On Manipulablity of Random Serial Dictatorship in Sequential Matching with Dynamic Preferences

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Abstract

We consider the problem of repeatedly matching a set of alternatives to a set of agents in the absence of monetary transfer. We propose a generic framework for evaluating sequential matching mechanisms with dynamic preferences, and show that unlike single-shot settings, the random serial dictatorship mechanism is manipulable.

Introduction

Consider the problem of assigning a set of indivisible objects to a set of agents, each having a private preference ordering over the objects. In various real-life applications the use of monetary transfer is prohibited: assigning dormitory rooms or college courses to students, teaching load among faculty, residents to hospitals, scarce medical resources and organs to patients, *etc.* (Sönmez and Ünver 2010a; 2010b; Dickerson, Procaccia, and Sandholm 2012). Despite a recent focus on designing mechanisms for problems with non-transferable utilities, little has been done on strategic behavior of agents in repeated assignment problems. Real-life applications are often situated in a temporal context with other decisions and stochastic events: assigning members to subcommittees each year, students to housing each year, *etc*.

In static settings, randomization is key in restoring strategyproofness and fairness in matching problems (Budish et al. 2013). The random serial dictatorship (RSD) is a randomized ordinal mechanism that is strategyproof, fair (in terms of equal treatment of equals), and ex post efficient (Abdulkadiroğlu and Sönmez 1998).

In this paper, we study matching problems in which a sequence of decisions must be made for agents whose preferences may change over time. Through our proposed generic framework for evaluating dynamic mechanisms, we investigate the incentive properties of the RSD mechanism in repeated assignment problems, and formally prove that RSD is susceptible to manipulation under dynamic preferences.

The Model

Consider a set of agents $N = \{1, \ldots, n\}$, and a set of alternatives $M = \{1, \ldots, m\}$, where $n \geq m$. Each agent has a strict preference over alternatives at time t, i.e. $a \succ_i^t b$. Since an agent's preferences can change, it may be the case that $a \succ_i^t b$ while $b \succ_i^{t+1} a$. Let $\mathcal{P}(M)$ or $\mathcal{P}$ denote the class of all strict linear preferences over M where $|\mathcal{P}| = m!$. Agent i's preference at time t is denoted by $\succ_i^t \in \mathcal{P}$, thus, $\succ^t = (\succ_1^t, \ldots, \succ_n^t) \in \mathcal{P}^n$ denotes the *preference profile* of agents at time t. We write $\succ_{-i}^t$ to denote $(\succ_1^t, \ldots, \succ_{i-1}^t, \succ_{i+1}^t, \ldots, \succ_n^t)$, and thus $\succ^t = (\succ_i^t, \succ_{-i}^t)$. A *matching*, $\mu^t : N \to M$, is a mapping from agents to objects, where $\mu^t(i) \in \mathcal{M}$ is agent i's assignment at time t. We let $\bar{\mu}^t$ denote a probability distribution over the set of possible (deterministic) matchings at time t . That is, $\bar{\mu}^t \in \Delta(\mathcal{M})$.

We consider a discrete-time *sequential matching process* as a sequence of matchings prescribed by a matching mechanism. Given a preference profile $\succ^t \in \mathcal{P}^n$, a *matching mechanism* $\pi(\mu | \succ^t)$ returns the probability of applying matching μ. Thus, the probability of agent i being allocated alternative x at time t is $p_i^t(x \,|\succ^t) = \sum_{\mu \in \mathcal{M}:\mu(i)=x} \pi(\mu \,|\succ^t)$, where $\sum_{i \in N} p_i^t(x | \succ^t) = 1$. The definition of a matching mechanism, π, incorporates randomized or deterministic matching mechanisms.

To evaluate a given matching mechanism π with ordinal preferences, we look at the expected probabilities of being allocated particular alternatives in the sequence of random matchings. More concretely, let o^ℓ be any alternative ranked in position ℓ. Given $\succ^t$, the expected probability that agent i receives alternatives with rankings as good as ℓ under matching mechanism π is defined recursively as

$$W_i^\pi(\succ^t, o^\ell) = \sum_{x=o^1}^{o^\ell} p_i^t(x | \succ^t) \times \quad (1)$$
$$\sum_{\mu \in \mathcal{M}} \sum_{\succ^{t+1} \in \mathcal{P}^n} \pi(\mu | \succ^t) P(\succ^{t+1} \mid \succ^t, \mu) W_i^\pi(\succ^{t+1}, o^\ell)$$

where $P(\succ^{t+1} \mid \succ^t, \mu^t)$ is the underlying transition kernel (as common knowledge) which denotes the probability that agents will transition to a state where they have joint preference $\succ^{t+1}$ after matching μ^t in a state with preference $\succ^t$.

Properties We are interested in analyzing the truthful properties of matching mechanisms in sequential matching problems. Inspired by the work of (Bogomolnaia and Moulin 2001), we define the required properties for analyzing random matchings in sequential settings based on the notion of first order stochastic dominance.

For two matching mechanisms π and π', *stochastic dominance* prescribes that given a transition model, for each rank ℓ, the expected probability that alternatives with rankings as good as ℓ get selected under π, is greater or equal to the expected probability that π' selects such alternatives.

Definition 1. *Given a transition model P, matching mechanism π **stochastically dominates** π', if at all states $\succ^t \in \mathcal{P}^n$, for all agents $i \in N$,*

$$\forall o^\ell \in M,\ W_i^\pi(\succ^t, o^\ell) \geq W_i^{\pi'}(\succ^t, o^\ell) \tag{2}$$

Global strategyproofness is an incentive requirement which states that under any possible transition of preference profiles, no agent can improve her sequence of random matchings by a strategic report. A one-shot matching mechanism coincides precisely with the random assignment problem (Abdulkadiroğlu and Sönmez 1998; Bogomolnaia and Moulin 2001).

Definition 2. *A matching mechanism is **globally strategyproof** iff for all transitions P, given any misreport $\hat{\succ}_i^t$ at time t such that $\hat{\succ}^t = (\hat{\succ}_i^t, \succ_{-i}^t)$, for all agents $i \in N$,*

$$\forall o^\ell \in M,\ W_i^\pi(\succ^t, o^\ell) \geq W_i^\pi(\hat{\succ}^t, o^\ell) \tag{3}$$

A matching is *Pareto efficient* if there is no other matching that makes all agents weakly better off and at least one agent strictly better off. A random matching is **ex post efficient** if it can be represented as a probability distribution over Pareto efficient deterministic matchings.

Sequential RSD

Random serial dictatorship randomly assigns a priority ordering to agents, and then the first agent in the ordering receives her favorite (the most preferred) alternative, the next agent receives his favorite alternative among the remaining ones, and so on. In single-shot settings, RSD is strategyproof, fair (in terms of equal treatment of equals), and ex post efficient (Abdulkadiroğlu and Sönmez 1998).

Theorem 1. *With fixed preferences, sequential RSD (a sequence of RSD induced matchings) is globally strategyproof, and yields a sequence of ex post efficient matchings.*

Proof. RSD is strategyproof in each period, thus, no agent can immediately benefit from misreporting. With fixed preferences, an agent's misreport does not affect the sequence of decisions, implying sequential RSD is strategyproof. □

While RSD satisfies strategyproofness at each period, we argue that a sequence of independent RSD induced random matchings (or *sequential RSD*) is prone to manipulation when agents have dynamic preferences.

Theorem 2. *With dynamic preferences, sequential RSD is not globally strategyproof.*

Proof. Consider 2 decision periods with deterministic preference dynamics known to agents. Let $\bar{\mu}^2_{\mu_1}$ denote the random matching at $t = 2$ after assignment μ_1 at $t = 1$. With truthful preferences (Table 1a), RSD induces the following random matching: $(\frac{1}{2}\mu_1, \frac{1}{2}\mu_2) = (\frac{1}{2}abc, \frac{1}{2}cba)$. If agent 3 misreports (Table 1b), the probability distribution would be $(\frac{2}{6}\mu_1, \frac{3}{6}\mu_2, \frac{1}{6}\mu_3) = (\frac{2}{6}abc, \frac{3}{6}cba, \frac{1}{6}acb)$. Assuming truthfulness in the second period, given a preference profile, identical decisions always result in identical next states. For each ranking position ℓ, we compute $W_3(\cdot, o^\ell)$ as shown in Table 2. For o^1 and o^2, the truthful revelation stochastically dominates the matching decisions when agent 3 is non-truthful. For strategyproofness we must show that for o^3, $1 \times \frac{3}{6}[\bar{\mu}^2_{\mu_1} + \bar{\mu}^2_{\mu_2}] \geq 1 \times [\frac{3}{6}(\bar{\mu}^2_{\mu_1}) + \frac{2}{6}(\bar{\mu}^2_{\mu_2}) + \frac{1}{6}(\bar{\mu}^2_{\mu_3})]$. By simple algebra, we see that for all preferences at the second period wherein agent 3's assignment under $\bar{\mu}^2_{\mu_3}$ stochastically dominates $\bar{\mu}^2_{\mu_2}$ the above inequality does not hold. Thus, sequential RSD is not globally strategyproof. □

(a) Truthful

$\succ_1$:	$a \succ c \succ b$
$\succ_2$:	$b \succ c \succ a$
$\succ_3$:	$a \succ c \succ b$

(b) Misreport

$\succ_1$:	$a \succ c \succ b$
$\succ_2$:	$b \succ c \succ a$
$\hat{\succ}_3$:	$a \succ b \succ c$

Table 1: Preferences revealed by three agents.

	$W_3(\succ^t, o^\ell)$	$W_3((\hat{\succ}_i^t, \succ_{-i}^t), o^\ell)$
o^1	$\frac{3}{6} \times \frac{3}{6}(\bar{\mu}^2_{\mu_1})$	$\frac{3}{6} \times \frac{3}{6}(\bar{\mu}^2_{\mu_1})$
o^2	$1 \times \frac{3}{6}[\bar{\mu}^2_{\mu_1} + \bar{\mu}^2_{\mu_2}]$	$\frac{5}{6} \times [\frac{3}{6}(\bar{\mu}^2_{\mu_1}) + \frac{2}{6}(\bar{\mu}^2_{\mu_2})]$
o^3	$1 \times \frac{3}{6}[\bar{\mu}^2_{\mu_1} + \bar{\mu}^2_{\mu_2}]$	$1 \times [\frac{3}{6}(\bar{\mu}^2_{\mu_1}) + \frac{2}{6}(\bar{\mu}^2_{\mu_2}) + \frac{1}{6}(\bar{\mu}^2_{\mu_3})]$

Table 2: Evaluation of the matching mechanism for agent 3.

Concluding Remarks

We showed that in dynamic settings, RSD is susceptible to manipulation. Due to the incompatibility of strategyproofness and efficiency, we only focused on incentive properties of random mechanisms. The aim, in the future, is to evaluate the efficiency of such mechanisms and design truthful mechanisms that satisfy some approximate notion of efficiency.[1]

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[1] More details on the overarching research is in (Hosseini, Larson, and Cohen 2015).