

## Modeling Human Ad Hoc Coordination

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### Abstract

Whether in groups of humans or groups of computer agents, collaboration is most effective between individuals who have the ability to coordinate on a joint strategy for collective action. However, in general a rational actor will only intend to coordinate if that actor believes the other group members have the same intention. This circular dependence makes rational coordination difficult in uncertain environments if communication between actors is unreliable and no prior agreements have been made. An important normative question with regard to coordination in these ad hoc settings is therefore how one can come to believe that other actors will coordinate, and with regard to systems involving humans, an important empirical question is how humans arrive at these expectations. We introduce an exact algorithm for computing the infinitely recursive hierarchy of graded beliefs required for rational coordination in uncertain environments, and we introduce a novel mechanism for multiagent coordination that uses it. Our algorithm is valid in any environment with a finite state space, and extensions to certain countably infinite state spaces are likely possible. We test our mechanism for multiagent coordination as a model for human decisions in a simple coordination game using existing experimental data. We then explore via simulations whether modeling humans in this way may improve human-agent collaboration.

Forming shared plans that support mutually beneficial behavior within a group is central to collaborative social interaction and collective intelligence (Grosz and Kraus 1996). Indeed, many common organizational practices are designed to facilitate shared knowledge of the structure and goals of organizations, as well as mutual recognition of the roles that individuals in the organizations play. Once teams become physically separated and responsiveness or frequency of communication declines, the challenge of forming shared plans increases. Part of this difficulty is fundamentally computational. In theory, coming to a fully mutually recognized agreement on even a simple action plan among two choices can be literally impossible if communication is even mildly unreliable, even if an arbitrary amount of communication is allowed (Halpern and Moses 1990; Lynch 1996).

This problem is well-studied within the AI literature (e.g., (Gmytrasiewicz and Durfee 1992)), though the core difficulties still manifest in contemporary research on “ad hoc

coordination”—collaborative multiagent planning with previously unknown teammates (Stone et al. 2010). However, surprisingly little is known about the strategies that *humans* use to overcome the difficulties of coordination (Thomas et al. 2014). Understanding how and when people try to coordinate is critical to furthering our understanding of human group behavior, as well as to the design of agents for human-agent collectives (Jennings et al. 2014). Existing attempts at modeling human coordination have focused either on unstructured predictive models (e.g., (Frieder, Lin, and Kraus 2012)) or bounded depth socially recursive reasoning models (e.g., (Gal and Pfeffer 2008; Yoshida, Dolan, and Friston 2008)), but there is reason to believe that these accounts miss important aspects of human coordination.

One concept that appears repeatedly in formal treatments of coordination but has not appeared meaningfully in empirical modeling is common knowledge. Two agents have common knowledge if both agents have infinitely nested knowledge of the other agent’s knowledge of a proposition, i.e. the first agent knows the second agent knows, the first agent knows the second agent knows the first agent knows, etc. Common knowledge has been shown to be necessary for exact coordination (Halpern and Moses 1990), and a probabilistic generalization of common knowledge, called common  $p$ -belief, has been shown to be necessary for approximate coordination (Monderer and Samet 1989). While these notions are clearly important normatively, it is not entirely clear how important they are empirically in human coordination. Indeed, supposing that humans are able to mentally represent an infinitely recursive belief state seems a priori implausible, and the need to represent and infer this infinite recursive belief state has also been a barrier to empirically testing models involving common knowledge.

Nevertheless, building on the existing normative results, a group of researchers recently designed a set of experiments to test whether people are able to recognize situations in which common knowledge might obtain (Thomas et al. 2014) (hereafter referred to as the “Thomas experiments”). These researchers argued that people do possess a distinct mental representation of common knowledge by showing that people will attempt to coordinate more often in situations where common knowledge can be inferred. However, this previous work did not formalize this claim in a model or rigorously test it against plausible alternative computational

models of coordination. This existing empirical work therefore leaves open several important scientific questions that a modeling effort can help address. In particular: How might people mentally represent common p-belief? Do people reason about graded levels of common p-belief, or just “sufficiently high” common p-belief? Finally, what computational processes could people use to infer common p-belief?

In this work we use a previously established fixed point characterization of common p-belief (Monderer and Samet 1989) to formulate a novel model of human coordination. In finite state spaces this characterization yields an exact finite representation of common p-belief, which we use to develop an efficient algorithm for computing common p-belief. This algorithm allows us to simulate models that rely on common p-belief. Because of the normative importance of common p-belief in coordination problems, our algorithm may also be independently useful for coordination in artificial multiagent systems. We show using data from the Thomas experiments that this model provides a better account of human decisions than three alternative models in a simple coordination task. Finally, we show via simulations based on the data from the Thomas experiments that modeling humans in this way may improve human-agent coordination.

## 1 Background

We first provide a description of the coordination task we will study in this paper: the well-known coordinated attack problem. We then provide an overview of the formal definitions of common knowledge and common p-belief, and their relationship to the coordinated attack problem.

### 1.1 Coordinated Attack Problem

The coordination task that we study in this paper is alternatively called the coordinated attack problem, the two generals problem, or the email game. The original formulation of this task was posed in the literature on distributed computer systems to illustrate the impossibility of achieving consensus among distributed computer processors that use an unreliable message-passing system (Halpern and Moses 1990), and the problem was later adapted by economists to a game theoretic context (Rubinstein 1989). Here we focus on the game theoretic adaptation, as this formulation is more amenable to decision-theoretic modeling and thus more relevant for modeling human behavior.<sup>1</sup>

In this task the world can be in one of two states,  $x = 1$  or  $x = 0$ . The state of the world determines which of two games two players will play together. The payoff matrices for these two games are as follows ( $a > c > \max(b, d)$ ):

$x = 1$	$A$	$B$	$x = 0$	$A$	$B$
$A$	$a, a$	$b, c$	$A$	$d, d$	$b, c$
$B$	$c, b$	$c, c$	$B$	$c, b$	$c, c$

The players receive the optimal payoff if they coordinate on both playing  $A$  when  $x = 1$ , but playing  $A$  is risky. Playing  $A$  is inferior to playing  $B$  if  $x = 0$  or if there is a mismatch

<sup>1</sup>Our exposition largely assumes familiarity with rudimentary game theory, and familiarity with measure-theoretic probability as it appears in incomplete information games.

between the players’ actions. Playing  $B$  is safe with a sure payoff of  $c$ . Thus in order for it to be rational to play  $A$ , a player must believe with sufficient confidence both that the world state is  $x = 1$  and that the other player will play  $A$ .

### 1.2 Common p-Belief

In order for a player to believe the other player will play  $A$  in this game, it is not enough for that player to believe that the other player knows  $x = 1$ . If the second player does not believe that the first player knows  $x = 1$ , then the second player will not try to coordinate. Therefore the first player must also at least believe that the second player believes the first player knows  $x = 1$ . However, it turns out even this amount of knowledge does not suffice. In fact, an infinite hierarchy of recursive belief is needed (Morris and Shin 1997). This infinite hierarchy of beliefs has been formalized using a construct called common p-belief, which we now define.

Using standard definitions from game theory, we define a two-player finite Bayesian game to be a tuple  $((\Omega, \mu, (\Pi_0, \Pi_1)), (\mathcal{A}_0, \mathcal{A}_1), (u_0, u_1))$  consisting of a finite state space  $\Omega = \{\omega_1, \dots, \omega_{|\Omega|}\}$ , a probability measure  $\mu$  defined over that state space, the information partition  $\Pi_i$  of each player  $i$ , the action set  $\mathcal{A}_i$  of each player, and the utility function  $u_i$  of each player. Elements of  $\Omega$  are called states, and subsets of  $\Omega$  are called events. For a given world state  $\omega$  and player  $i$  the partition of  $\Omega$ ,  $\Pi_i$ , uniquely specifies the beliefs of player  $i$  in the form of posterior probabilities.  $\Pi_i(\omega)$ , which indicates the unique element of  $\Pi_i$  containing  $\omega$ , can be thought of as the observation that player  $i$  receives when the true state  $\omega$  occurs. Specifically for any event  $E \subseteq \Omega$ ,  $\mu(E | \Pi_i(\omega))$  is the probability that player  $i$  assigns to  $E$  having occurred given  $\omega$  has occurred. As a shorthand, we write  $P_i(E | \omega) = \mu(E | \Pi_i(\omega))$ . Using another common shorthand, we will treat propositions and events satisfying those propositions interchangeably. For example, in the coordinated attack problem we will be interested in whether there is “common p-belief” that  $x = 1$ , which will refer to common p-belief in the event  $C = \{\omega \in \Omega : x(\omega) = 1\}$ , where  $x$  formally is a random variable mapping  $\Omega \rightarrow \{0, 1\}$ .

Following (Dalkiran et al. 2012), we say that player  $i$  *p*-believes<sup>2</sup> an event  $E$  at  $\omega$  if  $P_i(E | \omega) \geq p$ . An event  $E$  is said to be *p*-evident if for all  $\omega \in E$  and for all players  $i$ , player  $i$  *p*-believes  $E$  at  $\omega$ . In a slight divergence from the standard terminology of this literature, we say an event  $E$  is super-*p*-evident if for all  $\omega \in E$  and for all players  $i$ ,  $P_i(E | \omega) > p$  (the only difference being strict inequality). We say there is common *p*-belief in an event  $C$  at state  $\omega$  if there exists a *p*-evident event  $E$  with  $\omega \in E$ , and for all  $\omega' \in E$  and all players  $i$ , player  $i$  *p*-believes  $C$  at  $\omega'$ . Common knowledge is defined as common *p*-belief for  $p = 1$ .

A critically important result of (Monderer and Samet 1989) states that this definition of common p-belief is equivalent to a more intuitive infinitely recursive formulation. The importance of this definition of common p-belief is therefore that it provides a fixed point characterization of common p-

<sup>2</sup>We use an italicized “*p*” when referring to specific values of p-belief and a non-italicized “p” when referring to the terms in general.

belief strictly in terms of beliefs about events rather than directly in terms of beliefs about other players. When  $\Omega$  is finite, common p-belief can thus be represented in terms of a finite set of states, rather than an infinite hierarchy of beliefs.

## 2 Models

We now describe four strategies for coordination in the coordinated attack game we study. Two of these strategies involve the computation of p-evident events and common p-belief, which we will use to test whether human coordination behavior could be explained in terms of reasoning about p-evident events. The other two strategies serve as baselines.

### 2.1 Rational p-Belief

The first strategy we consider represents an agent who maximizes expected utility at an equilibrium point of the coordinated attack problem we study. The strategy is implemented as follows: player  $i$  plays action  $A$  if and only if the player believes with probability at least  $p^* = \frac{c-b}{a-b}$  that both players have common  $p^*$ -belief that  $x = 1$ . This strategy forms an equilibrium of the coordinated attack problem if  $p^* > P_i(x = 1)$  and if evidence that  $x = 1$  always leads to certain belief that  $x = 1$ , i.e.  $P_i(x = 1 | \omega) > P_i(x = 1) \Rightarrow P_i(x = 1 | \omega) = 1$  for all  $\omega$ . These conditions will be satisfied by the specific state spaces and payoffs we use to represent the Thomas experiments. We call this model the rational p-belief strategy. The proof that this strategy forms an equilibrium, including a derivation for the specific form of  $p^*$ , is included in our supplementary materials.<sup>3</sup>

### 2.2 Matched p-Belief

The second strategy we consider is a novel probabilistic relaxation of the rational p-belief strategy. Humans have been shown to exhibit a behavior called probability matching in many decision-making settings (Herrnstein 1961; Vulkan 2000). Probability matching consists of taking the probability  $p$  that a decision is the best decision available, and choosing to make that decision with probability  $p$ . While probability matching is not utility maximizing, it can be viewed as rational if players are performing sample-based Bayesian inference and if taking samples is costly (Vul et al. 2014). Motivated by this frequently observed behavior, we propose a model we call the matched p-belief strategy. A player  $i$  using this strategy chooses action  $A$  at  $\omega$  with probability  $p$  equal to the maximal common p-belief that the player perceives at  $\omega$ , i.e. the largest value such that  $i$  p-believes at  $\omega$  that there is common  $p$ -belief that  $x = 1$ .

### 2.3 Iterated Maximization

Next we consider a well-known model of boundedly rational behavior sometimes called a “level- $k$ ” depth of reasoning model. This family of models has been shown to be consistent with human behavior in a diversity of settings, including

<sup>3</sup>A version of our paper that includes the supplementary materials (as well as any post-publication corrections to the main text) is available in multiple locations online, including at <http://people.csail.mit.edu/pkrafft/papers/krafft-et-al-2016-modeling-human-ad-hoc-coordination.pdf>.

some coordination games (e.g., (Yoshida, Dolan, and Friston 2008)), and hence is a strong baseline. Since the term “level- $k$ ” is used for many slightly different models, we call our instantiation of this model the iterated maximization strategy. This strategy assumes that players have a certain fixed level of recursive social reasoning  $k$ . A player using the level- $k$  iterated maximization strategy chooses the action that maximizes that player’s expected utility when playing with a player using the level- $(k - 1)$  strategy. The level-0 player takes action  $A$  at  $\omega$  if  $P_i(x = 1 | \omega) > \frac{c-b}{a-b}$ . This level-0 strategy corresponds to the player maximizing expected utility assuming the player can control the actions of both players, or equivalently that the optimal joint action is taken according to that player’s beliefs. While in general the predictions of level- $k$  models depend strongly on the specification of the level-0 strategy, in informal exploration we found that the qualitative conclusions of our work are robust to whether we instead specify the level-0 strategy as always playing  $A$  or choosing between  $A$  and  $B$  uniformly randomly.

### 2.4 Iterated Matching

Finally, we also consider a less common depth of reasoning model that combines the iterated maximization strategy with probability matching behavior, which we call iterated matching. Like the iterated maximization strategy, this strategy assumes that players have a certain fixed level of recursive social reasoning  $k$ . However, instead of choosing the action that maximizes expected utility, a level- $k$  player using the iterated matching strategy chooses to take action  $A$  with probability equal to that player’s belief that  $x = 1$ , times the expected probability that a level- $(k - 1)$  companion player would play  $A$ . The level-0 player probability matches on  $P_i(x = 1 | \omega)$ .

## 3 Algorithms

In this section we present the algorithms we use to implement each of the models we consider. To the best of our knowledge the existing literature on common p-belief has yet to offer algorithms for computing common p-belief (or in our case the perceived maximal common p-belief) for a given world state and observation model. This computation is central to the rational and matched p-belief strategies. Hence we offer the first fully computational account of coordination via reasoning about p-evident events. Algorithms for iterated reasoning are straightforward and well-known.

The challenge in developing an algorithm for computing a player’s perception of the maximal common p-belief is avoiding enumeration over all exponentially many possible subsets of  $\Omega$ . While it is straightforward to evaluate whether a particular given event is p-evident, the definition of common p-belief requires only the existence of some such event. Computing perceived maximal common p-belief therefore requires jointly searching over values of  $p$  and over subsets of  $\Omega$ . We leverage the generic mathematical structure of p-evident events in finite state spaces in order to develop an exact algorithm that avoids enumerating all subsets. Our algorithm only requires a search that is polynomial in the size of the state space. Of course, the state spaces in many

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**Algorithm 1** common\_p\_belief( $C, i, \omega$ )

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 $E := \Omega; F := \Omega$ 
while  $P_i(F | \omega) > 0$  do
   $p := \text{evidence\_level}(F, C)$ 
   $E := F$ 
   $F := \text{super\_p\_evident}(E, C, p)$ 
 $p := \text{evidence\_level}(E, C)$ 
return  $p$ 
```

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**Algorithm 2** evidence\_level( $E, C$ )

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```
return  $\min_{\omega \in E} \text{min\_belief}(E, C, \omega)$ 
```

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problems are often themselves exponential in some underlying parameter variables, and hence future improvements on this algorithm would be desirable. Extensions to at least certain countably infinite or continuous state spaces are likely possible as well, such as perhaps by refining the search to only consider events that have non-zero probability given the player’s observations.

### 3.1 Computing Information Partitions

Our algorithms require access to the information partitions of each player. However, directly specifying the information partitions that people have in a naturalistic setting, such as in the data we use from the Thomas experiments, is difficult. Instead, we take the approach of specifying a plausible generative probabilistic world model, and we then construct the information partitions from this factored representation via a straightforward algorithm. The generative world model specifies the pieces of information, or “observations”, that each player receives. The algorithm for generating information partitions, which is specified precisely in our supplementary materials, consists of iterating over all combinations of random values of variables in the generative world model, treating each such combination as a state in  $\Omega$ , and for each player grouping together the states that yield identical observations.

### 3.2 Computing Common p-Belief

Algorithms 1-4 present the functions needed to compute perceived maximal common p-belief. Formally, given a player  $i$ , a particular state  $\omega$ , and an event  $C$ , Algorithm 1 computes the largest value  $p$  for which player  $i$   $p$ -believes that there is common  $p$ -belief in  $C$ . Note that it is insufficient to compute the largest value of  $p$  for which there is common  $p$ -belief in  $C$  at  $\omega$ , since in general at state  $\omega$  player  $i$  only knows that the event  $\Pi_i(\omega)$  has occurred, not that  $\omega$  specifically has occurred. Relatedly, note that while Algorithm 1 takes  $\omega$  as input for convenience, the algorithm only depends on  $\Pi_i(\omega)$ , and hence could be executed by a player.

Formal proofs of the correctness of these algorithms are included in our supplementary materials. The basic logic of the algorithms is to maintain a candidate p-evident event  $E$ , and to gradually remove elements from  $E$  to make it more p-evident until a point where player  $i$  believes the event to be impossible because the elements of the event are no longer

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**Algorithm 3** min\_belief( $E, C, \omega$ )

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return  $\min_{i \in \{0,1\}} \min(P_i(E | \omega), P_i(C | \omega))$ 
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**Algorithm 4** super\_p\_evident( $E, C, p$ )

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```
while  $E$  has changed do
  if  $\exists \omega \in E : \text{min\_belief}(E, C, \omega) \leq p$  then
     $E := E \setminus \{\omega\}$ 
return  $E$ 
```

---

consistent with that player’s observations. By only removing elements that either cause  $E$  to be unlikely or cause  $C$  to be unlikely, we are guaranteed to arrive at a more p-evident event at each iteration, and one that preserves belief in  $C$ . By starting with  $E$  as the entire state space, the final candidate event must be the largest, most p-evident event that player  $i$   $p$ -believes at state  $\omega$  in which  $C$  is common  $p$ -belief. This algorithm can also be viewed as traversing a unique nested sequence of maximally evident events (independent of  $i$  and  $\omega$ ) induced by  $C$  and the structure of  $(\Omega, \mu, (\Pi_0, \Pi_1))$ , halting at the first event in this sequence that does not include any elements of  $\Pi_i(\omega)$ .

The rational p-belief strategy consists of player  $i$  taking action  $A$  if  $\text{common\_p\_belief}(x = 1, i, \omega) > \frac{c-b}{a-b}$ . The matched p-belief strategy consists of player  $i$  choosing  $A$  with probability  $\text{common\_p\_belief}(x = 1, i, \omega)$ .

### 3.3 Iterated Reasoning

We now present our algorithms for the iterated reasoning strategies. For level  $k > 0$ , given a player  $i$  and a state  $\omega$ , the iterated maximization strategy computes

$$f_i^k(\omega) = \mathbb{1}\left(\sum_{\omega' \in \Pi_i(\omega)} P_i(\omega' | \omega) \left( P_i(x = 1 | \omega') f_{1-i}^{k-1}(\omega') a + P_i(x = 0 | \omega') f_{1-i}^{k-1}(\omega') d + (1 - f_{1-i}^{k-1}(\omega')) b \right) > c \right),$$

where  $\mathbb{1}(\cdot)$  is the indicator function, and  $f_i^0(\omega) = \mathbb{1}(P_i(x = 1 | \omega) > \frac{c-b}{a-b})$ . If  $f_i^k(\omega) = 1$ , then player  $i$  plays  $A$ , and otherwise player  $i$  plays  $B$ . For the iterated matching strategy,

$$q_i^k(\omega) = P_i(x = 1 | \omega) \cdot \sum_{\omega' \in \Pi_i(\omega)} P_i(\omega' | \omega) q_{1-i}^{k-1}(\omega').$$

$A$  is then played with probability  $q_i^k$ ,  $q_i^0 = P_i(x = 1 | \omega)$ .

## 4 Data

We now present the data we use for our empirical results. The dataset comes from the Thomas experiments (Thomas et al. 2014). These experiments presented participants with a stylized coordinated attack problem couched in a story about the butcher and the baker of a town. In their story, these merchants can either work together to produce hot dogs, or they can work separately to produce chicken wings and dinner rolls, respectively. The merchants can sell chicken wings and dinner rolls separately for a constant profit of  $c$  each on any day, but the profit of hot dogs varies from day-to-day. The

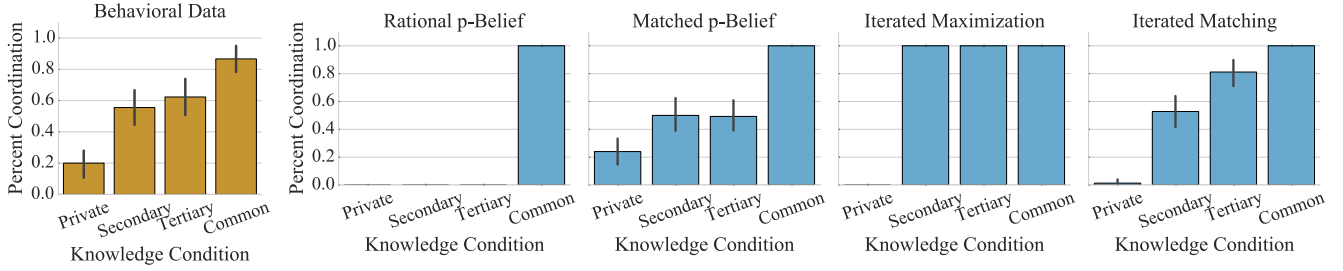


Figure 1: Data from the Thomas experiments and the predictions of each of the models we consider.

merchants make a profit of  $a$  each if  $x = 1$  on a particular day or  $d$  if  $x = 0$ . There is also a loudspeaker that sometimes publicly announces the prices of hot dogs, and a messenger who runs around the town delivering messages. The experiments had four different knowledge conditions that specified the information that participants received:

1. **Private Knowledge:** “The Messenger Boy [sic] has not seen the Butcher today, so he cannot tell you anything about what the Butcher knows.”
2. **Secondary Knowledge:** “The Messenger Boy says he stopped by the butcher shop before coming to your bakery. He tells you that the Butcher knows what today’s hot dog price is. However, he says that he forgot to mention to the Butcher that he was coming to see you, so the Butcher is not aware that you know today’s hot dog price.”
3. **Tertiary Knowledge:** “The Messenger Boy mentions that he is heading over to the butcher shop, and will let the Butcher know today’s price as well. The Messenger Boy will also tell the Butcher that he just came from your bakery and told you the price. However, the Messenger Boy will not inform the Butcher that he told you he would be heading over there. So, while the Butcher is aware that you know today’s price, he is not aware that you know that he knows that.”
4. **Common Knowledge:** “The loudspeaker broadcast the market price . . . The messenger boy did not come by. Because the market price was broadcast on the loudspeaker, the Butcher knows today’s price, and he knows that you know this information as well.”

After being shown this information as well as additional information indicating that  $x = 1$ , the participants were asked whether they would like to try to make hot dogs or not. The dataset from this experiment is visualized in Figure 1. Since the researchers provided evidence that the behavior of participants in their two-player experiments was invariant to payoffs, here we focus on their first payoff condition, in which  $a = 1.1$ ,  $b = 0$ ,  $c = 1$ , and  $d = 0.4$ .

We use this dataset to test whether the coordination strategies we have described are good models of human coordination in this setting. In order to be able to generate predictions for these models, we must determine a state space that represents the story in the Thomas experiments. We designed the following two probabilistic generative world models (one

for the messenger, and one for the loudspeaker) to be consistent with a reading of the knowledge conditions from those experiments. The  $\text{observe}(i, o)$  function indicates that player  $i$  observes  $o$ .

**Messenger:**

```

 $x \sim \text{Bernoulli}(\delta)$ 
 $\text{visit}_0 \sim \text{Bernoulli}(0.5)$ 
 $\text{visit}_1 \sim \text{Bernoulli}(0.5)$ 
 $\text{tell\_plan}_0 \sim \text{visit}_0 \wedge \text{Bernoulli}(0.5)$ 
 $\text{tell\_plan}_1 \sim \text{visit}_1 \wedge \text{Bernoulli}(0.5)$ 
if  $\text{visit}_0$ :
     $\text{observe}(0, x)$ 
    if  $\text{tell\_plan}_0$ :
         $\text{observe}(0, (\text{visit}_1, \text{tell\_plan}_1))$ 
if  $\text{visit}_1$ :
     $\text{observe}(1, (x, \text{visit}_0))$ 
    if  $\text{tell\_plan}_1$ :
         $\text{observe}(1, \text{tell\_plan}_0)$ 

```

**Loudspeaker:**

```

 $x \sim \text{Bernoulli}(\delta)$ 
 $\text{broadcast} \sim \text{Bernoulli}(0.5)$ 
if  $\text{broadcast}$ :
     $\text{observe}(0, x), \text{observe}(1, x)$ 

```

These models share a free parameter  $\delta$ . We take  $\delta = 0.25$ . This setting provides a closer fit to the empirical data than the maximum entropy setting of  $\delta = 0.5$ . We interpret statements that one player is “not aware” as meaning that the player could have been made aware, and assign a maximum entropy probability of 0.5 to these events.

The state spaces corresponding to these world models consist of the sets of all possible combinations of variables in the models’ generative processes:  $(x, \text{visit}_0, \text{visit}_1, \text{tell\_plan}_0, \text{tell\_plan}_1)$  for  $\Omega_{\text{messenger}}$  and  $(x, \text{broadcast})$  for  $\Omega_{\text{loudspeaker}}$ . The generative processes also uniquely specify probability measures over each state space. The knowledge conditions correspond to the following states. Private:  $(1, 1, 0, 1, 0) \in \Omega_{\text{messenger}}$ , Secondary:  $(1, 1, 1, 0, 1) \in \Omega_{\text{messenger}}$ , Tertiary:  $(1, 1, 1, 1, 0) \in \Omega_{\text{messenger}}$ , and Common Knowledge:  $(1, 1) \in \Omega_{\text{loudspeaker}}$ . The participants act as player 0 in all but the secondary condition. Due to high

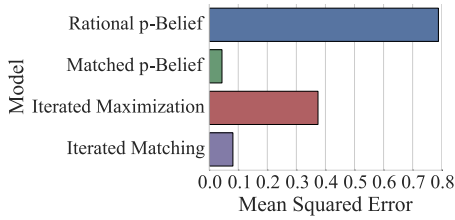


Figure 2: The mean-squared error of each model’s predictions on the Thomas experiments’ data.

ambiguity in the wording of the private knowledge condition, we considered two plausible readings. Either the messenger is communicating an intention to not visit the other player, or the messenger is being unhelpful in not offering any information about the messenger’s plan. By using the state  $(1, 1, 0, 1, 0)$  we assume the first interpretation. This interpretation results in a better empirical fit.

## 5 Results

We now present our empirical results. We first examine the predictions of each of the coordination strategies we consider given the generative processes representing the Thomas experiments. We then examine the extent to which a computer agent equipped with the best fitting model of human coordination is able to achieve higher payoffs in a simulated human-agent coordination problem. All of our code is available online at <https://github.com/pkrafft/modeling-human-ad-hoc-coordination>.

### 5.1 Model Comparison

To perform model comparison we compute the probability of choosing  $A$  that each model predicts given our formal representations of each of the four knowledge conditions. We then compare these predicted probabilities to the actual probabilities observed in the Thomas experiments. For the two iterated reasoning models we use a grid search over  $[0, 1, 2, 3, 4, 5]$  to find the best fitting  $k$  for each model (ultimately  $k = 1$  in iterated maximization and  $k = 3$  in iterated matching). The specific predictions of each model are shown in Figure 1. As shown in Figure 2, the matched p-belief model achieves the lowest mean-squared error. Qualitatively, the most striking aspect of the data that the matched p-belief model successfully captures is the similarity in the probability of coordination between the secondary and tertiary knowledge conditions. The two models that involve maximizing agents (rational p-belief and iterated maximization) both make predictions that are too extreme. The iterated matching model offers a competitive second place fit to the data, but it fails to capture the similarity between the middle two knowledge conditions.

The reason that the matched p-belief model makes good predictions for the two middle conditions is that the player in both of those conditions has the same amount of uncertainty appearing at some level of that player’s infinite recursive hierarchy of interpersonal beliefs. Common p-belief essentially represents a minimum taken over all these levels,

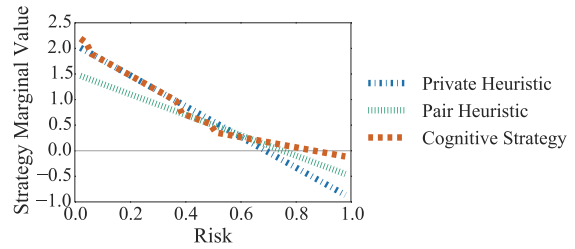


Figure 3: Performance of agents in our simulated human-agent coordination experiments. A strategy’s marginal value is the expected sum of payoffs the strategy obtained in each of the four knowledge conditions, minus the payoffs that could have been obtained by always playing  $B$ .

and thus the common p-belief in each of those two conditions is the same. The rational p-belief model is aware of the uncertainty at higher levels of recursive belief, but its predictions are too coarse due to the assumption of utility maximization. An interesting avenue for future work is to examine whether the rational p-belief model can be relaxed in any other way to allow for intermediate predictions, such as by allowing for heterogeneity in interpretations of the world model across agents. It is possible that the matched p-belief model is approximating such a case.

### 5.2 Human-Agent Coordination

Besides testing the fit of the models of human coordination that we have proposed, we are also interested in whether the best fitting model helps us improve outcomes in human-agent coordination. We use the data from the Thomas experiments to evaluate this possibility. For this task our computer agents implement what we call a “cognitive strategy”. An agent using the cognitive strategy chooses the action that maximizes expected utility under an assumption that the agent’s companion is using the matched p-belief model. These agents play the humans’ companion player in each of the four knowledge conditions of the Thomas experiments (player 1 in all but the Secondary condition). We evaluate the payoffs from the agents’ actions using the human data from the Thomas experiments. In this simulation we vary the payoffs  $(a, b, c, d)$ , and we assume that the humans would remain payoff invariant across the range of payoffs that we use. This assumption is reasonable given that participants in the Thomas experiments displayed payoff invariance in the two-agent case. We vary the payoffs according to risk, taking the payoffs to be  $(1, 0, p^*, 0)$  for a particular risk level  $p^* = \frac{c-b}{a-b}$ . As shown in Figure 3, we find that having the matched p-belief model of human coordination may help in human-agent coordination. We compared to two baseline strategies: an agent using a “private heuristic” who always coordinates if the agent knows  $x = 1$ , and an agent using a “pair heuristic” who always coordinates if the agent knows that both the agent and the human know  $x = 1$ . The private heuristic achieves good performance for low risk levels, and the pair heuristic achieves good performance for high risk levels. The cognitive strategy achieves good performance

at both low and high levels of risk, and only has negative marginal value over always playing the safe action B at very high levels of risk.

## 6 Discussion

In the present paper we focused on laying the groundwork for using common p-belief in AI and cognitive modeling. In particular, we developed an efficient algorithm for the inference of maximal perceived common p-belief, we showed that the coordination strategy of probability matching on common p-belief explains certain surprising qualitative features of existing data from a previous human experiment, and we showed that this model may also help improve agent outcomes in human-agent coordination. This work has three main limitations. Due to the small amount of human data we had and the lack of a held-out test set, our empirical results are necessarily only suggestive. While the data are inconsistent with the rational p-belief model and the iterated maximization model, the predictions of the iterated matching model and the matched p-belief model are both reasonably good. The strongest evidence we have favoring the matched p-belief model is this model's ability to produce equal amounts of coordination in the secondary and tertiary knowledge conditions as well as a low amount with private knowledge and a high amount with common knowledge. No iterated reasoning model under any formulation we could find of the Thomas experiments was able to capture the equality between the two middle conditions while maintaining good predictions at the extremes. Two other important limitations of our work are that the coordination task we consider did not involve intentional communication, and that the state and action spaces of the task were simple. While these features allowed us to easily test the predictions of each of our alternative models, it would be interesting to see how the models we considered would compare in more complex environments. A related interesting direction for future work is the application of inference of common p-belief through reasoning about p-evident events to artificial distributed systems, such as for developing or analyzing bitcoin/blockchain-like protocols, synchronizing remote servers, or distributed planning in ad hoc multi-robot teams.

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