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Controlled School Choice with Soft Bounds and Overlapping Types

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Abstract

School choice programs are implemented to give students/parents an opportunity to choose the public school the students attend. Controlled school choice programs need to provide choices for students/parents while maintaining distributional constraints on the balance on the composition of students, typically in terms of socioeconomic status. Previous works show that setting soft-bounds, which flexibly change the priorities of students based on their types, is more appropriate than setting hard-bounds, which strictly limit the number of accepted students for each type. We consider a case where soft-bounds are imposed and one student can belong to multiple types, e.g., "financially-distressed" and "minority" types. We first show that when we apply a model that is a straightforward extension of an existing model for disjoint types, there is a chance that no stable matching exists. Thus, we propose an alternative model and an alternative stability definition, where a school has reserved seats for each type. We show that a stable matching is guaranteed to exist in this model, and develop a mechanism called Deferred Acceptance for Overlapping Types (DA-OT). The DA-OT mechanism is strategy-proof and obtains the student-optimal matching within all stable matchings. Computer simulation results illustrate that the DA-OT outperforms an artificial cap mechanism, where the number of seats for each type is fixed.

Introduction

The theory of two-sided matching has been developed and has been applied to various markets in practice.[1] Also, there is a growing interest for this topic among AI and multiagent systems researchers, e.g., handling optimization problems in kidney exchange (Awasthi and Sandholm 2009), compactly representing preferences (Pini, Rossi, and Venable 2014), handling partial preferences (Drummond and Boutilier 2013), and so on.

In many application domains, various distributional constraints are often imposed on an outcome. For example, regional maximum quotas are imposed in the Japan Residency Matching Program, which organizes matching between medical residents and hospitals. To avoid placing too many doctors in urban areas and causing doctor shortage in rural areas, the Japanese government now imposes a regional maximum quota on each region of the country (Kamada and Kojima 2014). Regional maximum quotas are utilized in various contexts, such as Chinese graduate school admissions, Ukrainian college admissions, Scottish probationary teacher matching, among others (Kamada and Kojima 2014). Furthermore, there are many matching problems in which minimum quotas are imposed (Fragiadakis et al. 2012; Goto et al. 2014). School districts may need at least a certain number of students in each school in order for the school to operate, as in college admissions in Hungary (Biró et al. 2010). The cadet-branch matching program organized by United States Military Academy imposes minimum quotas on the number of cadets who can be assigned to each branch (Sönmez and Switzer 2013).

Yet another type of distributional constraint dealt with in this paper is diversity constraints in school choice programs. Such programs are implemented to give students/parents an opportunity to choose the public school the students attend. However, a school is required to satisfy balance on the composition of students, typically in terms of socioeconomic status. Controlled school choice programs need to provide choices for students/parents while maintaining distributional constraints.

A seminal work by Abdulkadiroglu and Sonmez (2003) proposes using the Deferred Acceptance (DA) mechanism in school choice programs. Kojima (2012) considers a model where there are two types of students, i.e., minority and majority, and shows that setting hard-bounds for the number of majority students may hurt the minority students. To overcome this shortcoming, Hafalir, Yenmez, and Yildirim (2013) propose soft-bounds for the number of minority students, i.e., schools give higher priority to minority students up to a certain number. Kominers and Sönmez (2012) consider a model where each seat/slot of one school has a different priority ranking for students. This model can represent certain types of affirmative action and can be considered a generalization of (Hafalir, Yenmez, and Yildirim 2013). Ehlers et al. (2014) generalize the model in (Hafalir, Yenmez, and Yildirim 2013) to cases where the

[1]See Roth and Sotomayor (1990) for a comprehensive survey of many results in this literature.

number of student types can be more than two.

Our work is based on Ehlers et al. (2014) and examines a more general case where one student can belong to multiple types, e.g., a student belongs to both 'financially-distressed" and "minority" types. Although we can assume all types are disjoint by enumerating all combinations of types and consider each combination a different type (e.g., financially-distressed minority, financially-distressed majority, financially-sound minority, and financially-sound majority), setting an appropriate quota for each finely divided type would be difficult. Furthermore, when the number of types increases, considering all combinations and setting appropriate goals for each type combination becomes impractical. As far as the authors aware, no previous work has considered a model where a student can belong to multiple types with soft-bounds.

In this paper, we first show a model that is a straightforward extension of the model used in Ehlers et al. (2014). It turns out that we cannot guarantee the existence of a stable matching in this model. Thus, we propose an alternative model and an alternative stability definition, where a school has reserved seats for each type, and stability is defined based on the number of students who are assigned to the reserved seats. We show that a stable matching is guaranteed to exist and develop a mechanism called Deferred Acceptance for Overlapping Types (DA-OT) that obtains a stable matching. The DA-OT mechanism is strategy-proof and obtains the student-optimal matching within all stable matchings. We also show computer simulation results, which illustrate that the DA-OT outperforms an artificial cap mechanism, where the number of reserved seats of each type is fixed.

Model for controlled school choice program

Basic model

We first show a basic model that is a straightforward extension of the one presented in Ehlers et al. (2014). A market is a tuple $(S, C, T, \tau, X, \succ_S, \succ_C, q_C, p_{C,T})$, where each component is defined as follows:

- finite number of students $S = \{s_1, s_2, \ldots, s_n\}$,
- finite number of schools $C = \{c_1, c_2, \ldots, c_m\}$,
- type space $T = \{t_1, t_2, \ldots, t_k\}$,
- type function $\tau : S \to 2^T$, where $\tau(s)$ is the set of types to which student s belongs,
- set of contracts $X = S \times C$, where $(s, c) \in X$ means student s is matched to school c,
- students' preference profile $\succ_S = (\succ_{s_1}, \ldots, \succ_{s_n})$, where $\succ_s$ is the strict preference relation of student $s \in S$ over contracts related to s, i.e., $(s, c) \succ_s (s, c')$ means that student s strictly prefers school c over school c',
- schools' priority profile $\succ_C = (\succ_{c_1}, \ldots, \succ_{c_m})$, where each $\succ_c$ is the strict priority ranking of school $c \in C$ over contracts related to c, i.e., $(s, c) \succ_c (s', c)$ means that student s has a higher priority ranking than student s' to be enrolled at school c,
- vector of maximum quotas $q_C = (q_c)_{c \in C}$, where q_c is the maximum quota (capacity) of school $c \in C$,
- vector of soft-bounds $p_{C,T} = (p_{c_1,T}, \ldots, p_{c_m,T})$, where each $p_{c,T} = (p_{c,t_1}, \ldots, p_{c,t_k})$ represents type specific soft-bounds, i.e., $p_{c,t}$ is a (non-binding) target quota of students with type t that school c is supposed to accept.

We assume $|\tau(s)| \geq 1$, i.e., a student can belong to multiple types. This is the only essential difference between our model and the one in Ehlers et al. (2014), which assumes $|\tau(s)| = 1$. To be more precise, the model in Ehlers et al. (2014) applies two parameters, i.e., a floor and a ceiling, that specify a non-binding target range. In this paper, to simplify our model and to make theoretical analysis more tractable, we use only one parameter that corresponds to the floor for each school and each type.[2]

For notational simplicity, we assume each contract $(s, c) \in X$ is acceptable for both s and c. This assumption is not crucial and the results obtained in this paper still hold when this assumption is relaxed, i.e., when a student (or a school) considers some schools (or students) unacceptable. We also assume $\sum_{c \in C} q_c \geq n$ holds, i.e., the total capacity of the schools is large enough to accept all the students. We assume $\sum_{t \in T} p_{c,t} \leq q_c$ holds for all $c \in C$, i.e., target quotas can be satisfied without violating the maximum quota.

For $X' \subseteq X$, let X'_s denote $\{(s, c) \in X' \mid c \in C\}$, and X'_c denote $\{(s, c) \in X' \mid s \in S\}$. Also, let $X'_{c,t}$ denote $\{(s, c) \in X' \mid s \in S, t \in \tau(s)\}$.

We say X' is feasible if $|X'_s| = 1$ and $|X'_c| \leq q_c$ hold for all $s \in S$ and $c \in C$. We call a feasible set of contracts a **matching**. Let $\mathcal{X}$ denote a set of matchings. $X' \in \mathcal{X}$ is **student-optimal** within $\mathcal{X}$ if $X'_s \succ_s X''_s$ or $X'_s = X''_s$ hold for all $X'' \in \mathcal{X}$ and $s \in S$. A mechanism is a function that takes a profile of students' preferences as input and returns matching X'. We say a mechanism is **strategy-proof** if no student ever has any incentive to misreport her preference, regardless what the other students report.

Let us introduce two conditions that compose stability.

Definition 1 (fairness). We say student s has justifiable envy toward $s' \neq s$ in matching X', where $(s, c), (s', c') \in X'$, if the following conditions hold: $(s, c') \succ_s (s, c)$, $(s, c') \succ_{c'} (s', c')$, and $\forall t \in \tau(s') \setminus \tau(s)$, $|X'_{c',t}| > p_{c',t}$ (or $\tau(s') \setminus \tau(s) = \emptyset$). We say that matching X' is fair if no student has justifiable envy. We say a mechanism is fair if it always gives a fair matching.

Basically, student s can have justifiable envy toward another student s', when s would rather be matched to school c' than her current school c, and she has a higher priority ranking at c' than student s'. However, if s' belongs to type t (and s does not belong to it), and the number of type t students accepted to c' is less than or equal to $p_{c',t}$, the envy of s toward s' cannot be justified.

Definition 2 (nonwastefulness). We say student s claims an empty seat of c' in matching X', where $(s, c) \in X'$, if the following conditions hold: $(s, c') \succ_s (s, c)$ and $|X'_{c'}| < q_{c'}$.

[2]In principle, extending our model and mechanism to handle a floor and a ceiling for each type is rather straightforward. Since these quotas are soft-bounds, they can be treated in a similar way. However, this change makes the notations and descriptions extremely verbose. Thus, we decided to apply a simpler model.

Also, we say student s claims an empty seat of c' by type in matching X', where $(s, c) \in X'$, if the following conditions hold: $(s, c') \succ_s (s, c)$ and $\exists t \in \tau(s), |X'_{c',t}| < p_{c',t}$. We say that matching X' is nonwasteful if no student claims an empty seat or claims an empty seat by type. We say a mechanism is nonwasteful if it always gives a nonwasteful matching.

We say a matching is **stable** if it is fair and nonwasteful. When $|\tau(s)| = 1$ for all $s \in S$, our definition of stability becomes equivalent to fairness under the soft-bounds and nonwastefulness used in Ehlers et al. (2014).

Next, let us show a case where no stable matching exists.

Example 1. *Assume* $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, $T = \{t_1, t_2, t_3\}$, $\tau(s_1) = \{t_3\}$, $\tau(s_2) = \{t_1, t_2\}$, $\tau(s_3) = \{t_1\}$, *and* $\tau(s_4) = \{t_2\}$. *Also assume* $q_{c_1} = 2$, $p_{c_1,t_1} = 1$, $p_{c_1,t_2} = 1$, $p_{c_1,t_3} = 0$, $q_{c_2} = 1$, *and* $q_{c_3} = 1$. *We assume* $p_{c_2,t}$ *and* $p_{c_3,t}$ *are* 0 *for all* t. *The priorities of all schools are identical, i.e.,* $\succ_c$: $(s_1, c), (s_2, c), (s_3, c), (s_4, c)$. *Also,* $\succ_{s_2}$: $(s_2, c_2), (s_2, c_1), (s_2, c_3)$. *The preferences of the other students are identical, i.e.,* $\succ_s$: $(s, c_1), (s, c_2), (s, c_3)$.

Let us show that there exists no stable matching in this situation. First, let us assume s_2 is assigned to c_1. If s_3 or s_4 is assigned to c_1, s_1 has justifiable envy. If no other student is assigned to c_1, s_1 claims an empty seat of c_1. Thus, s_1 must be assigned to c_1. Then, s_3 or s_4 must be assigned to c_2. However, s_2 has justifiable envy toward the student assigned to c_2. Thus, s_2 cannot be assigned to c_1.

Next, let us assume s_2 is not assigned to c_1. Then both s_3 and s_4 are assigned to c_1. Otherwise, they can claim an empty seat by type. Assume s_2 is assigned to c_2. Then s_1 must be assigned to c_3. However, since s_1 has justifiable envy toward s_2, s_2 cannot be assigned to c_2. Thus, let us assume s_2 is assigned to c_3. However, then s_2 has justifiable envy toward s_3 or s_4. If s_2 is not assigned to any school, there exists school c whose maximum quota is not satisfied. Then s_2 claims an empty seat of c. Thus, there exists no matching that is fair and nonwasteful.

New model

Let us consider a slightly modified model. The model is quite similar to the previous model. One major difference is that school c provides distinct **reserved seats** for each type, and each contract explicitly states the fact that a student is assigned to a particular seat of a school. The preferences/priorities of the students/schools, as well as stability requirements, are defined based on these contracts.

To be more precise, we represent a market as a tuple $(S, C, T, \tau, X, \succ_S, \succ_C, q_C, p_{C,T})$. The definitions of S, C, T, τ, and q_C are identical to the previous model. School c provides distinct reserved seats for each type. Contract $x \in X$ is represented as (s, c, t), which describes the fact that student s is assigned to type t seat of school c. Thus, X is given as $\{(s, c, t) \mid s \in S, c \in C, t \in \tau(s)\}$.

We assume student s has a strict preference over contracts related to her. Thus, s has a preference over seats of the same school. This assumption is natural if we assume a school provides a different program for different seats, e.g., a student who is assigned to a "financially-distressed" seat can obtain a scholarship, and a student who is assigned to an "English as a second language" seat can attend an English language class. Also, this definition allows a case such that $(s, c, t) \succ_s (s, c', t) \succ_s (s, c, t') \succ_s (s, c', t')$, i.e., student s prefers type t seat over type t' seat, and if the types of seats are the same, she prefers school c over school c'.[3]

We also assume school c has a strict priority ranking over contracts related to it. Thus, when both s and s' have types t and t', there is a chance that $(s, c, t) \succ_c (s', c, t)$ and $(s', c, t') \succ_c (s, c, t')$ hold, i.e., the relative ordering of s and s' can be different for different seats. This assumption is also natural, e.g., for "financially-distressed" seats, s has a higher priority ranking than s' since s is more financially distressed, while for "standard" seats, s' has a higher priority ranking than s since s' has a better SAT score.

For any subset of contracts $X' \subseteq X$, let X'_s denote $\{(s, c, t) \in X' \mid c \in C, t \in T\}$, and X'_c denote $\{(s, c, t) \in X' \mid s \in S, t \in T\}$. Also, let $X'_{c,t}$ denote $\{(s, c, t) \in X' \mid s \in S\}$. Note that $|X'_{c,t}|$ means the number of type t students accepted to c in the previous model. Here, $|X'_{c,t}|$ means the number of students accepted to type t seats of school c. The actual number of type t students accepted to c can exceed $|X'_{c,t}|$, since a student who has another type (as well as t) might be assigned to a different seat of school c.

The definition of feasibility in this model is identical to the original model. We modify the definitions of fairness and nonwastefulness as follows.

Definition 3 (fairness). We say student s has justifiable envy toward $s' \neq s$ in matching X', where $(s, c, t), (s', c', t') \in X'$ and $(s, c', t'') \in X \setminus X'$, if the following conditions hold: $(s, c', t'') \succ_s (s, c, t)$, $(s, c', t'') \succ_{c'} (s', c', t')$, and either **(fr-i)** $t' = t''$ or **(fr-ii)** $|X'_{c',t'}| > p_{c',t'}$.

As in the original definition, basically, student s can have justifiable envy toward another student s', when s prefers (s, c', t'') over her current contract, and (s, c', t'') has a higher priority ranking at c' than (s', c', t'). However, if $t' \neq t''$ and the number of type t' contracts accepted to c' is less than or equal to $p_{c',t'}$, this envy cannot be justified.

Definition 4 (nonwastefulness). We say student s claims an empty seat of c' in matching X', where $(s, c, t) \in X'$ and $(s, c', t') \in X \setminus X'$, if $(s, c', t') \succ_s (s, c, t)$, and one of the following conditions holds:

(nw-i) $|X'_{c'}| < q_{c'}$, or

(nw-ii) $c = c', (s, c, t') \succ_c (s, c, t)$, and $|X'_{c,t}| > p_{c,t}$.

We say student s claims an empty seat of c' by type in matching X', where $(s, c, t) \in X'$ and $(s, c', t') \in X \setminus X'$, if $(s, c', t') \succ_s (s, c, t)$ and the following condition holds:

(nw-iii) $|X'_{c',t'}| < p_{c',t'}$.

Conditions (**nw-i**) and (**nw-iii**) are basically identical to the original definition. Condition (**nw-ii**) can be considered

[3]If a student is indifferent between different seats of the same school, we can artificially extend her preference and apply our mechanism. Our mechanism is still strategy-proof and obtains a stable matching. However, we can no longer guarantee student-optimality. See Abdulkadiroglu, Pathak, and Roth (2009) for discussion on handling indifference in students' preferences.

as s has justifiable envy toward her own contract (s, c, t), which is accepted while she prefers (s, c, t') and it has a higher priority ranking at c than (s, c, t).

One main reason that no stable matching exists in the previous example is that s_1 and s_2 are *complimentary* for c_1, i.e., if c_1 accepts s_2, it also needs to accept s_1, while if c_1 does not accept s_2, it cannot accept s_1. By introducing type-specific contracts, such a complementary relation disappears. Accepting contract (s_2, c_1, t_1) (or (s_2, c_1, t_2)) does not give any advantage for (s_1, c_1, t_3) to be accepted. As shown in the next section, a stable (i.e., fair and nonwasteful) matching always exists.

Deferred Acceptance mechanism for Overlapping Types

In this section we propose a strategy-proof mechanism called Deferred Acceptance mechanism for Overlapping Types (DA-OT).

The mechanism is built upon choice functions $Ch_S : 2^X \to 2^X$ and $Ch_C : 2^X \to 2^X$. Ch_S is a choice function of students S, which is a union of individual choice functions, i.e., $Ch_S(X') := \bigcup_{s \in S} Ch_s(X')$. In the same way, Ch_C is a choice function of schools C, which is a union of individual choice functions, i.e., $Ch_C(X') := \bigcup_{c \in C} Ch_c(X')$. The choice function of student s is simple. $Ch_s(X')$ returns $\{(s, c, t)\} \in X'_s$, where (s, c, t) is s's most preferred contract in X'_s (or $\emptyset$ if X'_s is $\emptyset$). The choice function of school c is more complicated and is defined as follows. Here, $Y_{c,t} := \{(s, c, t) \in Y \mid s \in S\}$ and $Z_{c,t} := \{(s, c, t) \in Z \mid s \in S\}$.

Definition 5 (Choice function of school c). $Ch_c(X')$ is defined as follows:

Step 1 $Y \leftarrow \emptyset$, $Z \leftarrow X'_c$.

Step 2 For each $t \in T$, repeat the following procedure: If $|Y_{c,t}| = p_{c,t}$ or $|Z_{c,t}| = 0$ then go to the procedure for next t. Otherwise, choose $(s, c, t) \in Z$ with the highest priority ranking in Z based on $\succ_c$. Move (s, c, t) from Z to Y.

Step 3 Repeat the following procedure: If $|Y| = q_c$ or $|Z| = 0$, return Y. Otherwise, choose $(s, c, t) \in Z$ with the highest priority ranking in Z based on $\succ_c$. Move (s, c, t) from Z to Y.

The DA-OT is one instance of the generalized Gale-Shapley mechanism presented in Hatfield and Milgrom (2005), which is defined as follows.

Definition 6 (Deferred Acceptance mechanism for Overlapping Types (DA-OT)).

Step 1 $Re \leftarrow \emptyset$.

Step 2 $X' \leftarrow Ch_S(X \setminus Re)$, $X'' \leftarrow Ch_C(X')$.

Step 3 If $X' = X''$, then return X', otherwise, $Re \leftarrow Re \cup (X' \setminus Ch_C(X'))$, go to Step 2.

Here Re represents the rejected contracts. A student cannot choose a contract in Re. First, students propose a set of contracts that are most preferred and not rejected so far X'. Then schools choose X'', which is a subset of X'. If no contract is rejected, the mechanism terminates. Otherwise, the rejected contracts are added to Re, and the mechanism repeats the same procedure.

About the complexity of the DA-OT, Re in Definition 6 expands monotonically. Thus, Step 2 is repeated at most $|X|$ times where X is the set of all contracts. The time required to calculate $Ch_S(X')$ or $Ch_C(X')$ is at most linear in $|X'|$. Overall, the time complexity of the DA-OT is $O(|X|^2)$.

We show an example to illustrate the execution of the DA-OT.

Example 2. *We consider the situation basically identical to Example 1. We assume the preference of s_2 is given as:*

$$\succ_{s_2}: \ (s_2, c_2, t_1),\ (s_2, c_2, t_2),\ (s_2, c_1, t_1),\ (s_2, c_1, t_2),\ (s_2, c_3, t_1),\ (s_2, c_3, t_2).$$

Also, the preferences of all schools are identical, and it is defined as

$$\succ_c: \ (s_1, c, t_3),\ (s_2, c, t_1),\ (s_2, c, t_2),\ (s_3, c, t_1),\ (s_4, c, t_2).$$

First, each student chooses her most preferred contract; $X' = \{(s_1, c_1, t_3), (s_2, c_2, t_1), (s_3, c_1, t_1), (s_4, c_1, t_2)\}$. $Ch_{c_1}(X')$ is $\{(s_3, c_1, t_1), (s_4, c_1, t_2)\}$. (s_1, c_1, t_3), which has the highest priority ranking, is rejected since (s_3, c_1, t_1) and (s_4, c_1, t_2) are selected at Step 2 in Definition 5.

Then s_1 chooses her second preferred contract; $X' = \{(s_1, c_2, t_3), (s_2, c_2, t_1), (s_3, c_1, t_1), (s_4, c_1, t_2)\}$. $Ch_{c_2}(X')$ is $\{(s_1, c_2, t_3)\}$, since (s_1, c_2, t_3) has a higher priority ranking than (s_2, c_2, t_1).

Next s_2 chooses her second preferred contract, i.e., (s_2, c_2, t_2), but this contract is also rejected. Thus, s_2 chooses her third preferred contract, i.e., (s_2, c_1, t_1); $X' = \{(s_1, c_2, t_3), (s_2, c_1, t_1), (s_3, c_1, t_1), (s_4, c_1, t_2)\}$. $Ch_{c_1}(X')$ is $\{(s_2, c_1, t_1), (s_4, c_1, t_2)\}$, since (s_2, c_2, t_1) has a higher priority ranking than (s_3, c_1, t_1).

Then s_3 chooses her second preferred contract; $X' = \{(s_1, c_2, t_3), (s_2, c_1, t_1), (s_3, c_2, t_1), (s_4, c_1, t_2)\}$. $Ch_{c_2}(X')$ is $\{(s_1, c_2, t_3)\}$, since (s_1, c_2, t_3) has a higher priority ranking than (s_3, c_2, t_1).

Finally, s_3 chooses her third preferred contract; $X' = \{(s_1, c_2, t_3), (s_2, c_1, t_1), (s_3, c_3, t_1), (s_4, c_1, t_2)\}$. Then, no contract is rejected and the mechanism terminates.

Note that in the original model, s_1 has justifiable envy toward s_4 because c_1's target quotas are satisfied by accepting s_2. This is not the case in our new model, since s_2 is accepted for a type t_1 seat and s_4 is accepted for a type t_2 seat.

Properties of DA-OT

In this section, we first introduce another stability condition called Hatfield-Milgrom (HM)-stability (Hatfield and Milgrom 2005). Next, we show HM-stability is equivalent to our stability (Lemma 1). In Hatfield and Milgrom (2005), it is shown that if Ch_C satisfies three properties, i.e., the irrelevance of rejected contracts, the law of aggregate demand, and the substitutes condition, then a HM-stable matching always exists. Also, the generalized Gale-Shapley mechanism is strategy-proof and obtains the student-optimal matching in all the HM-stable matchings. We show Ch_C satisfies these properties (Lemma 2). From these lemmas, we obtain our main theorem.

Definition 7 (HM-stability). We say matching X' is HM-stable if there exists no $x \in X \setminus X'$ such that $x \in Ch_S(X' \cup \{x\})$ and $x \in Ch_C(X' \cup \{x\})$.

Lemma 1. *Matching X' is HM-stable iff it is fair and nonwasteful.*

Proof. We first show that HM-stability implies fairness and nonwastefulness. Let us assume X' is not fair, i.e., there exists s, s', where $(s, c, t), (s', c', t') \in X'$, $(s, c', t'') \in X \setminus X'$, $(s, c', t'') \succ_s (s, c, t)$, and $(s, c', t'') \succ_{c'} (s', c', t')$ hold. It is clear that $(s, c', t'') \in Ch_s(X' \cup \{(s, c', t'')\})$ holds. Let us examine the choice function of school c'. First, let us assume $t' = t''$ (Condition (**fr-i**) in Definition 3) holds. Then, (s, c', t') is chosen before (s', c', t') since $(s, c', t') \succ_{c'} (s', c', t')$ holds. Therefore, $(s, c', t') \in Ch_{c'}(X' \cup \{(s, c', t')\})$ holds. Thus, X' is not HM-stable. Next, let us assume $|X'_{c',t'}| > p_{c',t'}$ (Condition (**fr-ii**) in Definition 3) holds. From the fact that $(s', c', t') \in Ch_{c'}(X')$, (s', c', t') is accepted at Step 3 in Definition 5, or there exists another contract (s'', c', t') that is accepted at Step 3 in Definition 5 and $(s', c', t') \succ_{c'} (s'', c', t')$. Thus, when calculating $Ch_{c'}(X' \cup \{(s, c', t'')\})$, if (s, c', t'') is not accepted at Step 2, it should be accepted at Step 3, since (s, c', t'') is chosen before (s', c', t') or (s'', c', t'). Thus, $(s, c', t'') \in Ch_{c'}(X' \cup \{(s, c', t'')\})$ holds; X' is not HM-stable.

Next, let us assume s claims an empty seat of c'. Thus, $(s, c, t) \in X'$, $(s, c', t') \in X \setminus X'$, and $(s, c', t') \succ_s (s, c, t)$ hold. It is clear that $(s, c', t') \in Ch_s(X' \cup \{(s, c', t')\})$. Also, if $|X_{c'}| < q_{c'}$, i.e., Condition (**nw-i**) in Definition 4, holds, it is clear that $(s, c', t') \in Ch_{c'}(X' \cup \{(s, c', t')\})$. Furthermore, if $c = c'$, $(s, c, t') \succ_c (s, c, t)$, i.e., Condition (**nw-ii**) in Definition 4, holds, then either (s, c, t) is accepted at Step 3 in Definition 5, or there exists another contract (s', c, t) that is accepted at Step 3 in Definition 5 and $(s, c, t) \succ_c (s', c, t)$. Thus, when calculating $Ch_c(X' \cup \{(s, c, t')\})$, if (s, c, t') is not accepted at Step 2, it should be accepted at Step 3 since (s, c, t') is chosen before (s, c, t) or (s', c, t). Thus, $(s, c, t') \in Ch_c(X' \cup \{(s, c, t')\})$ holds; X' is not HM-stable.

Finally, let us assume s claims an empty seat of c' by type. Thus, $(s, c', t') \in X \setminus X'$ and $(s, c', t') \succ_s (s, c, t)$ holds. It is clear that $(s, c', t') \in Ch_s(X' \cup \{(s, c', t')\})$. Also, $|X'_{c',t'}| < p_{c',t'}$, i.e., Condition (**nw-iii**) in Definition 4, holds. Then, $(s, c', t') \in Ch_{c'}(X' \cup \{(s, c', t')\})$ holds since (s, c', t') should be accepted at Step 2; X' is not HM-stable.

Next, let us show that fairness and nonwastefulness imply HM-stability. Assume matching X' is not HM-stable, i.e., there exists $(s, c, t) \in X \setminus X'$ such that $(s, c, t) \in Ch_s(X' \cup \{(s, c, t)\})$ and $(s, c, t) \in Ch_c(X' \cup \{(s, c, t)\})$ hold. Let us assume $(s, c', t') \in X'$. It is clear that $(s, c, t) \succ_s (s, c', t')$ holds since $(s, c, t) \in Ch_s(X' \cup \{(s, c, t)\})$.

Assume $Ch_c(X' \cup \{(s, c, t)\}) = Ch_c(X') \cup \{(s, c, t)\}$, i.e., no contract is rejected as the consequence of accepting (s, c, t). Then $|X'_c| < q_c$ must hold and X' is wasteful from Condition (**nw-i**) in Definition 4. Thus, let us assume $(s', c, t'') \in Ch_c(X')$ and $(s', c, t'') \notin Ch_c(X' \cup \{(s, c, t)\})$ hold, i.e., (s', c, t'') is rejected as the consequence of accepting (s, c, t).

First, let us consider the case where $|X'_{c,t}| < p_{c,t}$. Then, s claims an empty seat of c by type since Condition (**nw-iii**) in Definition 4 holds. Next, let us consider the case where $s = s'$ (and $t'' = t'$). Then, $(s, c, t) \succ_c (s, c, t')$ and $|X'_{c,t'}| > p_{c,t'}$ holds. Thus, s claims an empty seat of c since Condition (**nw-ii**) in Definition 4 holds. Finally, let us consider the case where $s \neq s'$. Then, either (i) $t = t''$ and $(s, c, t) \succ_c (s', c, t)$, or (ii) $(s, c, t) \succ_c (s', c, t'')$ and $|X'_{c,t''}| > p_{c,t''}$ holds. In case (i) s has justifiable envy toward s' since $(s, c, t) \succ_c (s', c, t)$, and Condition (**fr-i**) in Definition 3 holds. In case (ii), s has justifiable envy toward s' since $(s, c, t) \succ_c (s', c, t)$, and Condition (**fr-ii**) in Definition 3 holds. □

Lemma 2. *Ch_C satisfies the following properties.*

Irrelevance of rejected contracts: For any $X' \subseteq X$ and $x \in X \setminus X'$, $Ch_C(X') = Ch_C(X' \cup \{x\})$ if $x \notin Ch_C(X' \cup \{x\})$.

Law of aggregate demand: For any $X', X'' \subseteq X$ with $X' \subseteq X''$, $|Ch_C(X')| \leq |Ch_C(X'')|$.

Substitutes condition: For any $X', X'' \subseteq X$ with $X' \subseteq X''$, $X' \setminus Ch_C(X') \subseteq X'' \setminus Ch_C(X'')$.

Proof. It is obvious that the irrelevance of rejected contracts holds from Definition 5. To show the law of aggregate demand and the substitutes condition, it is sufficient to show that each individual choice function Ch_c satisfies the following conditions:

(a) For any $X' \subseteq X$ and $x \in X \setminus X'$, $|Ch_c(X')| \leq |Ch_c(X' \cup \{x\})|$.

(b) For any $X' \subseteq X$ and $x \in X \setminus X'$, $X' \setminus Ch_c(X') \subseteq (X' \cup \{x\}) \setminus Ch_c(X' \cup \{x\})$.

From Definition 5, it is clear that if $|X'_c| \leq q_c$, then $Ch_c(X') = X'$ holds. Also, if $|X'_c| \geq q_c$, then $|Ch_c(X')| = q_c$ holds. In words, if the number of contracts related to c is less than or equal to q_c, then no contract is rejected. Also, if the number of contracts related to c is more than or equal to q_c, then the number of accepted contracts equals q_c. Thus, it is clear that condition (a) holds.

Assume condition (b) is not satisfied, i.e., there exists $X' \subseteq X$, $x \in X \setminus X'$, $x' \in X' \setminus Ch_c(X')$, such that $x' \in Ch_c(X' \cup \{x\})$, i.e., x' is rejected from c when X' is present, but it is accepted when $X' \cup \{x\}$ is present. There can be multiple contracts that satisfy the above conditions. Without loss of generality, we assume x' has the highest priority ranking among such contracts. It is clear that x must be a contract related to c and $x \in Ch_c(X' \cup \{x\})$. If $|X'_c| \leq q_c$, no contract is rejected thus $|X'_c| > q_c$ holds. Furthermore, $|Ch_c(X')| = |Ch_c(X' \cup \{x\})| = q_c$ holds. When calculating $Ch_c(X' \cup \{x\})$, there are four possibilities: (i) both x and x' are accepted at Step 2 in Definition 5, (ii) x is accepted at Step 2 while x' is accepted at Step 3, (iii) x is

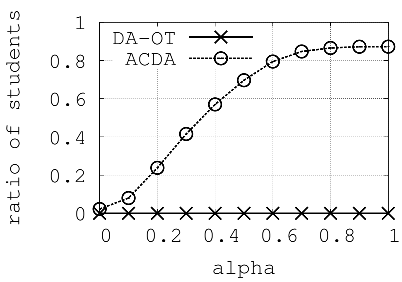

Figure 1: Ratio of claiming students

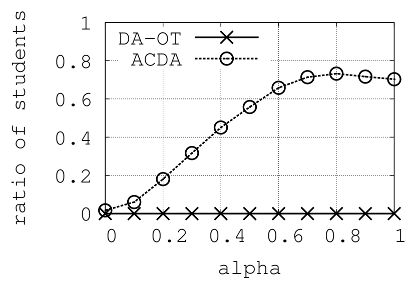

Figure 2: Ratio of students with envy

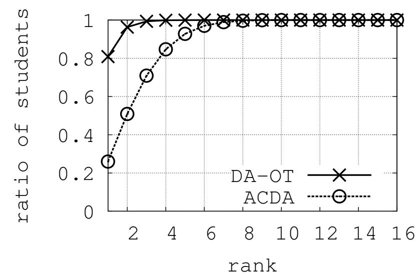

Figure 3: CDFs of students' welfare at $\alpha = 0.5$

accepted at Step 3 while x' is accepted at Step 2, or (iv) both are accepted at Step 3.

In cases (i) or (iii), when $X' \cup \{x\}$ is present, $x' = (s, c, t')$ is accepted at Step 2 since $|Y_{c,t'}| < p_{c,t'}$ holds. Then, when only X' is present, x' must be chosen before $|Y_{c,t'}|$ becomes equal to $p_{c,t'}$. Thus, x' must be accepted also in this situation. This is a contradiction. Thus, cases (i) and (iii) are not possible. Also, in cases (ii) or (iv), when x' is accepted at Step 3 when $X' \cup \{x\}$ is present, $|Y| < q_c$ holds. Then, when only X' is present, x' must be chosen before $|Y|$ becomes equal to q_c. Thus, x' must be accepted also in this situation. This is a contradiction. Thus, cases (ii) and (iv) are not possible. □

Theorem 1. *A stable matching always exists, and the DA-OT is strategy-proof and obtains a stable matching, which is student-optimal in all the stable matchings.*

Proof. Hatfield and Milgrom (2005) show that when Ch_C satisfies the three properties, a HM-stable matching always exists and the generalized Gale-Shapley mechanism is strategy-proof and obtains the student-optimal matching in all the HM-stable matchings. HM-stability is equivalent to our stability (Lemma 1). The DA-OT is one instance of the generalized Gale-Shapley mechanism. Also, Ch_C satisfies these properties (Lemma 2). □

Evaluation

This section evaluates our newly developed DA-OT by computer simulation. As far as the authors aware, the DA-OT is the only known strategy-proof mechanism that is guaranteed to obtain a stable matching in our problem setting. More specifically, the DA-OT can flexibly allocate seats across types, i.e., it is nonwasteful. Also, the DA-OT is fair across types, i.e., Condition **(fr-i)** in Definition 3 covers a case where s and s' have different types. If we give up nonwastefulness and fairness across types, we can use the following method. We fix the number of seats for each type. Let $q_{c,t}$ denote the maximum quota for type t contracts of school c. We set $q_{c,t}$ such that $p_{c,t} \leq q_{c,t}$ holds for all $c \in C, t \in T$, and $\sum_{t \in T} q_{c,t} = q_c$ holds for all $c \in C$. Then, we apply the standard DA mechanism, assuming school c is divided into multiple sub-schools $c_{t_1}, c_{t_2}, \ldots$ with maximum quotas $q_{c,t_1}, q_{c,t_2}, \ldots$, respectively. We call this mechanism Artificial Cap Deferred Acceptance mechanism (ACDA). The ACDA is wasteful, but no student claims an empty seat by type. Also, it removes justifiable envy among the students with the same type, i.e., if $\tau(s) = \tau(s')$, then s and s' do not have justifiable envy among themselves.

We compare the DA-OT and ACDA as follows. We consider a market with $n = 256$ students, $m = 8$ schools, and $k = 4$ types. For each $c \in C$, we set q_c to 48, and for each type $t \in T$, we set $p_{c,t}$ to 4. For each student $s \in S$, we set $|\tau(s)|$, i.e., the number of s's types, to 2. We generate students' preferences as follows. We create one common $m \times |T|$ matrix V^*, where each element $v^*_{c,t}$ is uniformly drawn from $[0, 1]$ at random. Then, for each student $s \in S$, we create private $m \times |T|$ matrix V^s, where each element $v^s_{c,t}$ is uniformly drawn from $[0, 1]$ at random. Then we construct cardinal utilities over all m schools and T types for student s as $\alpha V^* + (1 - \alpha)V^s$, for some $\alpha \in [0, 1]$, i.e., the cardinal utility of contract (s, c, t) is $\alpha v^*_{c,t} + (1 - \alpha)v^s_{c,t}$. We then convert these cardinal utilities into an ordinal preference relation for each student. The higher the value of α is, the more correlated the students' preferences are. The priority ranking of each school c is drawn uniformly at random. We create 100 problem instances for each parameter setting.

Figure 1 shows the ratio of students who claim an empty seat. The x-axis denotes the value of α, and the y-axis denotes the average ratio of students who claim an empty seat. Since the DA-OT is nonwasteful, no student claims an empty seat. We observe that students are more likely to claim an empty seat in the ACDA as α increases, e.g., more than 80% of the students claim an empty seat when $\alpha \geq 0.6$. This is because as students' preferences become more correlated, the competition among the students becomes more severe.

Figure 2 shows the ratio students with justifiable envy. The x-axis is the same as in Fig. 1. Since the OT-DA is fair, no student has justifiable envy. As in Fig. 1, students are more likely to have justifiable envy in ACDA as α increases, e.g., more than 70% of the students have justifiable envy when $\alpha \geq 0.7$.

Figure 3 illustrates the students' welfare by plotting the cumulative distribution functions (CDFs) of the average number of students matched with their i-th or higher ranked contract when α is 0.5. Hence, a steep upper trend line is desirable. The DA-OT performs much better than the ACDA for all α. In Fig. 3, we can see For example, 80% of the students obtain their first choice, and 96% of students obtain their first or second choice in the DA-OT. While in the ACDA, only 26% of students obtain their most preferred contracts, and 51% of students obtain their most or second preferred contracts. Setting artificial caps decreases students' welfare, since it abandons too much flexibility.

Conclusion

In this paper, we developed a new model for controlled school choice programs with soft-bounds, in which a student can belong to multiple types. We first presented a model that is a straightforward extension of an existing model for disjoint types. We proved that there exists a case where no matching is stable in this model and developed an alternative model. We showed that a stable matching is guaranteed to exist in this alternative model and developed the DA-OT mechanism, which is strategy-proof and obtains the student-optimal matching within all the stable matchings. We also showed computer simulation results, which illustrate the DA-OT outperforms the ACDA, in which the number of seats for each type is fixed.

Our future works include introducing other type specific constraints, such as ceilings, and developing a more general mechanism that can simultaneously handle different types of distributional constraints, e.g., regional minimum quotas.

Acknowledgments

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