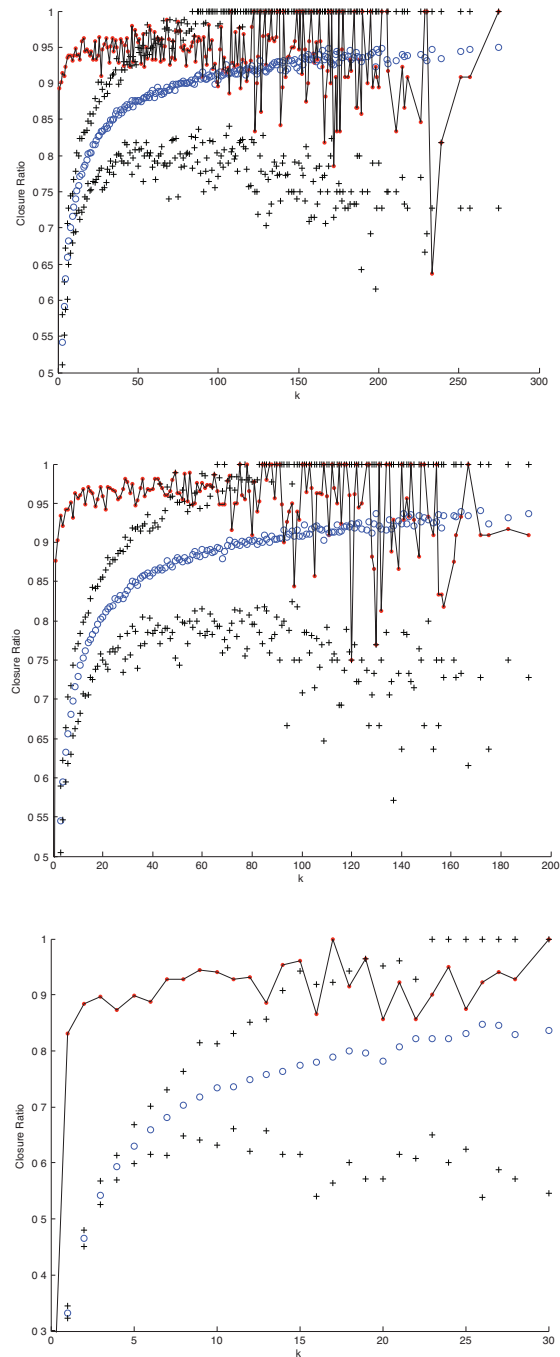

Figure 4: The connected dots indicate the actual value of f_k, the circles indicate the average closure ratio among the 100 simulations, and the plus signs indicate the error bars. Results for 3 μ-celebrities are shown. The trend is similar for all other μ-celebrities
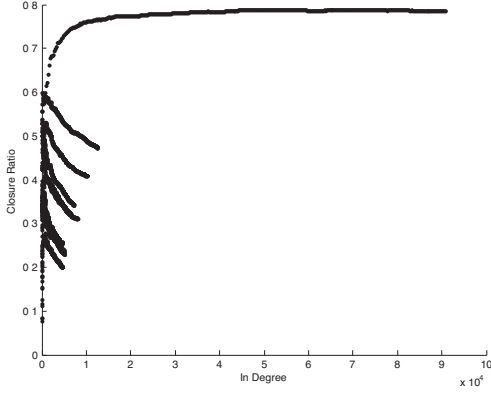
Figure 5: Results from the preferential attachment simulation with $N = 200,000$, $\alpha = .3$, and $D = 10$. The figure shows the closure ratio as a function of edge arrival order of the 10 nodes with highest in-degree.

of directed closure in the data. First, for nodes whose in-degrees are at the level of μ-celebrities, the closure ratio saturates to a constant f as edges arrive over time. Second, this constant f is quite different for different μ-celebrities, and it is not closely related to the total in-degree of the μ-celebrity.

We now compare this with the predictions of a sequence of increasingly complex models. We begin with a very basic model — a variant of the standard *preferential attachment* process, defined as follows (Albert and Barabási 2002; Newman 2003):

- Fix $\alpha \in [0,1]$, and $D, N \in \mathbb{N}$. The graph will have N nodes labeled $0, 1, 2, ..., N-1$.

- Initially (at $t = 0$) the graph consists of node labeled 1 with an edge pointing to the node labeled 0.

- At each time step ($t = j$) node j will join the graph with D edges directed to nodes chosen from a distribution on $1, 2, ...j - 1$. The endpoint of each edge is chosen in the following way: With probability α the endpoint is chosen uniformly at random from $\{1, 2, ..., j - 1\}$. With probability $1 - \alpha$ the endpoint is chosen at random from a probability distribution which weights nodes by their current in-degree.

We run this process with different values of α, D, and N and find that preferential attachment does not achieve the desired results for μ-celebrities. In our simulations, only nodes with very large in-degree have a reasonably large closure ratio, while for other nodes it is essentially zero. For those nodes with very large in-degree, the closure ratio saturates to a constant f as edges arrive, and the value of f is different for different nodes. However, the value of f is monotonically increasing as the final in-degree of node increases (See Figure 5).

Through a heuristic calculation we now estimate the expected closure fraction of a node in a graph generated by the preferential attachment process.

Let E_t be the total number of edges at time t, N_t be the total number of nodes at time t, $d_t(j)$ be the in-degree of node j at time t,

$$F_t(j) = \{x : \exists e = (x, j) \text{ at time } t\},$$

$$d_t(S) = \sum_{x \in S} d_t(x), \text{ and}$$

$$S_t(j) = \alpha \frac{|F_t(j)|}{N_t} + (1 - \alpha) \frac{d_t(F_t(j))}{E_t}.$$

Note that $S_t(j)$ is the probability that a particular edge from node $t + 1$ is directed to a node k such that there is an edge from k to j. In other words it is the probability that an edge from node $t + 1$ is directed to a node that points to j.

Fix a node j and an edge e coming out of node $t + 1$. We would like to calculate the probability of the following event V: There is another edge $e' = (t + 1, x)$ created before e such that x points to j (i.e $\exists$ edge $g = (x, j)$). We will use $C_{t,e}(j)$ to denote the probability of this event V. Note that we do not know which of the D edges coming out of $t + 1$ the edge e is, or what the destination of e is. Note that if e is the first edge coming out of $t + 1$ then the event V cannot happen; if e is the second edge coming out of $t + 1$ then $C_{t,e}(j) = S_t(j)$, if e is the third edge coming out of $t + 1$ then $C_{t,e}(j) = [1 - (1 - S_t(j))^2]$, and more generally if e is the dth edge coming out of $t + 1$ then $C_{t,e}(j) = [1 - (1 - S_t(j))^{d-1}]$. Since it is equally likely that e is any of the D edges coming out of $t + 1$ we write

$$
\begin{aligned}
C_{t,e}(j) &= \frac{1}{D}[1 - (1 - S_t(j))] + \frac{1}{D}[1 - (1 - S_t(j))^2] + \\
&\quad \cdots + \frac{1}{D}[1 - (1 - S_t(j))^{D-1}] \\
&= 1 - \frac{1 - (1 - S_t(j))^D}{D S_t(j)}.
\end{aligned}
$$

If we knew that edge e pointed to node j then the event V exactly says that e exhibits closure. Therefore if we want to know the probability that e exhibits closure given that $e = (t + 1, j)$ we would need to calculate $P(V|e = (t + 1, j))$. For the sake of our approximation, we use the unconditional probability $P(V) = C_{t,e}(j)$ instead as our estimate of the probability that e exhibits closure. Note that the quantity $C_{t,e}(j)$ only depends on j and t, so we define $C_t(j) = 1 - \frac{1-(1-S_t(j))^D}{DS_t(j)}$. In general, a given edge $e = (x, y)$ exhibits closure with a probability of approximately $C_{x-1}(y)$. If $\lim_{t \to \infty} C_t(j) = L < \infty$ then, for a large enough T, if $t > T$ then $C_t(j) \approx L$. In other words, if $t > T$ the probability that an edge coming out of node t directed to node j exhibits closure is approximately L, which in turn is approximately $C_t(j)$. Therefore, if $\lim_{t \to \infty} C_t(j) = L < \infty$ and our parameter N is large enough then $C_t(j) \approx C_{N-1}(j)$ for $t > T$. Hence, if N is large enough the final closure ratio of node j is approximately $C_{N-1}(j)$.

In Figure 6 we show that despite the approximations made in this argument, the calculation is a close fit to the actual closure ratios.
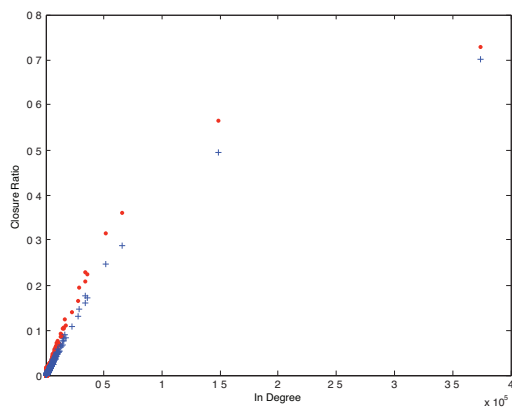
142

Figure 6: The actual closure ratio of each node j generated by the preferential attachment model with parameters $N = 200,000$, $\alpha = .3$, and $D = 10$ (dots) and its approximation by $C_{N-1}(j)$ (plus signs).

Preferential Attachment with Fitness

The fact that preferential attachment produces very few nodes with non-trivial closure ratios, and that these closure ratios are closely tied to the in-degrees, indicates the need for a more complex model. One alternative would be the use of *copying models* (Kumar et al. 2000; Vazquez 2000), where nodes explicitly copy links from other nodes that have already joined the network. Such a mechanism builds copying into the model, generally with a tunable parameter that could be used to control quantities such as the closure ratio. However, we would like to understand whether non-trivial closure ratios — and in particular, high levels of diversity in closure ratios — can also appear in networks arising from models that do not explicitly define copying as a mechanism. As a first step in this direction, we investigate an extension of preferential attachment incorporating the idea that different nodes may have different levels of inherent *fitness* or *attractiveness*, which affects how strongly they attract links (Bianconi and Barabási 2001).

Here is how this model works:

- Fix $\alpha \in [0, 1]$, and $D, N \in \mathbb{N}$. The graph will have N nodes labeled $0, 1, 2, ..., N - 1$.

- Each node also has a fitness parameter $f_i \in (0, 1)$ chosen uniformly at random.

- Initially (at $t = 0$) the graph consists of node labeled 1 with an edge pointing to the node labeled 0.

- At each time step ($t = j$) node j will join the graph with D edges directed to nodes chosen from a distribution on $1, 2, ...j - 1$. The endpoint of each edge is chosen in the following way: With probability α the endpoint is chosen uniformly at random from $\{1, 2, ..., j-1\}$. With probability $1-\alpha$ the endpoint is chosen at random from a probability distribution which weights each node i by $d_i f_i$, where d_i is the node's current in-degree.
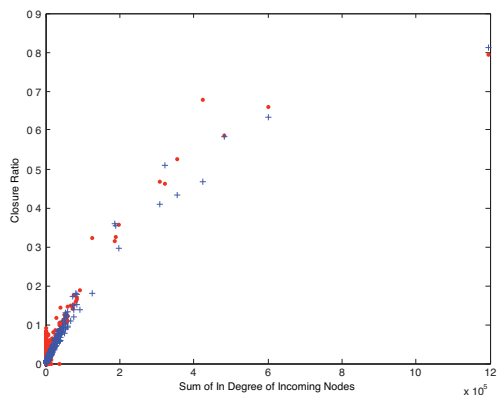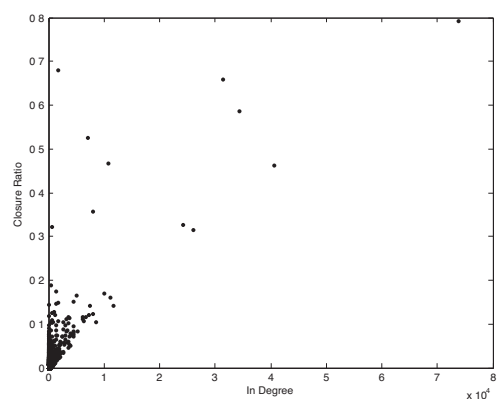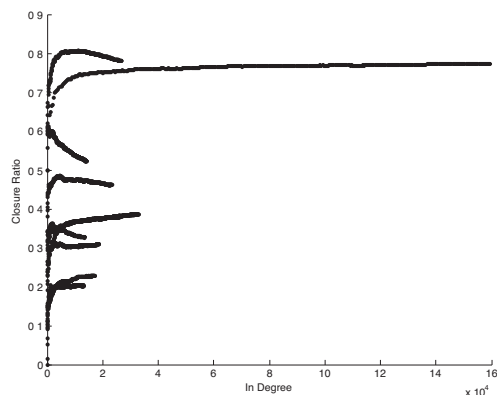

Figure 7: Results from the preferential attachment with fitness simulation with $N = 200,000$, $\alpha = .3$, and $D = 10$. The top figure shows the closure ratio as a function of in-degree of the 10 nodes with highest in-degree. The bottom function shows the final closure ratio of each node j (dots) and its approximation by $C_{N-1}(j)$ (plus signs).
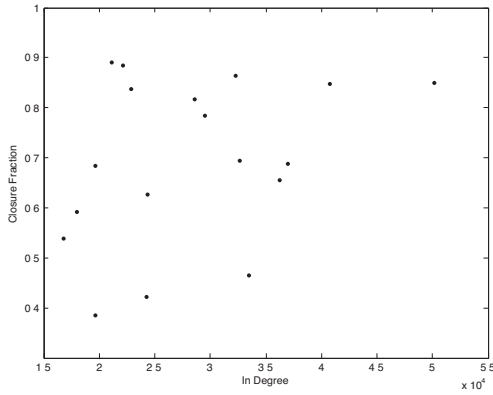
143

Figure 8: Closure ratio as a function of In-Degree.

Figure 9: Closure ratio as a function of the Sum of In-Degree of Incoming Nodes.

We run simulations of preferential attachment with fitness, with different parameters, and find an improvement from the simple preferential attachment model. A node's final closure ratio is not correlated with the final in-degree of the node, which matches what we found in our data set. However, just like in the simple preferential attachment model, very few nodes have a closure fraction that is non-trivially larger than 0 (see Figure 7). In particular, for the nodes that would correspond to μ-celebrities, the fraction is basically zero. This is not consistent with the data, which shows that μ-celebrities can have very large closure ratios.

We find that the heuristic calculation for the closure ratio we derived for the preferential attachment model is very accurate for preferential attachment with fitness as well. Furthermore, from the calculation we see that for a node j the term $d_t(F_{N-1}(j))$ (the sum of the in-degree of nodes that point to j) is the most important in determining the closure ratio when α is small. For preferential attachment with fitness, the closure ratio of a node j is much more correlated with $d_t(F_{N-1}(j))$ than with the in-degree of j (see Figure 7). This is also the case for the μ-celebrities in our data set (see Figures 8 and 9), which means that in determining a user's closure ratio, the more important variable seems to be not the number of followers the user has but the total number of followers of those who follow the user.

Preferential Attachment with Communities

The previous model, incorporating fitness, manages to produce heterogeneity in the closure ratios, but it still only produces very few nodes for which the closure ratios are non-trivial. We now present a model in which many nodes will have non-trivial closure ratios.

The model is *preferential attachment with communities*: we assume that each node belongs to a particular community of nodes, and the node is more likely to attach to nodes from its own community than to nodes from other communities. Specifically:

- Fix $\alpha \in [0, 1], \beta \in [.5, 1]$, and C, D, and $N \in \mathbb{N}$. The graph will have N nodes labeled $0, 1, 2, ..., N - 1$ and there will be C communities.
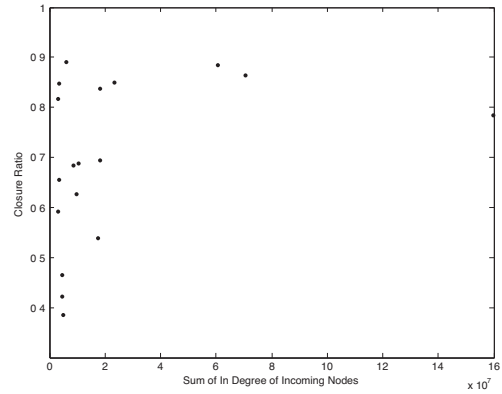
- Initially (at $t = 0$) the graph consists of the C communities, each with two nodes, one pointing at the other.

- At each time step ($t = j$) node j will join the graph and will be assigned a community uniformly at random. Then j will create D edges directed to nodes chosen from a distribution on $1, 2, ...j - 1$. The endpoint of each edge is chosen in the following way: With probability β the endpoint will be a node from the same community as j; with probability $1 - \beta$ the endpoint will be chosen from any of $1, 2, ...j - 1$. With probability α the endpoint will be chosen preferentially (i.e. at random from a probability distribution which weights nodes by their current in-degree) and with probability $1 - \alpha$ the endpoint will be chosen uniformly at random from the set of nodes already determined.

Simulations with different parameters show that this model generates nodes whose closure ratios converge as in-degree increases (see Figure 10), and the final fraction is not closely related to the in-degree as it was in the case of simple preferential attachment. Furthermore, the nodes that would correspond to a μ-celebrity level of in-degree can have reasonably large closure ratios.

It is also interesting to note that the sum of a node's followers' in-degrees, an important parameter in the previous two models, still plays a role here, but with a twist: as Figure 11 shows, a node's closure ratio is more closely correlated with the sum of in-degrees of the followers *from its own community* than with the sum of the in-degrees of all its followers. It would be interesting to explore this quantity on the Twitter data, using different approximations of community structure in Twitter.

Conclusion

We have studied the process of directed closure in information networks, developing a definition and methodology for evaluating it, and providing evidence for directed closure in the follower network of Twitter. We also found that the extent of directed closure varies considerably between the sets
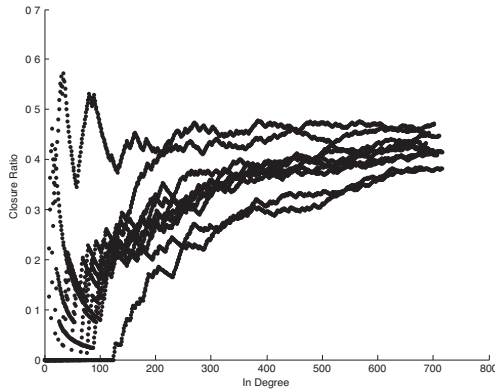
Figure 10: The closure ratio as a function of in-degree for the 10 nodes with highest in-degree. Preferential attachment with communities simulation with $N = 200,000$, $\alpha = .3$, $\beta = .8$, $C = 1,000$, and $D = 10$.
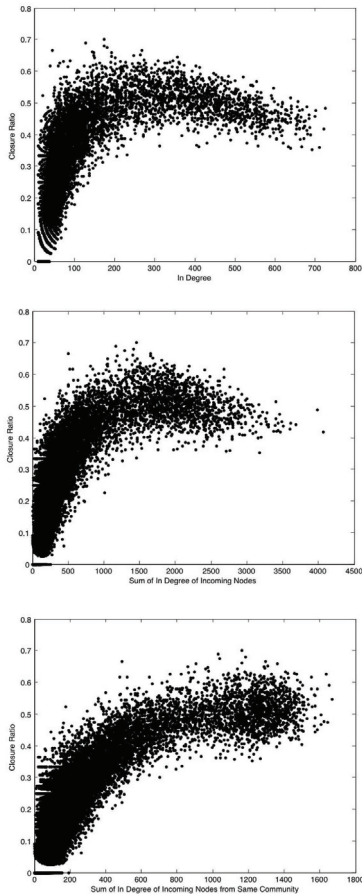

Figure 11: Results from the preferential attachment with communities simulation with $N = 200,000$, $\alpha = .3$, $\beta = .8$, $C = 1000$, and $D = 10$.

of followers of different popular users. A sequence of models generalizing the principle of preferential attachment provide some explanation for our findings, and identify a more subtle parameter — the sum of the in-degrees of one's followers — that is related to the extent of directed closure.

It is an interesting direction for further work to try understanding better the causes of heterogeneity in the closure ratios of micro-celebrities on Twitter, and the extent to which identifying communities in the Twitter network structure can help evaluate the more detailed predictions of preferential attachment with communities. It will also be interesting to explore comparative analyses of these measures on other information networks.

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