

Achieving Stable and Fair Profit Allocation with Minimum Subsidy in Collaborative Logistics

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Abstract

With the advent of e-commerce, logistics providers are faced with the challenge of handling fluctuating and sparsely distributed demand, which raises their operational costs significantly. As a result, horizontal cooperation are gaining momentum around the world. One of the major impediments, however, is the lack of stable and fair profit sharing mechanism. In this paper, we address this problem using the framework of computational cooperative games. We first present cooperative vehicle routing game as a model for collaborative logistics operations. Using the axioms of Shapley value as the conditions for fairness, we show that a stable, fair and budget balanced allocation does not exist in many instances of the game. By relaxing budget balance, we then propose an allocation scheme based on the normalized Shapley value. We show that this scheme maintains stability and fairness while requiring minimum subsidy. Finally, using numerical experiments we demonstrate the feasibility of the scheme under various settings.

Introduction

With the prevalence of e-commerce in almost all categories of retail industry, great pressure for changes has been mounted for logistics service providers (LSPs). An obvious challenge is the increasingly unpredictable and fluctuating demands. To adapt to this new trend, we are beginning to see various form of horizontal cooperations among LSPs around the world. Such horizontal cooperations help to reduce empty mileage, increase quality of service, enable strategic inventory repositioning, and increase environmental sustainability. Despite these obvious benefits, many LSPs still choose not to join the collaboration, mainly due to the lack of fair allocation mechanisms (Cruijssen, Cools, and Dullaert 2007). In this work, we seek to address this problem using the framework of cooperative game theory.

To help readers understand the context of the problem and our motivation, we first explain how horizontal cooperation works in urban (sometimes known as last-mile) logistics. In our setting, there are multiple independent LSPs owning vehicle fleets and customer demands, and agree to form coalitions by sharing their demands as well as fleet capacities to deliver such demands. A designated coalition

manager then generates plans for assigning demands to vehicles according to some mutually agreed operational rules. Finally, the coalition manager also decides how revenue should be split among coalition members (e.g., if one of A's customers is served by B's vehicle, how should A and B split the revenue). From the regulator's point of view, this is highly desirable as coalitions can lead to higher vehicle utilization, thus reducing road congestion and pollution. However, coalition is not always desirable for individual LSPs, as they might be concerned about privacy, competitiveness, and most importantly, the fairness of how benefits are distributed among coalition members. In other words, for an individual LSP, even when it could benefit from a coalition, if it perceives that other members receive more benefits than they deserve out of this coalition, it might decide to leave the coalition.

Summarizing the above problem context, we can see that there are two important decision criteria that we need to satisfy: 1) coalitions must bring sufficient benefits to prevent agents from leaving; and 2) the distribution of benefits has to be perceived as fair by agents. An ideal theoretical framework that considers both decision making criteria is cooperative game theory, which studies interactions of players in an environment where they can form coalitions to generate *surplus* that can then be allocated back to their members. Formally speaking, the solution of the game is a vector of payment amounts for all agents, and the properties we are seeking are stability and fairness. When a solution is said to be stable, no subset of agents should have incentives to break away and operate independently. Solutions satisfying stability criterion are said to be in the *core*, which is a well-studied solution concept. The fairness criterion, on the other hand, is characterized by the notion of *Shapley value* (Shapley 1953), which is a unique vector of payments that reflects the relative importance of individual agents. It essentially ensures that the payment received by an agent reflects the normalized value he brings to all potential coalitions.

The above idea of combining the core and the Shapley value seems straightforward, however, its implementation could potentially be very difficult and challenging. In addressing these challenges, we make the following contributions. First, we show that under our model, stability can always be achieved, i.e., the core is nonempty. The Shapley value, however, might not be in the core. To address

this, we alter the game structure by introducing subsidy into the system. In other words, we relax the budget balance property and formulate this implementation problem as an optimization problem, to find an allocation scheme that is stable and in proportions to the Shapley value, while requiring minimum subsidy. We then propose δ -BSA, a simple iterative-based scheme that satisfies these requirements. Second, when evaluating the value of a particular coalition, we actually have to solve the vehicle routing problem that pools all demand points and the fleet depots (which are owned by individual LSPs in this coalition). To address this, we present an MILP model for computing coalition values, which is an extension of VRP to multiple depots. By using synthetic data, we show the feasibility of the proposed scheme by comparing required subsidies to the surplus generated by grand coalitions.

Related Works

Cooperative games in the context of single-vehicle routing problems (VRP) can be traced back to the 1980s in the operations research literature. Games in this category include variants of traveling salesman games (Fishburn and Pollak 1983; Potters, Curiel, and Tijds 1992) and Chinese postman games (Hamers et al. 1999). In these games, players are associated with edges or vertices of the graph, and they are concerned with fair allocations of the cost of optimal routes. In the context of multi-vehicle routing games, Göthe-Lundgren, Jörnsten, and Värbrand (1996) studied the routing of multiple vehicles from a single depot, and is concerned with the allocation of the cost to the customers. Krajewska et al. (2008) studied the multi-depot VRP, and define each player to represent a single VRP; and Shapley value is used as the allocation scheme which may not be stable.

Profit sharing has also been studied as a cooperative game in the context of cooperative truckload delivery (CTLD) by Hezarkhani, Slikker, and Van Woensel (2015) where a set of formal properties have been proposed that have the ability to capture the notions of fairness and/or competitiveness with regard to allocations in CTLD situations, as well as a solution that satisfies such properties.

Among many solution concepts in cooperative game theory, the Shapley value (Shapley 1953) is one of the most well-known and well-adopted axiomatization approach. By formulating a set of axioms that uniquely characterizes a solution concept, one hopes to demonstrate the “reasonableness” of the solution. In other words, if players agree that these axioms are reasonable, then they should accept the solution concept uniquely defined by them. One of the interesting aspects of this approach is that a solution concept can be characterized by seemingly very different sets of axioms, like in the case of Shapley value (Young 1985; Hart and Mas-Colell 1989; Chun 1989). Examples of axiomatizations of other solution concepts include variants of the core (Peleg 1985) and the nucleolus (Sudhölter 1997). In this paper, we demonstrate the reasonableness of our proposed scheme by describing desirable properties from the core and the Shapley value that are satisfied. We leave the full axiomatizations of the scheme to future works.

Most formulations of computational cooperative games involve hard optimization problems; in our case, the VRP. This issue is aggravated by the fact that the naive approach to compute the Shapley value involves solving $O(2^n)$ instances of hard problems where n is the number of players. As such, exact methods can only be applied in special cases (Conitzer and Sandholm 2004; Michalak et al. 2013), while approximations are inevitable for more general cases (Fatima, Wooldridge, and Jennings 2007; 2008; Soufiani et al. 2014).

Since computing the core and the Shapley value is intractable in most real-world applications, there are also efforts in looking into relaxed versions of these concepts. Such approach is further justified by the fact that most agents are computational bounded anyway, and they can only compute value functions approximately. Sandholm and Lesser (1997) explored such relaxations. Similarly, Li and Conitzer (2015) explored the non-exact solution concepts when only noisy estimates are available for the value function.

Cooperative Vehicle Routing Game

In this section, we define the cooperative vehicle routing game (CVRG), which extends classical cooperative games by having coalition values computed by solving multi-depot vehicle routing problems. Formally speaking, a cooperative game is characterized by the pair (N, φ) , where N is the set of n agents and $\varphi : 2^N \rightarrow \mathbb{R}$ is the characteristic function that assigns a real value $\varphi(S)$ to every subset $S \subseteq N$. The set S contains members of a coalition, while $\varphi(S)$ represents the value of such coalition.

In the following, we define N and φ in the context of CVRG. Let $N = \{1, \dots, n\}$ be the set of players (providers) who operate on a directed graph $G = (V, E)$. The vertex set $V = D \cup M$ consists of both the set of depots ($D = \{d_1, \dots, d_n\}$) and the set of requests ($M = M_1 \cup \dots \cup M_n$), where player i 's depot and request set are denoted as d_i and M_i . All M_i are disjoint as well. The edge set E is defined as: $\{(u, v) | \forall u \in D, \forall v \in M\} \cup \{(u, v) | \forall u, v \in M, u \neq v\} \cup \{(u, v) | \forall u \in M, \forall v \in D\}$. The cost of traversing an edge $e \in E$ is assumed to be the same for all players and denoted as $c(e)$. For $i \in N$ and $j \in M_i$, $q(j) > 0$ denotes the quantity of goods requested by j from i , where the revenue for fulfilling the request is $r(j) > 0$. Vehicles are assumed to be homogeneous with capacity Q , and each customer demand is assumed to be at most Q (so that no split delivery is required). A coalition $S \subseteq N$ is a subset of players, where N is called the grand coalition. Given a coalition S , we define the following restricted graph $G_S = (V_S, E_S)$ based on G :

- $V_S = D_S \cup M_S$, where $D_S = \{d_i \in D | i \in S\}$ and $M_S = \bigcup_{i \in S} M_i$,
- $E_S = \{(u, v) \in E | u, v \in V_S\}$.

The value (profit) of the grand coalition, denoted as $\varphi(N)$, is the objective function value obtained by solving the following multi-depot vehicle routing problem (MDVRP), while the value of a nonempty coalition S , $\varphi(S)$, is obtained by solving the same problem over the restricted graph G_S . By definition, $\varphi(\emptyset) = 0$.

MDVRP Formulation

For all $e \in E$ and $d \in D$, let x_e^d be the binary decision variable where x_e^d is set to 1 if and only if e is traversed by a vehicle starting from depot d , and 0 otherwise. Let y_e be the amount of goods carried by a vehicle traversing edge e . The value of the grand coalition is obtained by solving:

$$\varphi(N) = \max \sum_{j \in M} \sum_{u \in V} \sum_{d \in D} x_{uj}^d r(j) - \sum_{e \in E} \sum_{d \in D} x_e^d c(e),$$

subject to

$$\sum_{u \in V} x_{uv}^d = \sum_{u \in V} x_{vu}^d, \quad \forall v \in V, d \in D \quad (1)$$

$$\sum_{u \in V} \sum_{d \in D} x_{uv}^d \leq 1, \quad \forall v \in M \quad (2)$$

$$x_{dv}^d = 0, \quad \forall v \in M, \forall d, d' \in D, d \neq d' \quad (3)$$

$$x_{ud}^d = 0, \quad \forall u \in M, \forall d, d' \in D, d \neq d' \quad (4)$$

$$y_{dv} \leq Qx_{dv}^d, \quad \forall d \in D, v \in M \quad (5)$$

$$y_{uv} = \sum_{d \in D} x_{uv}^d \left(\sum_{u' \in V} y_{u'u} - q(u) \right), \quad \forall u \in M, v \in V \quad (6)$$

$$x_e^d \in \{0, 1\}, y_e \in \mathbb{R}_{\geq 0}, \quad \forall e \in E, d \in D$$

In the above formulation, (1) represents flow conservation constraints, (2) constrains that a customer is visited at most once; (3) and (4) constrain that all vehicles return to the same depot they originate from; and finally (5) and (6) represent the capacity constraints.

Equivalent VRP Formulation

While the above formulation is used in our implementation, the following equivalent formulation is used in the analysis. Given a coalition $S \subseteq N$, we define an S -partition as the set $z^S = \{z_i^S\}_{i \in S}$ such that $\bigcup_{i \in S} z_i^S = M_S$, and for all $i, j \in S, i \neq j, z_i^S \cap z_j^S = \emptyset$. In other words, an S -partition divides and assigns the collective requests of coalition S to its members such that each request is assigned to exactly one member. Next, for all $i \in N$ and $w \subseteq M$, we define the function $\text{VRP}_i(w)$ as the objective value obtained from solving the MDVRP with a single depot $d_i \in D$ and the request set w . An equivalent definition of $\varphi(S)$, in terms of VRP_i and S -partition, is given by

$$\varphi(S) = \max_{z^S} \sum_{i \in S} \text{VRP}_i(z_i^S) = \sum_{i \in S} \text{VRP}_i(\mathbf{z}_i^S),$$

where \mathbf{z}^S denotes an S -partition that gives the maximum $\varphi(S)$ value.

The Core and Shapley Value

An allocation $\pi = (\pi_1, \dots, \pi_n)$ describes the payoff received by each provider in the grand coalition. We are concerned with the following two solution concepts: the core

and the Shapley value. The core is defined as:

$$\text{Core}(\varphi) = \left\{ (\pi_1, \dots, \pi_n) \in \mathbb{R}_{\geq 0}^n \mid \sum_{i \in N} \pi_i = \varphi(N) \wedge \sum_{i \in S} \pi_i \geq \varphi(S), \forall S \subseteq N \right\},$$

that is, the set of allocations satisfying budget-balance and stability simultaneously. A budget balanced allocation is one where the total values allocated equals the values generated by the coalition. A stable allocation is one that satisfies sub-group rationality, where no sub-coalition is able to generate higher values (than those allocated) by leaving the coalition.

The Shapley value (Shapley 1953; Roth 2005), $(\text{Sh}_i)_{i \in N}$, on the other hand, is an allocation where for each player $i \in N$, $\text{Sh}_i(\varphi)$ is given by

$$\sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} (\varphi(S \cup \{i\}) - \varphi(S)),$$

which allocates the profit according to respective marginal contribution that each player brings to the coalition. Together with budget balance, the Shapley value is characterized by the following properties¹:

- **Symmetry.** Two players $i, i' \in N$ are symmetric if for all $S \subset N$ where $i, i' \notin S, \varphi(S \cup \{i\}) = \varphi(S \cup \{i'\})$. An allocation π is symmetric if for any symmetric players i and $i', \pi_i = \pi_{i'}$.
- **Dummy player.** An allocation π satisfies this property if, for any dummy player i , i.e., $\varphi(S \cup \{i\}) = \varphi(S)$ for all $S \subset N$, we have $\pi_i = 0$.
- **Additivity.** Let $\langle N, \varphi_1 \rangle, \langle N, \varphi_2 \rangle, \langle N, \varphi_3 \rangle$ be cooperative games such that $\forall S \subseteq N, \varphi_3(S) = \varphi_1(S) + \varphi_2(S)$. And let π_1, π_2, π_3 be the allocations for the games respectively under an allocation scheme. The scheme is additive if $\pi_3 = \pi_1 + \pi_2$.

While a core allocation removes the incentive to deviate due to higher profit, the Shapley value obviates perceived unfairness. It is known that in a general cooperative game, the core may or may not be empty. And even if it is not empty, the Shapley value may or may not lie in the core. On the contrary, if the core is empty, the Shapley value is necessarily not stable. Recall that the purpose of this paper is to propose an allocation scheme that is both stable and fair, in addition to being budget balanced. In this regard, we use the three axioms above as the conditions for fairness. In the following, we define classes of games with well-known properties that are relevant to our purpose, and in the next section, we show the extent to which these properties are present in CVRG.

Definition. A cooperative game has nonempty core iff it is balanced (Shapley 1967). We denote by **Bal** the class of balanced games.

¹In his original paper (Shapley 1953), Shapley presents symmetry, carrier and additivity as the axioms for the value. Most later works, however, divide the carrier property into budget balance and dummy player.

Definition. A cooperative game $\langle N, \varphi \rangle$ is totally balanced iff every subgame of $\langle N, \varphi \rangle$ is balanced. We denote by **TBal** the class of totally balanced games.

Definition. A cooperative game $\langle N, \varphi \rangle$ is convex iff $\forall S, T \subseteq N, \varphi(S) + \varphi(T) \leq \varphi(S \cup T) + \varphi(S \cap T)$. Furthermore, if $\langle N, \varphi \rangle$ is convex, then $\text{Sh}(\varphi) \in \text{Core}(\varphi)$. We denote by **Conv** the class of convex games.

The relationship between the classes of games is given as follows:

$$\text{Conv} \subset \text{TBal} \subset \text{Bal}.$$

Properties of CVRG

In this section, we establish some of the properties of CVRG. The first property is superadditivity. A cooperative game $\langle N, \varphi \rangle$ is superadditive if for all $S, T \in N$ such that $S \cap T = \emptyset, \varphi(S) + \varphi(T) \leq \varphi(S \cup T)$.

Proposition 1. CVRG is superadditive.

Proof. Let $S, T \subseteq N, S \cap T = \emptyset$. Let \mathbf{z}^S and \mathbf{z}^T be S and T -partition with values $\varphi(S)$ and $\varphi(T)$. As S and T are disjoint, the set $\mathbf{z}^S \cup \mathbf{z}^T$ is an W -partition, where $W = S \cup T$. Let $\mathbf{z}^W = \mathbf{z}^S \cup \mathbf{z}^T$. We have $\varphi(S) + \varphi(T) = \sum_{i \in S} \text{VRP}_i(\mathbf{z}_i^S) + \sum_{i \in T} \text{VRP}_i(\mathbf{z}_i^T) = \sum_{i \in W} \text{VRP}_i(\mathbf{z}_i^W) \leq \sum_{i \in W} \text{VRP}_i(\mathbf{z}_i^W) = \varphi(W)$. \square

Since φ is superadditive, the grand coalition achieves the highest value and thus, from a social perspective, is the most desirable outcome. Superadditive games also ensure that individual rationality can always be achieved in any coalition. Next, we establish that the core of CVRG is nonempty by showing that it is balanced.

Proposition 2. CVRG \in Bal.

Proof. We start by introducing a class of games called market games. A market game is a cooperative game $\langle N, \phi \rangle$, where for each $S \subseteq N$,

$$\phi(S) = \max_{\alpha} \left\{ \sum_{i \in S} u_i(\alpha_i) \mid \alpha \in \prod_{i \in S} \mathbb{R}_{\geq 0}^{\beta}, \sum_{i \in S} \alpha_i = \sum_{i \in S} \gamma_i \right\}$$

where u_i is a concave utility function of player i . The vector $\gamma_i \in \mathbb{R}_{\geq 0}^{\beta}$ can be viewed as the initial distribution of continuous goods of β types among the players. When players form a coalition, they redistribute their collective goods among themselves in such a way that maximizes the sum of their utility. Market games have been shown to be balanced (Shapley and Shubik 1969). Notice the close resemblance of market games to the VRP formulation of CVRG. The differences are: (1) In CVRG, we are dealing with discrete requests instead of continuous goods, and (2) the utility of players are discrete set functions (obtained by solving VRP) instead of concave continuous functions. In the following we show that, given a CVRG, we can construct an equivalent market game, thus showing that CVRG is balanced.

Let $\langle N, \varphi \rangle$ be a cooperative game. For any subset of all requests $w \subseteq M$, we define the indicator vector $\mathbf{1}^w$ as $(\mathbf{1}_j^w)_{j \in M} \in \{0, 1\}^{|M|}$ such that

$$\mathbf{1}_j^w = \begin{cases} 1 & \text{if } j \in w, \\ 0 & \text{otherwise.} \end{cases}$$

The initial distribution of requests can then be denoted by $\mathbf{1}^{M_1}, \dots, \mathbf{1}^{M_n}$ for each player $1, \dots, n$ respectively, and a player i 's utility function can be defined as the set function $\text{VRP}_i : \{0, 1\}^{|M|} \rightarrow \mathbb{R}$. Now, let $\text{VRP}_i^+ : [0, 1]^{|M|} \rightarrow \mathbb{R}$ be the concave closure of VRP_i . A concave closure is an extension of a set function to continuous domain (other extensions include multilinear, convex, and Lovász extensions). It has the properties that (1) it agrees with the original function on discrete inputs, i.e., $\text{VRP}_i(\mathbf{1}^w) = \text{VRP}_i^+(\mathbf{1}^w)$ for all $w \in M$, and (2) it is concave. We now define a cooperative game $\langle N, \varphi^+ \rangle$, where for all $S \subseteq N$, the value $\varphi^+(S)$ is defined as

$$\max_{\alpha} \left\{ \sum_{i \in S} \text{VRP}_i^+(\alpha_i) \mid \alpha \in \prod_{i \in S} \mathbb{R}_{\geq 0}^{|M|}, \sum_{i \in S} \alpha_i = \mathbf{1}^{M_S} \right\}.$$

By property (2), $\langle N, \varphi^+ \rangle$ is a market game and thus is balanced. The maximization problem above admits integer solution when the cost function c satisfies triangle inequality. Due to space constraint, the proof of this is omitted. Together with property (1), this establishes that $\langle N, \varphi \rangle$ and $\langle N, \varphi^+ \rangle$ are equivalent. \square

Since every subgame of a CVRG is still a CVRG, by Proposition 2, it is balanced, and we have the following corollary.

Corollary 1. CVRG \in TBal.

Unfortunately, CVRG is a non-convex game, as shown in the counterexamples in the experiment section. Proposition 3 below introduces a special case for which CVRG is convex. The conditions given in Proposition 3 are however very restrictive, and would probably never be satisfied in real-world instances. Simply relying on the Shapley value and hoping that it lies in the core, therefore, is not sufficient in the context of our application, where stability is considered as the most important property.

Proposition 3. Let CVRG⁻ be cooperative vehicle routing games where

1. all providers have the same depot location,
2. vehicles have unlimited capacity, and
3. the edges have uniform cost.

Then, CVRG⁻ \in Conv.

Proof. (Sketch) Let $A, B, C \subset N, A \cap B \cap C = \emptyset$ be disjoint subcoalitions of a given CVRG⁻. And let $h(A \cup B)$ and $h(B \cup C)$ be the sets of optimal routes obtained by solving the MDVRP for subcoalitions $A \cup B$ and $B \cup C$ respectively. Let $h(A \cup B) + h(B \cup C)$ be the superimposition of the two sets of routes. Now if we remove all loops involving the requests of subcoalition B from $h(A \cup B) + h(B \cup C)$, and put it in the set $h'(B)$, we end up with two valid suboptimal routes $h'(A \cup B \cup C)$ and $h'(B)$ for $A \cup B \cup C$ and B respectively. These suboptimal routes are valid precisely because of the conditions given above. The value of the superimposition is $\varphi(A \cup B) + \varphi(B \cup C)$ since both are optimal routes. This is less than or equal to the value of the routes

$h'(A \cup B \cup C)$ plus the value of the routes $h'(B)$ by the property of the described procedure. Since the later two are sub-optimal routes, we have $\varphi(A \cup B) + \varphi(B \cup C) \leq \text{val}(h'(A \cup B \cup C)) + \text{val}(h'(B)) \leq \varphi(A \cup B \cup C) + \varphi(B)$. \square

δ -Balanced Stable-Fair Allocation (δ -BSA)

Since CVRG has been shown to be non-convex in general above, the Shapley value may not be stable. In other words, the existence of a fair, stable, and budget balanced allocation is not guaranteed in general. In the following, we propose an allocation scheme that returns the Shapley value if it is stable, otherwise, the “best” allocation is returned by relaxing the budget balance property.

The proposed scheme, denoted by δ -BSA, tries to find an allocation that are both stable and fair by allowing some external subsidy into the system. Here, we seek an allocation that minimizes subsidy required to achieve the properties of both the core and the Shapley value (except budget balance). The central idea of δ -BSA revolves around the normalized Shapley value $(\text{Sh}_i^0)_{i \in N}$, which is defined as

$$\text{Sh}_i^0(\varphi) = \frac{\text{Sh}_i(\varphi)}{\sum_{i' \in N} \text{Sh}_{i'}(\varphi)},$$

with respect to a cooperative game (N, φ) . It is used as the measure of relative contributions of players in the grand coalition. As such, any amount of profit to be distributed should be done so in proportion to this value. From this perspective, the Shapley value is simply the distribution of the exact profit of the grand coalition in proportion to the normalized values. By relaxing budget balance, we allow the distributed amount to exceed the actual profit. In this section, we formally state the problem of achieving stability with minimal subsidy. We also point out when subsidy becomes infeasible.

Algorithm 1: COMPUTE δ -BSA

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Input:  $\langle N, \varphi \rangle, \text{Sh}^0(\varphi)$ 
begin
1   $\pi \leftarrow 0, (T, \Delta, f) \leftarrow (\emptyset, 0, 0)$ 
2   $(T, \Delta, f) \leftarrow \text{UPDATE}(\pi)$ 
3  while  $\Delta > 0$  do
4    for  $i \in N$  do
5       $\pi_i \leftarrow \pi_i + (\text{Sh}_i^0(\varphi) \times 1/f \times \Delta)$ 
6     $(T, \Delta, f) \leftarrow \text{UPDATE}(\pi)$ 
7   $\delta \leftarrow \sum_{i \in N} \pi_i - \varphi(N)$ 
   return  $(\pi, \delta)$ 
UPDATE( $\pi$ ):
begin
8   $T \leftarrow \arg \max_S (\varphi(S) - \sum_{i \in S} \pi_i)$ 
9   $\Delta \leftarrow \varphi(T) - \sum_{i \in T} \pi_i$ 
10  $f \leftarrow \sum_{i \in T} \text{Sh}_i^0(\varphi)$ 
   return  $(T, \Delta, f)$ 

```

The procedure for computing δ -BSA is given in Algorithm 1. In each iteration, using the UPDATE() function (line

8-10), it finds a coalition that is least stable, i.e., one with the largest discrepancy between allocation and profit (line 2, 6). The idea then, is to increase the allocation for this coalition by Δ such that the discrepancy is eliminated (line 5). Players not in the coalition will receive increments as well to keep the overall allocation π in proportion to the normalized Shapley value. The value f (line 10) represents the contribution of the chosen coalition, and since it receives an increment of Δ , each player i (whether inside or outside of the coalition) receives an increment of $\text{Sh}_i^0(\varphi) \times \Delta/f$, where Δ/f is the total amount added to the system in each iteration. Note that after the first iteration, the value of π is always in proportion to the normalized Shapley value. Furthermore, if the game is superadditive, then in the first iteration, the grand coalition will be chosen, since it has the highest profit, and the first allocation computed is the Shapley value, which is returned in the next iteration if it is stable. The value δ is the subsidy needed to realize the resulting allocation. In the following, we establish the properties of δ -BSA. The first property is on minimum subsidy.

Proposition 4. *Algorithm 1 gives a stable allocation that is in proportion to the normalized Shapley value with minimum subsidy.*

Proof. At each iteration of Algorithm 1, a coalition is chosen, and the amount Δ is added to the coalition so that its allocation equals its profit (line 9). Let S be the coalition chosen at the last iteration, and π the final allocation, then we have $\sum_{i \in S} \pi_i = \varphi(S)$. Let π' be an allocation that is in proportion to the normalized Shapley value, but requiring less subsidy than π , in other words, $\sum_{i \in N} \pi'_i < \sum_{i \in N} \pi_i$. If f is the contribution of coalition S , then we have $\sum_{i \in S} \pi'_i = f(\sum_{i \in N} \pi'_i) < f(\sum_{i \in N} \pi_i) = \sum_{i \in S} \pi_i = \varphi(S)$, which means that π' is necessarily unstable. \square

Note that the argument above does not use the fact that the coalition chosen at each iteration has the maximum discrepancy (line 8). In fact, the property still holds if at each iteration, we choose an arbitrary coalition with positive Δ instead. The difference is, this will require more iterations. Next, we establish that δ -BSA satisfies stability and the axioms of the Shapley value except budget balance. Stability, symmetry and dummy player are satisfied exactly while additivity is satisfied approximately.

Proposition 5. *δ -BSA satisfies the properties of stability, symmetry, and dummy player.*

Proof. Stability. Algorithm 1 terminates when the condition in line 3 is violated, at which point $\Delta \leq 0$. By the definition of Δ given in the UPDATE() function, this implies that at termination, we have $\Delta = \max_S (\varphi(S) - \sum_{i \in S} \pi_i) \leq 0$, in other words, $\forall S \in N, \sum_{i \in S} \pi_i \geq \varphi(S)$.

Symmetry. Since the Shapley value is symmetric, two players who are symmetric have the same Shapley value. Next, note that by definition, for two players $i, i' \in N$, $\text{Sh}_i^0(\varphi) = \text{Sh}_{i'}^0(\varphi)$ iff $\text{Sh}_i(\varphi) = \text{Sh}_{i'}(\varphi)$, that is, two players have the same normalized Shapley value if and only if they have the same Shapley value. To establish the symmetry of δ -BSA, therefore, we just need to show that if two

players $i, i' \in N$ have the same normalized Shapley value, $\text{Sh}_i^0(\varphi) = \text{Sh}_{i'}^0(\varphi)$, then Algorithm 1 assigns the same value to both. This can be straightforwardly inferred from step 5. At each iteration, the amount of value added to a player i depends on $\text{Sh}_i^0(\varphi)$, $1/f$ and Δ . Since the latter two are the same for all players in any iteration, two players $i, i' \in N$ with the same normalized Shapley value will get the same increment. And since all players start with the same initial value, we have $\pi_i = \pi_{i'}$ in each iteration and at termination.

Dummy player. Similarly, the Shapley value of a dummy player i is zero, implying that its normalized value is also zero. Step 5 of Algorithm 1 shows that, this player receives zero increment in each iteration. And since its initial value is zero, π_i is zero at termination. \square

Proposition 6. *Let $\langle N, \varphi_1 \rangle$, $\langle N, \varphi_2 \rangle$ and $\langle M, \varphi_3 \rangle$ be three games with the same players such that $\forall S \subseteq N, \varphi_3(S) = \varphi_1(S) + \varphi_2(S)$. And let π^1, π^2 and π^3 be their respective allocations under δ -BSA. Then $\forall i \in N, |\pi_i^1 + \pi_i^2 - \pi_i^3| \leq \max\{\delta_1 + \delta_2, \delta_3\}$, where δ_1, δ_2 and δ_3 are the subsidies for the games.*

Proof. Proposition 4 establishes that δ -BSA allocates payoffs in proportions to the normalized Shapley value. In other words, if π is the allocation for a cooperative game $\langle N, \varphi \rangle$ under δ -BSA, then $\forall i \in N$,

$$\pi_i = \text{Sh}_i^0(\varphi) \times [\varphi(N) + \delta],$$

where δ is the subsidy computed by Algorithm 1. Now, let $\langle N, \varphi_1 \rangle$, $\langle N, \varphi_2 \rangle$ and $\langle M, \varphi_3 \rangle$ be arbitrary games defined as above. Since the Shapley value is additive, we have

$$\text{Sh}_i(\varphi_1) + \text{Sh}_i(\varphi_2) = \text{Sh}_i(\varphi_3),$$

and by definition,

$$\text{Sh}_i^0(\varphi_1)\varphi_1(N) + \text{Sh}_i^0(\varphi_2)\varphi_2(N) = \text{Sh}_i^0(\varphi_3)\varphi_3(N),$$

for all $i \in N$. If δ_1, δ_2 and δ_3 are the subsidies for the games respectively under δ -BSA, then $|\pi_i^1 + \pi_i^2 - \pi_i^3|$

$$\begin{aligned} &= |\text{Sh}_i^0(\varphi_1) \times [\varphi_1(N) + \delta_1] + \text{Sh}_i^0(\varphi_2) \times \\ &\quad [\varphi_2(N) + \delta_2] - \text{Sh}_i^0(\varphi_3) \times [\varphi_3(N) + \delta_3]| \\ &= |\text{Sh}_i^0(\varphi_1)\varphi_1(N) + \text{Sh}_i^0(\varphi_2)\varphi_2(N) - \text{Sh}_i^0(\varphi_3)\varphi_3(N) + \\ &\quad \text{Sh}_i^0(\varphi_1)\delta_1 + \text{Sh}_i^0(\varphi_2)\delta_2 - \text{Sh}_i^0(\varphi_3)\delta_3| \\ &= |\text{Sh}_i^0(\varphi_1)\delta_1 + \text{Sh}_i^0(\varphi_2)\delta_2 - \text{Sh}_i^0(\varphi_3)\delta_3| \\ &\leq \max\{\delta_1 + \delta_2, \delta_3\}. \end{aligned}$$

\square

Finally, we have to establish the feasibility of the subsidy: subsidy only makes sense when it doesn't exceed the surplus of the grand coalition. In other words, when $\varphi(N) - \sum_{i \in N} \varphi(\{i\}) > \delta$. Here, we define the surplus to be the difference between the profit of the grand coalition and the sum of individual profits. But this is only an approximation. To be more precise, for the latter, we should instead use the value generated by the best collection of stable, fair, and budget balanced disjoint subcoalitions. But computing

this value is, in itself, a very challenging problem, which we will address in future works. Rather, in our experiments below, we will empirically examine the characteristics on the problem instances under which subsidy is feasible.

Numerical Experiments

In this section, we present the numerical results of computing δ -BSA for random instances of CVRG. For each instance, the locations of the depots and requests are generated uniformly over a 1×1 2D plane. The revenue from serving a request is generated according to a normal distribution, the quantity of goods according to a uniform distribution, while the cost of traveling is a factor of the Euclidean distance. The value function for each instance is generated using an exact solver based on a CPLEX implementation of the MDVRP given in the first formulation. We first present a small example of an instance of CVRG together with the solution concepts discussed in this paper. This example illustrates the case when the Shapley value is not in the core. It also shows how δ -BSA is able to resolve this conflict by relaxing the budget balance constraint.

In this example, the number of players is 4 and the total number of requests is 12, with the distribution of $\{3, 5, 2, 2\}$. The quantity of goods requested is randomly generated in the range of 1-3, and the vehicle capacity is set to 3. The mean of the revenue is 10 with standard deviation 20, while the travel cost is 25 times the Euclidean distance. The results are shown in Figure 1. Figure 1(a) shows the optimal routes for subcoalition $\{1, 3\}$, while Figure 1(b) shows the optimal routes of the grand coalition. Together, they illustrate the superadditivity property of CVRG, how surplus values are generated, and the attractiveness of the grand coalition from the social perspective. While subcoalition $\{1, 3\}$ cannot serve some of their requests profitably (which is also the case for most subcoalitions in this instance), by forming the grand coalition, all requests could be served profitably. Figure 1(c) compares values based on different solution concepts. The first bars are the values of the function φ ; the second bars are the marginal contributions of each subcoalition $S \subseteq N$ to the grand coalition, i.e., $\varphi(N) - \varphi(N \setminus S)$; the third bars are values derived from an allocation in the core; the fourth bars are the Shapley value; and the last bars are the δ -BSA values. To illustrate why a core allocation may not be fair, we can compare players 2 and 3 under this allocation. Both players have roughly the same marginal contributions, yet the payoff to player 3 is much higher. The value of player 2 is zero because it cannot serve any of its requests profitably. Its marginal contribution comes from the fact that these requests become profitable in the grand coalition. At the same time, the Shapley value allocation is not stable because subcoalition $\{0, 1, 2\}$ has the incentive to leave since its own value exceeds its Shapley value. δ -BSA, on the other hand, is both stable and fair (allocations are in proportion to Shapley values). For this instance, it's also feasible as the subsidy is much smaller than the surplus generated by the grand coalition.

Finally, we try to identify the key factor that determines the feasibility of subsidy. Intuitively speaking, coalition only makes sense when players are dependent on each other; in

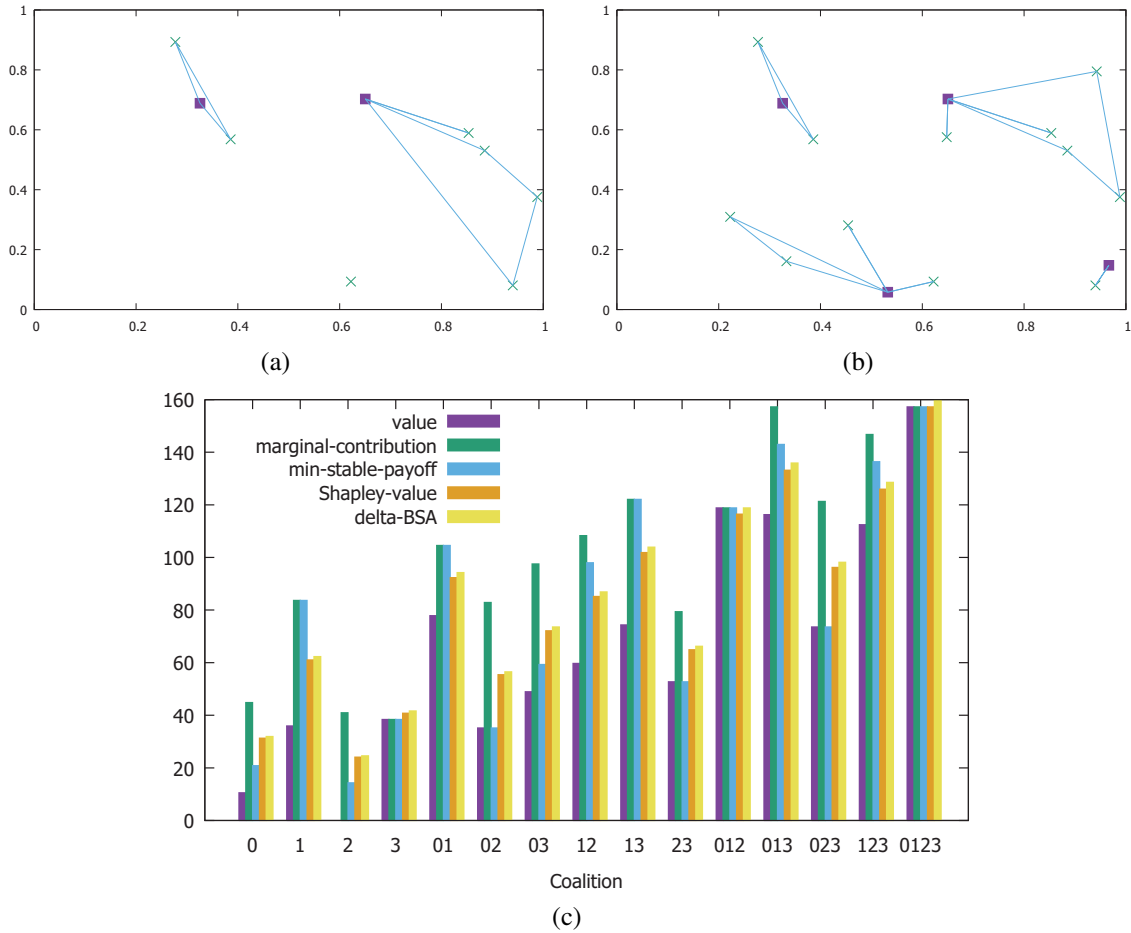


Figure 1: An example of CVRG: (a) The optimal routes for subcoalition $\{1, 3\}$, (b) The optimal routes for the grand coalition, (c) The value function and the values of different solution concepts.

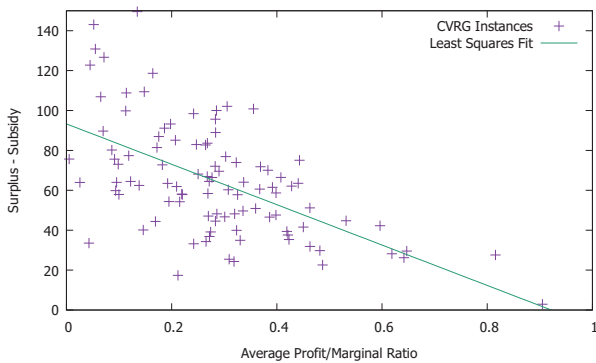


Figure 2: Measuring the feasibility of δ -BSA.

other words, we expect the value of coalition to be low (thus making subsidy less likely to be feasible) when players are relatively independent. To test our hypothesis, we define the following two variables to be measured: (1) σ , which measures how independent individual players are, is defined as the average ratio of individual profit to marginal

contribution, i.e., for each instance $\langle N, \varphi \rangle$, $\sigma = 1/|N| \times \sum_{i \in N} \varphi(\{i\}) / (\varphi(N) - \varphi(N \setminus \{i\}))$; and (2) p , which measures the difference between surplus and subsidy under δ -BSA; $p = \varphi(N) - \sum_{i \in N} \varphi(\{i\}) - \delta$.

100 instances of CVRG are generated with the following parameters: The number of players is set to 8 and total requests to 24, with a random initial distribution. The quantity of goods requested is between 1-3 with vehicle capacity set to 3. The mean and standard deviation of revenues are 5 and 10 respectively, while the travel cost is 12 times the Euclidean distance.

Figure 2 shows the results of the experiment by plotting p against σ . It confirms our conjecture that higher player independence (σ) leads to lower feasibility (p).

Conclusions

In this paper, we presented the cooperative vehicle routing game (CVRG) as the solution model, and established some of its properties to help understand the viability of urban collaborative logistics delivery operations. We showed that under budget balance, stability and fairness can always be achieved individually, but not always simultaneously. We

then proposed δ -BSA, an allocation scheme that guarantees both stability and fairness while requiring minimum subsidy. In our numerical experiments, we demonstrated the effectiveness of our proposed allocation scheme, and we also established player independence as one major contributing factor in determining the feasibility of giving out subsidy. The following are potential future works: (1) scaling δ -BSA to larger (real-world) problem instances by considering computationally bounded solution concepts, (2) related to the above, designing (approximate) methods for efficient computation of δ -BSA, and finally (3) providing full axiomatization of the proposed allocation scheme.

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