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Efficient Computation of Semivalues for Game-Theoretic Network Centrality

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Abstract

Solution concepts from cooperative game theory, such as the Shapley value or the Banzhaf index, have recently been advocated as interesting extensions of standard measures of node centrality in networks. While this direction of research is promising, the computation of game-theoretic centrality can be challenging. In an attempt to address the computational issues of game-theoretic network centrality, we present a generic framework for constructing game-theoretic network centralities. We prove that all extensions that can be expressed in this framework are computable in polynomial time. Using our framework, we present the first game-theoretic extensions of weighted and normalized degree centralities, impact factor centrality, distance-scaled and normalized betweenness centrality, and closeness and normalized closeness centralities.

Introduction

Determining whether a given node or edge in a network is more significant (or *central*) than another is an important research topic, studied in fields like social network analysis, biology, and computer science. To this end, researchers have proposed a range of *centrality measures* that aim to numerically characterise the centrality of a node or edge. Among the numerous centrality measures proposed in the literature, the four most fundamental and prominent are *degree centrality*, *closeness centrality*, *betweenness centrality* (Freeman 1979), and *eigenvector centrality* (Bonacich 1972).

Recently, solution concepts from the field of cooperative game theory (e.g., the Shapley value (Shapley 1953), the Banzhaf power index (Banzhaf 1965), and, more generally, semivalues (Dubey, Neyman, and Weber 1981)) have been advocated as measures of network centrality extending standard centrality measures (Gómez et al. 2003; Amer, Giménez, and Magaña 2012). The basic idea behind game-theoretic network centrality is to consider network nodes as players in a cooperative game (where players form coalitions and divide between them the payoff from cooperation). In this setting, game-theoretic solution concepts of payoff division become metrics for network analysis. We refer the reader to Tarkowski et al. (2014) for an overview on the topic. This approach has been considered for a variety of applications, such as social and organisational network analysis (Suri and Narahari 2010), biological networks (Kotter et al. 2007), and covert networks (Michalak et al. 2013a).

Many studies of the complexity of game-theoretic solution concepts have been carried out over the past decade (see Chalkiadakis, Elkind, and Wooldridge (2012)). Unfortunately, for many representations of games, the solution concepts of most interest are computationally intractable (typically, $\#P$-complete). However, only a few studies to date have investigated whether negative complexity results carry over when these solution concepts are applied to network centrality. Among those, Michalak et al. (2013a) established that computing the Shapley value-based centrality measure for connectivity games on graphs is $\#P$-complete. It is also known that various other cooperative games on graphs that could be used as centrality measures are challenging (Nebel 2011; Aziz and de Keijzer 2014). [1]

Fortunately, positive results have also been established. In particular, it is possible to compute in polynomial time various Shapley value-based centrality measures that extend standard degree and closeness centrality (Michalak et al. 2013b). A similar positive result was obtained for the standard betweenness centrality (Szczepański, Michalak, and Rahwan 2012). However, many other centrality measures have not been extended to game-theoretic centrality, nor is the computational complexity of such extensions known.

In this paper, we address this issue. We present a general framework in which it is possible to construct a variety of game-theoretic network centralities (and, in particular, extensions of standard centrality measures) such that they can be computed in polynomial time. Using this key result, we are able to present game-theoretic extensions of weighted and normalized degree centralities, impact factor centrality, distance-scaled and normalized betweenness centrality, and closeness and normalized closeness centralities. As a corollary of our general result, all of these centrality measures can be computed in polynomial time.

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[1]In principle, any cooperative game described on a graph and an associated solution concept can be considered as a game-theoretic centrality measure.

Preliminaries

A *network* is a graph $G = (V, E)$, where V is a set of nodes and E a subset of unordered pairs (v, u) of nodes in V. A *path* π_{st} from node s to node t in a graph G is an ordered set $(v_0, v_1, \ldots, v_n)$ such that $v_0 = s$ and $v_n = t$ and $(v_i, v_{i+1}) \in E$ for all i with $1 \leq i < n$. We define the set of *neighbours* of a node v and a subset C of nodes by $E(i) = \{j : (i, j) \in E\}$ and $E(C) = \bigcup_{i \in C} E(i) \setminus C$, respectively. We refer to the *degree* of a node v by $deg(v) = |E(v)|$. The *distance* from a node s to a node t is denoted by $dist(s, t)$ and is defined as the size of the shortest path between s and t. The distance between a node v and a subset of nodes $C \subseteq V$ is denoted by $dist(C, v) = \min_{u \in C} dist(u, v)$. Paths, neighbours, and distances are prominent in definitions of *network centrality measures*, i.e., functions that associate with each node a real value that represents its centrality, and we will refer to them as *(graph) items.*

A cooperative game consists of a set $N = \{1, 2, \ldots, n\}$ of *players* and a *characteristic function* $\nu : 2^N \to \mathbb{R}$, which assigns to each *coalition* $C \subseteq N$ of players a real value (or payoff) indicating its performance, where $\nu(\emptyset) = 0$. A *cooperative game in characteristic function form* is a pair (N, ν), but we will refer to it simply by ν. In our network setting we will always consider the set of players as the set of nodes V in a graph G. The *grand coalition* is the set N of all players.

A basic research problem in cooperative game theory is how to divide the payoff from cooperation (usually the value of the grand coalition) among the players. *Semivalues* (Dubey, Neyman, and Weber 1981) represent an important class of solutions to this problem. To define semivalues, let us denote by $\text{MC}(C, i)$ the marginal contribution of the player i to the coalition C, i.e., $\text{MC}(C, i) = \nu(C \cup \{i\}) - \nu(C)$. Let $\beta : \{0, 1, \ldots, |N| - 1\} \to [0, 1]$ be a function such that $\sum_{k=0}^{|N|-1} \beta(k) = 1$. Intuitively, when we calculate the expected marginal contribution of a node, $\beta(k)$ will be the probability that a coalition of size k is chosen for this node to join. This is why $\beta(k)$ is defined on values ranging from 0 to $|N| - 1$. Since a node i cannot join a coalition that it is already in, we only need to look at coalitions not containing i. Given β, the semivalue $\phi_i(\nu)$ for a player i in cooperative game ν is:

$$\phi_i(\nu) = \sum_{0 \leq k < |V|} \beta(k) \mathbb{E}_{C_k}[\text{MC}(C_k, i)], \qquad (1)$$

where C_k is the random variable of all possible coalitions of size k drawn with uniform probability form the set $N \setminus \{i\}$, and $\mathbb{E}_{C_k}[\cdot]$ is the expected value operator for the random variable C_k.

The Shapley value (Shapley 1953) and the *Banzhaf index* of power (Banzhaf 1965) are two prominent and well-known examples of semivalues. They are defined by β-functions $\beta^{Shapley}$ and $\beta^{Banzhaf}$, respectively:

$$\beta^{Shapley}(i) = \frac{1}{|N|} \quad \text{and} \quad \beta^{Banzhaf}(i) = \frac{\binom{|N|-1}{i}}{2^{|N|-1}}.$$

For the purposes of defining a cooperative game on any network, we will systematically associate characteristic functions with graphs through *representation functions*.

Definition 1 (Representation function) *A* representation function *is a function ψ that maps every graph $G = (V, E)$ onto a cooperative game (N, ν_G) with $N = V$.*

Now, solution concepts like semivalues can be used in the network setting by applying them to the characteristic function that a network represents.

Definition 2 (Game-theoretic centrality measure) *Formally, we define a game-theoretic centrality measure as a pair (ψ, ϕ) consisting of a representation function ψ and a solution concept ϕ.*

Example 1 *Let us consider a game-theoretic centrality measure $(\psi_R, \phi^{Shapley})$. We say that ψ_R is a representation function since it associates a coalitional game with any graph $G = (V, E)$, i.e. every graph represents a cooperative game. We have $\psi_R(G) = (V, \nu_G)$, where V is the set of nodes and $\nu_G : 2^V \to \mathbb{R}$ is the characteristic function. Let ν_G be the ranking of groups of nodes in G based on classical group betweenness centrality. In other words, ψ_R is simply the group betweenness centrality for any graph. For a specific graph G, the importance of each node $u \in V$ according to $(\psi_R, \phi^{Shapley})$ is evaluated by the Shapley value of the game $\psi_R(G)$, i.e. $\phi_u^{Shapley}(\nu_G)$. Since we started off with a standard centrality measure (betweenness centrality) and applied a game-theoretic solution concept to it (the Shapley value), we call the resulting centrality measure a game-theoretic extension of betweenness centrality.*

A Motivation for Game-Theoretic Centrality

In this section, we provide a brief motivation for game-theoretic network centrality. To this end, we list a number of applications of this approach, discuss how game theory accounts for synergies within networks, and conclude with a brief motivating example.

First, let us begin with real-world applications for which game-theoretic centrality has been advocated. Suri and Narahari (2010) applied a variant of Shapley-value degree centrality to study a co-authorship network of 8361 researchers from the field of high-energy Physics. They show that their centrality measure achieves better results than maximum degree heuristics—a well-known heuristic in the literature for the top-k node problem for information diffusion. Szczepański, Michalak, and Wooldridge (2014) use their game-theoretic centrality in order to analyse a citation network of 2084055 publications, 2244018 citation relationships and 22954 communities (that represent journals, conference proceedings, etc.). This innovative approach not only considers the importance of individual authors, but also that of the communities to which they belong. Lindelauf, Hamers, and Husslage (2013), Skibski et al. (2014) and Michalak et al. (2013a) use game-theoretic centrality in order to study key nodes in terrorist networks. This type of analysis is imperative to understanding the hierarchy of such organisations and for the efficient deployment of investigation resources.

The main motivation for applying cooperative game-theory to the field of network centrality is that it considers the functioning of nodes *jointly*, rather than in separation.

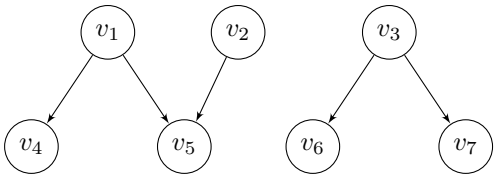

Figure 1: A graph presenting domination within a network. An edge (s, t) indicates that node s dominates node t. Which nodes should be considered most dominant?

Even if some nodes are relatively unimportant to the network individually, the synergy that results from their joint work may make them very important. By the same token, if we consider the contribution of some graph entities (e.g., paths within the network) made by a node to be relatively large, if many other nodes contribute these same entities, then then the contribution of this node is not as important as we originally surmised.

Example 2 *As an example, we will consider the β-measure (Brink and Gilles 2000) and how it handles synergies. Consider the context of ranking nodes in a directed network, $D = (V, E)$, according to their domination (control power) over other nodes. We say that a node s controls the node t if $(s, t) \in E$. In this context, a natural centrality measure (referred to in the literature as the score measure) of the node s is its outdegree, i.e., $\sigma_D(s) = |\{t \in V : (s, t) \in E\}|$. Similarly, we can consider the centrality of a group, C, as the number of nodes that this group controls, i.e., $\sigma_D(C) = |\{t \in V : \exists_s (s, t) \in E\}|$. However, if we consider coalitions of nodes, then if there is a crossover between the nodes controlled by members of the coalition, then we have negative synergy. In other words, the members together control fewer nodes than the simple sum of their individual centralities.*

Consider now the representation function $\psi(D) = \sigma_D$ and the centrality measure $(\psi, \phi^{Shapley})$. Brink and Gilles show that this centrality measure is equal to $\beta_D(s) = \sum_{t \in E(s)} \frac{1}{|E(t)|}$, where $E(v)$ is the neighbourhood of the node v. As compared to σ_D, we see that this centrality measure divides the resulting reward (i.e., raise in centrality) from controlling a node evenly between all nodes that dominate it.

If we look at Figure 1, then which nodes in this network should we consider to be most dominant (i.e., central)? According to the score measure, both v_1 and v_3 are equally dominant. However, we see that node v_4 is dominated by two nodes rather than one. This—depending on the application—could indicate one of a number of things:

- *v_4 is easy to dominate, and thereby does not contribute as much to the "domination power" of a node;*
- *The utility from dominating an already dominated node can be smaller; or*
- *Only one node can control any other node, and in the case of multiple dominators there is some probability distribution over which node is actually in control.*

The β-measure has the correct approach, ranking v_3 as the most dominant node, v_1 as second, followed by v_2 and then by the rest of the nodes. The gain in rank from dominating v_5 is equal to $\frac{1}{2}$ for both v_1 and v_2. In other words, each node—in expectation—dominates half of v_5.

The Class $\mathscr{M}$ of Representation Functions

In this section, we define a class $\mathscr{M}$ of representation functions. It will later be central in identifying which game-theoretic centralities are polynomially computable. Let us assume that the characteristic function ν_G represented by graph $G = (V, E)$ (as given by a representation function $\psi(G) = \nu_G$) can be computed by counting certain types of graph entities. These entities can be, for instance, nodes, paths, or shortest paths. For the purposes of this paper, we will call them *items* and denote them by ϑ. The set of all items in the particular graph G is denoted by Θ_G. We will also partition Θ_G into pairwise disjoint *groups*: $\Theta_1, \Theta_2 \ldots, \Theta_l, \ldots, \Theta_{h(G)}$. The number of these groups depends on the graph in question, and we will denote it by $h(G)$.

Example 3 *Let us consider a graph $G = (V, E)$ and a characteristic function ν_G, where the value of any subset of nodes is proportional to the number of shortest paths that pass through it. Let us denote the set of all shortest paths in G by $SP(G)$. There can be many different shortest paths between each pair of nodes $s, t \in V$. We can therefore partition $SP(G)$ into groups that contain all the shortest paths between the same pair of nodes. Thus, in this setting, we have the set of items $\Theta_G = SP(G)$, $h(G) = V^2$ disjoint groups that we denote by Θ_{st} and index by pairs of nodes.*

In the class $\mathscr{M}$, the value $\nu_G(C)$ of the coalition $C \subseteq V$ depends on the graph items associated with it. We will distinguish *positive*, *negative*, or *neutral association relations* between nodes and items, and we will denote them by R, $\widetilde{R}$, and N, respectively. Formally, $R, \tilde{R}, N \subseteq V \times \Theta_G$. For instance, a shortest path π_{st} can be negatively related to the nodes s and t, positively related to all other nodes that it visits and neutrally related with all nodes that it does not visit.

We will denote by $\widetilde{R}(u, \vartheta)$ the fact that item $\vartheta \in \Theta_G$ is *negatively* related with node $u \in V$. By this we mean that when u joins a coalition C it takes away the value of ϑ, provided it has been previously contributed by any of the nodes in C. Similarly, by $R(u, \vartheta)$ we denote that ϑ is *positively* related with u. Whenever u joins a coalition C, it contributes the value of ϑ to this coalition (provided that ϑ is not already positively or negatively related to some node in C). Finally, $N(u, \vartheta)$ conveys the property that u neither contributes nor takes away the value of ϑ. We have $N(u, \vartheta)$ if and only if neither $\widetilde{R}(u, \vartheta)$ nor $R(u, \vartheta)$.

By now we have introduced relations between nodes and items. These definitions can be naturally extended to sets of nodes. In particular, for each coalition C and item ϑ, we have three mutually exclusive and exhaustive possibilities:

$$\begin{aligned}
&\widetilde{R}(C, \vartheta) \text{ if and only if } \exists_{v \in C} \widetilde{R}(v, \vartheta),\\
&R(C, \vartheta) \text{ if and only if } \exists_{v \in C} R(v, \vartheta) \text{ and } \neg\exists_{u \in C} \widetilde{R}(u, \vartheta),\\
&N(C, \vartheta) \text{ if and only if } \forall_{v \in C} N(v, \vartheta).
\end{aligned}$$

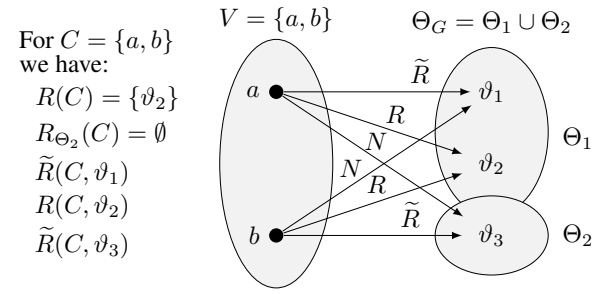

Figure 2: An example of notation used to describe relations between nodes and items.

For clarity, we will use the following notation for the relation R (and similarly for $\widetilde{R}$ and N),

$$\begin{aligned} R(C) &= \{\vartheta \in \Theta_G : R(C, \vartheta)\}, \\ R_{\Theta_l}(C) &= \{\vartheta \in \Theta_l : R(C, \vartheta)\}, \\ R^{-1}(\vartheta) &= \{C \subseteq V : R(C, \vartheta)\}, \\ R^{-1}_{\#k}(\vartheta) &= \{C \subseteq V : R(C, \vartheta) \text{ and } |C| = k\}. \end{aligned}$$

Thus, $R(C)$ denotes the set of items assigned to coalition C, $R_{\Theta_l}(C)$ the set of items in group Θ_l that is assigned to C, $R^{-1}(\vartheta)$ the set of coalitions to which item ϑ is assigned, and $R^{-1}_{\#k}(\vartheta)$ the set of coalitions of size k to which ϑ is assigned. Note that in the case of a singleton coalition we write $R(u)$ instead of $R(u)$ to avoid clutter.

Given the above framework, we will now present the class of characteristic functions that can be expressed in terms of graph items. For a characteristic function ν in this class, we will say that a set of items, relations and functions *models* ν.

Definition 3 (Model of a characteristic function) *Given a graph G, a* model *of a characteristic function ν_G is a tuple*

$$M_G = (\Theta_1, \ldots, \Theta_{h(G)}, R, \widetilde{R}, f, g),$$

where $\Theta_1, \ldots, \Theta_{h(G)}$ is a partition of a set Θ_G of items; R, $\widetilde{R}$ and $N = \neg R \wedge \neg \widetilde{R}$ are positive, negative and neutral association relations, respectively; and $f : \Theta_G \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ are functions, such that the following properties hold:

$(M1)$ $\nu_G(C) = g(|C|) \sum_{\vartheta \in R(C)} f(\vartheta)$;

$(M2)$ $\forall_{1 \le l \le h(G)} \forall_{\vartheta_i, \vartheta_j \in \Theta_l} f(\vartheta_i) = f(\vartheta_j)$;

$(M3)$ $\forall_{1 \le l \le h(G)} \forall_{\vartheta_i, \vartheta_j \in \Theta_l} |N^{-1}_{\#k}(\vartheta_i)| = |N^{-1}_{\#k}(\vartheta_j)|$;

$(M4)$ $\forall_{1 \le l \le h(G)} \forall_{\vartheta_i, \vartheta_j \in \Theta_l} |R^{-1}_{\#k}(\vartheta_i)| = |R^{-1}_{\#k}(\vartheta_j)|$.

If properties $(M2)$, $(M3)$, and $(M4)$ hold, then for all $\vartheta \in \Theta_l$ we use the following notation:

$$f(\Theta_l) = f(\vartheta), \;\; |N^{-1}_{\#k}(\Theta_l)| = |N^{-1}_{\#k}(\vartheta)|, \;\; |R^{-1}_{\#k}(\Theta_l)| = |R^{-1}_{\#k}(\vartheta)|.$$

Also note that Properties $(M3)$ and $(M4)$ imply that $|\widetilde{R}^{-1}_{\#k}(\vartheta_i)| = |\widetilde{R}^{-1}_{\#k}(\vartheta_j)|$ and we also abuse notation by writing $|\widetilde{R}^{-1}_{\#k}(\Theta_l)|$.

In short, the above definition ensures that the relations R, $\widetilde{R}$ (and implicitly N) in the model $M_G = (\Theta_1, \ldots, \Theta_{h(G)}, R, \widetilde{R}, f, g)$ (of the characteristic function ν_G) indeed satisfy the negative, positive and neutral properties discussed at the beginning of this section. In particular, Property $(M1)$ states that the value of a coalition is the product of a function of its size and the sum of the contributions of its members over all positively related items. The rest of the properties state that groups must group together similar items. Specifically, Property $(M2)$ states that the value of every item in a group must be the same. Moreover, Properties $(M3)$ and $(M4)$ state that the set of coalitions of size k to which any item is related (by relation R, N, and consequently by $\widetilde{R}$) is the same for items in the same group.

Now, we are ready to formally define the class $\mathscr{M}$ of representation functions:

Definition 4 (The class $\mathscr{M}$) *$\mathscr{M}$ is the class of representation functions such that for all $\psi \in \mathscr{M}$, and for every graph G, there exists a model M_G of $\psi(G)$.*

Example 4 *Let us consider Figure 2. We want to find an $M_G = (\Theta_1, \Theta_2, R, \widetilde{R}, f, g)$ such that it models $\nu_G(\{a\}) = 1, \nu_G(\{b\}) = 1, \nu_G(\{a, b\}) = 2$. We define $f(\vartheta_1) = f(\vartheta_2) = 1$, $f(\vartheta_3) = 2$ (satisfying $(M2)$) and $g(k) = k$. Property $(M1)$ holds, since $\nu_G(\{a\}) = \nu_G(\{b\}) = g(1)f(\vartheta_2)$ and $\nu_G(\{a, b\}) = g(2)f(\vartheta_2)$. However, property $(M3)$ is not satisfied since $|N^{-1}_{\#1}(\vartheta_1)| \neq |N^{-1}_{\#1}(\vartheta_2)|$. This can be fixed by defining three groups of singletons instead. In fact, a partition of singletons will always satisfy properties $(M2)$, $(M3)$ and $(M4)$. The motivating for groups will become apparent in the next chapter. We will show how restricting the number of groups (without restricting the number of items) will yield polynomial computation.*

Computing Semivalue-based Centralities

For each representation function ψ and semivalue ϕ, we define the computational problem SEMIVALUE(ψ, ϕ):

SEMIVALUE(ψ, ϕ)
Given: Graph $G = (V, E)$, node $u \in V$
Problem: Compute $\phi_u(\psi(G))$

In general, this problem is intractable and it is not immediately obvious what restrictions would yield polynomial computations. Even if we restrict ourselves to the class of representation functions $\mathscr{M}$, there are still functions in this class for which SEMIVALUE(ψ, ϕ) is intractable (for example, if $h(G)$ is exponential in the size of $|V|$). However, we will define a subclass $\mathscr{M}^*$ of $\mathscr{M}$ such that SEMIVALUE(ψ, ϕ) can be computed in polynomial time for all $\psi \in \mathscr{M}^*$.

Definition 5 (The subclass $\mathscr{M}^*$ of $\mathscr{M}$) *$\mathscr{M}^*$ is the subclass of representation functions in $\mathscr{M}$ for which additionally the following properties hold:*

$(M1^*)$ $\exists_{n \in \mathbb{N}} \forall_G \exists_{M_G}$ *M_G models $\psi(G)$ and $h(G) \le O(|V|^n)$;*

$(M2^*)$ *f and g can be computed in time polynomial in $|V|$;*

$(M3^*)$ *If $\psi(G)$ is modelled by $(\Theta_1, \ldots, \Theta_{h(G)}, R, \widetilde{R}, f, g)$, then the function that for each G returns*

$(f, g, |R_{\Theta_l}(u)|, |\widetilde{R}_{\Theta_l}(u)|, |N^{-1}_{\#k}(\Theta_l)|, |R^{-1}_{\#k}(\Theta_l)|)$ *is computable in time polynomial in* $|V|$.

Below, we show that the class $\mathscr{M}^*$ is sufficiently broad to capture all fundamental network centrality measures and their variants known in the literature. Furthermore, the properties $(M1^*)$ through $(M3^*)$ allow us to compute the expected marginal contributions of a node by iterating over a polynomial number of groups of items (Θ_l) and sizes of coalitions ($|C|$) rather than all items ($\vartheta \in \Theta_G$) and all coalitions ($C \subseteq V$). Our key result is as follows:

Theorem 1 *For all* $\psi \in \mathscr{M}^*$ *and all semivalues* ϕ, SEMIVALUE(ψ, ϕ) *can be solved in polynomial time.*

Proof: In order to solve SEMIVALUE(ψ, ϕ) in polynomial time, we will use equation (1) and show how to compute $\mathbb{E}_{C_k}[\text{MC}(C_k, u)]$ in polynomial time for all k. To this end, we need to consider the contributions of the node u to C_k through the various items that u is in relation with (as defined by $R(u)$, $\widetilde{R}(u)$ and $N(u)$).

We will denote by $\text{MC}(C, u, \vartheta)$ the marginal contribution of node u to C through the item ϑ (which we will define below). In effect, we have

$$\text{MC}(C, u) = \sum_{\vartheta \in \Theta_G} \text{MC}(C, u, \vartheta).$$

Two computational issues arise. First, we cannot immediately use this result, as it would require iteration over all coalitions. Second, iterating over a possibly exponential number of items within this sum is quite troubling. To address the first issue, we will denote by $\text{MC}(k, u, \vartheta)$ the sum of contributions of u through ϑ to all coalitions of size k. To address the second, for all groups Θ_l we will consider instead $\text{MC}(k, u, \Theta_l) = \sum_{\vartheta \in \Theta_l} \text{MC}(k, u, \vartheta)$. We get:

$$\begin{aligned}\mathbb{E}_{C_k}[\text{MC}(C_k, u)] &= \sum_{1 \le l \le h(G)} \mathbb{E}_{C_k}[\text{MC}(C_k, u, \Theta_l)] \\ &= \sum_{1 \le l \le h(G)} \frac{\text{MC}(k, u, \Theta_l)}{\binom{|V|-1}{k}}. \end{aligned} \quad (2)$$

The remainder of the proof will focus on defining $\text{MC}(C, u, \vartheta)$ and computing $\text{MC}(k, u, \vartheta)$. Polynomial computation of $\text{MC}(k, u, \Theta_l)$ will follow. A node u can contribute (possibly negative) value to a coalition C through $\vartheta \in \Theta_l$ in one of three mutually exclusive ways. These are:

[1] $R(u, \vartheta)$ and $N(C, \vartheta)$,

[2] $\widetilde{R}(u, \vartheta)$ and $R(C, \vartheta)$, and

[3] $R(u, \vartheta)$ or $N(u, \vartheta)$, and $R(C, \vartheta)$.

For all other cases $\text{MC}(C, u, \vartheta) = 0$. Since the remainder of the proof hinges on the understanding of these cases, let us illustrate them on our running example from Figure 2.

Example 5 *Let* $S = \{a\}$. *We will compute the contribution of node* b *to coalition* S. *We have* $\text{MC}(S, b, \vartheta_1) = 0$, *since* $\widetilde{R}(S, \vartheta_1)$ *(meaning there is nothing to take away). Evidently, none of the cases* [1], [2], [3] *are satisfied. We also have* $\widetilde{R}(b, \vartheta_3)$ *and* $N(S, \vartheta_3)$, *which implies* $\text{MC}(S, b, \vartheta_3) = 0$. *A contribution can only be made through the item* ϑ_2. *We have:* $R(b, \vartheta_2)$ *and* $R(S, \vartheta_2)$, *which satisfies condition* [3]. *It is not immediately obvious why a contribution is made, since* ϑ_2 *is already positively related with* S. *However, it is necessary to keep in mind that the size of the coalition will change. Due to the function* g, *this means that the value added by item* ϑ_2 *will now be* $g(k+1)f(\vartheta_2)$ *instead of* $g(k)f(\vartheta_2)$. *We say, then, that* b *contributes the value* $g(k+1)f(\vartheta_2) - g(k)f(\vartheta_2)$ *through the item* ϑ_2 *to coalition* S.

We compute $\text{MC}(k, u, \vartheta)$ in these three cases. In equations (3), (4) and (5) we will use properties $(M2)$ and $(M3)$ to give a closed formula that depends just on Θ_l and the size of the coalition, but not on ϑ itself. These formulas can be computed in polynomial time due to $(M3^*)$.

For the first case, $\text{MC}(C, u, \vartheta) = g(k+1)f(\vartheta)$. There are $|N^{-1}_{\#k}(\vartheta)|$ coalitions of size k such that $N(C, \vartheta)$. Thus, for $\vartheta \in \Theta_l$ we have:

$$\begin{aligned}\text{MC}^{[1]}(k, u, \vartheta) &= g(k+1)f(\vartheta)|N^{-1}_{\#k}(\vartheta)| \\ &= g(k+1)f(\Theta_l)|N^{-1}_{\#k}(\Theta_l)|. \end{aligned} \quad (3)$$

For the second case, $\text{MC}(C, u, \vartheta) = -g(k)f(\vartheta)$. There are $|R^{-1}_{\#k}(\vartheta)|$ coalitions of size k such that $R(C, \vartheta)$. We have:

$$\begin{aligned}\text{MC}^{[2]}(k, u, \vartheta) &= -g(k)f(\vartheta)|R^{-1}_{\#k}(\vartheta)| \\ &= -g(k)f(\Theta_l)|R^{-1}_{\#k}(\Theta_l)|. \end{aligned} \quad (4)$$

For the third case, $\text{MC}^{[3]}(C, u, \vartheta) = \Delta^k f(\vartheta)$, where $\Delta^k = g(k+1) - g(k)$. There are exactly $|R^{-1}_{\#k}(\vartheta)|$ coalitions of size k such that $R(C, \vartheta)$. We therefore have:

$$\text{MC}^{[3]}(k, u, \vartheta) = \Delta^k f(\vartheta)|R^{-1}_{\#k}(\vartheta)| = \Delta^k f(\Theta_l)|R^{-1}_{\#k}(\Theta_l)|. \quad (5)$$

Since $\text{MC}^{[i]}(k, u, \vartheta)$ depends on the group Θ_l such that $\vartheta \in \Theta_l$, equivalently we write $\text{MC}^{[i]}(k, u, \Theta_l)$. Using equations (3), (4) and (5), we find that $\text{MC}(k, u, \Theta_l)$ equals

$$\begin{aligned}\text{MC}(k, u, \Theta_l) = &\sum_{\vartheta \in R_{\Theta_l}(u)} \text{MC}^{[1]}(k, u, \vartheta) \\ + &\sum_{\vartheta \in \widetilde{R}_{\Theta_l}(u)} \text{MC}^{[2]}(k, u, \vartheta) + \sum_{\vartheta \in R_{\Theta_l}(u) \cup N_{\Theta_l}(u)} \text{MC}^{[3]}(k, u, \vartheta). \end{aligned}$$

Using equations (3), (4) and (5) this term can be rewritten as

$$\begin{aligned}\text{MC}(k, u, \Theta_l) = &|R_{\Theta_l}(u)|\text{MC}^{[1]}(k, u, \Theta_l) \\ &+|\widetilde{R}_{\Theta_l}(u)|\text{MC}^{[2]}(k, u, \Theta_l) \\ &+ \big(|R_{\Theta_l}(u)| + |N_{\Theta_l}(u)|\big)\text{MC}^{[3]}(k, u, \Theta_l). \end{aligned} \quad (6)$$

Observe that the summations over the potentially exponentially growing sets $R_{\Theta_l}(u)$, $\widetilde{R}_{\Theta_l}(u)$, and $N_{\Theta_l}(u)$ fall out of the equation. Moreover, in virtue of property $(M3^*)$, this term can be computed in polynomial time.

Finally, equations (6), (2) and (1) yield the following closed formula for the semivalue of node v, which also furnishes us with a polynomial algorithm:

$$\phi_v(\nu) = \sum_{0 \le k < |V|} \beta(k) \sum_{1 \le l \le h(G)} \frac{\text{MC}(k, u, \Theta_l)}{\binom{|V|-1}{k}}. \quad (7)$$

This concludes the proof. □

Corollary 1 *If $\psi \in \mathscr{M}^*$, then there exist m_1, m_2, and m_3 such that f can be computed in time $O(|V|^{m_1})$; g in time $O(|V|^{m_2})$; and $|N_{\#k}^{-1}(\Theta_l)|$, $|R_{\#k}^{-1}(\Theta_l)|$, $|R_{\Theta_l}(u)|$ and $|\widetilde{R}_{\Theta_l}(u)|$ can be computed in time $O(|V|^{m_3})$. We can solve* SEMIVALUE(ψ, ϕ) *in time* $O\big(h(G)|V|^{m_1} + |V|^{m_2+1} + h(G)|V|^{m_3+1}\big)$.

Corollary 2 *For all $\psi \in \mathscr{M}^*$ we can compute the Shapley value-based and Banzhaf centralities in polynomial time.*

An Algorithm for Semivalue-based Centrality

The Algorithm 1 (SEMI) is a direct implementation of equation (7). In lines 5-7, 8-10 and 11-12, it computes the contribution of v to C_k associated with the expressions $\text{MC}^{[1]}(k, v, \Theta_l)$, $\text{MC}^{[2]}(k, v, \Theta_l)$ and $\text{MC}^{[3]}(k, v, \Theta_l)$, respectively.

Algorithm 1: (**SEMI**) The semivalue-based centrality

Input: Graph $G = (V, E)$, node $v \in V$, model $M_G = (\{\Theta_i\}_1^{h(G)}, R, \widetilde{R}, f, g)$, function β, functions c_G^N and c_G^R, precomputed vectors $\{|R_{\Theta_l}(\{v\})| : 1 \le l \le h(G) \wedge v \in V\}$, and $\{|\widetilde{R}_{\Theta_l}(\{v\})| : 1 \le l \le h(G) \wedge v \in V\}$

Output: semivalue, $\phi_v(\Psi(G))$, for the node v

1 $\phi_v \leftarrow 0$;
2 **for** $k \leftarrow 0$ **to** $|V| - 1$ **do**
3 $\text{MC}_k \leftarrow 0$;
4 **for** $l \leftarrow 1$ **to** $h(G)$ **do**
  // ********** [$R(v, \vartheta)$ and $N(C, \vartheta)$] **********
5 $|N_{\#k}^{-1}(\Theta_l)| \leftarrow c_G^N(\vartheta \in \Theta_l, k)$;
6 $\text{MC}^{[1]} \leftarrow g(k+1) f(\Theta_l) |N_{\#k}^{-1}(\Theta_l)|$;
7 $\text{MC}_k \leftarrow \text{MC}_k + |R_{\Theta_l}(\{v\})| \text{MC}^{[1]}$;
  // ********** [$\widetilde{R}(v, \vartheta)$ and $R(C, \vartheta)$] **********
8 $|R_{\#k}^{-1}(\Theta_l)| \leftarrow c_G^R(\vartheta \in \Theta_l, k)$;
9 $\text{MC}^{[2]} \leftarrow g(k) f(\Theta_l) |R_{\#k}^{-1}(\Theta_l)|$;
10 $\text{MC}_k \leftarrow \text{MC}_k - |\widetilde{R}_{\Theta_l}(\{v\})| \text{MC}^{[2]}$;
  // ****** [$R(v, \vartheta)$ or $N(v, \vartheta)$, and $R(C, \vartheta)$] ******
11 $\text{MC}^{[3]} \leftarrow \big(g(k+1) - g(k)\big) f(\Theta_l) |R_{\#k}^{-1}(\Theta_l)|$;
12 $\text{MC}_k \leftarrow \text{MC}_k + |R_{\Theta_l}(\{v\}) \cup N_{\Theta_l}(\{v\})| \text{MC}^{[3]}$;
13 $\text{MC}_k \leftarrow \frac{\beta(k)}{\binom{|V|-1}{k}} \text{MC}_k$;
14 $\phi_v \leftarrow \phi_v + \text{MC}_k$;

Recall that polynomial computation of $|N_{\#k}^{-1}(\Theta_l)|$ and $|R_{\#k}^{-1}(\Theta_l)|$ was one of our assumptions in $\mathscr{M}^*$. The functions for calculating these values are represented in the pseudocode by c_G^N and c_G^R and must be provided in the input. The values $|R_{\Theta_l}(\{v\})|$ and $|\widetilde{R}_{\Theta_l}(\{v\})|$ must be precomputed for all $1 \le l \le h(G)$ and $v \in V$. Given those precomputations, $|N_{\Theta_l}(\{v\})|$ can also be computed in constant time. Due to Corollary 1, the time complexity of this algorithm is $O\big(h(G)|V|^{m_1} + |V|^{m_2+1} + h(G)|V|^{m_3+1}\big)$.

Centrality measure	$f(v)$	$g(\lvert C\rvert)$	Complexity
degree$^{(*)}$ (Everett and Borgatti 1999)	1	1	$O(\lvert V\rvert^2)$
weighted degree (Newman 2004)	$\frac{1}{deg(v)}$	1	$O(\lvert V\rvert^2)$
impact factor (Bollen et al. 2005)	1	$\frac{1}{\lvert C\rvert}$	$O(\lvert V\rvert^2)$
normalized degree (Everett and Borgatti 1999)	1	$\frac{1}{\lvert V\rvert - \lvert C\rvert}$	$O(\lvert V\rvert^2)$

$^{(*)}$(Michalak et al. 2013b) presented an algorithm for the Shapley Value-based degree centrality. Our algorithm applies to all semivalues.

Table 1: The f and g functions for various degree centralities

Computing Semivalues for Classic Centrality

In this section, we show that our framework has strong practical applications. It identifies efficiently computable game-theoretic extensions of such fundamental centrality measures as degree, betweenness, and closeness. Moreover, it provides a general and ready-to-use procedure to construct polynomial-time algorithms. Since our approach is general, it does not always yield optimal algorithms.[3] However, it constitutes a starting point for investigating them.

Parameterised degree centrality

In this section, we define in our framework the class of cooperative games, where a node's value is based on its degree. We consider the general parameterised group degree centrality of a coalition C in graph G, which is defined by:

$$\psi^D(G)(C) = \nu_G^D(C) = g(|C|) \sum_{v \in E(C)} f(v),$$

where $E(C)$ is the set of neighbours of C, and f and g are polynomially computable parameters such that $deg(u) = deg(v)$ implies $f(u) = f(v)$ for all $u, v \in V$. Using this characteristic function, we build game-theoretic extensions of degree, weighted degree, impact factor and normalized degree centralities (Table 1).

Proposition 1 *The parameterised representation function ψ^D (for polynomially computable parameters f and g) belongs to $\mathscr{M}^*$. Consequently,* SEMIVALUE(ψ^D, ϕ) *can be solved in polynomial time and the corresponding semivalue-based centrality can be computed in time $O(|V|^2 + |V|^{m_1+1} + |V|^{m_2+1})$.*

Proof: We will prove that for every ν_G^D we can find an $M_G = (\{\Theta_i\}_1^{h(G)}, R, \widetilde{R}, N, f, g)$ that models it, proving $\psi^D(G)$ is in the class $\mathscr{M}$. Next, we will show that all properties necessary for $\mathscr{M}^*$ are also satisfied. To this end, for all graphs G we define the set of items $\Theta_G = V$. The two relations, $R, \widetilde{R} \subseteq V \times V$, will be as follows:

$$R(v, u) \quad \text{if and only if} \quad \exists_{(v,u) \in E}\, v \neq u$$
$$\widetilde{R}(v, u) \quad \text{if and only if} \quad v = u.$$

And we define the partitions of Θ_G by $\Theta_i = \{v \in V : deg(v) = i\}$. Hence, we can rewrite $\nu_G^D(C) = g(|C|)\sum_{\vartheta \in R(C)} f(\vartheta)$, satisfying property $(M1)$. $(M2)$ is satisfied due to our assumption on f and all the examples in Table 1 satisfy it as well. Next, we consider the relations with respect to coalition sizes k and for $v \in \Theta_G$ we have:

$$|N_{\#k}^{-1}(v)| = \begin{cases} 0 & \text{if } |V| - 1 - deg(v) < k \\ \binom{|V|-1-deg(v)}{k} & \text{otherwise.} \end{cases}$$

$$|R_{\#k}^{-1}(v)| = \begin{cases} 0 & \text{if } |V| = k \\ \binom{|V|}{k} - |N_{\#k}^{-1}(v)| - \binom{|V|-1}{k-1} & \text{otherwise.} \end{cases}$$

Note that for all Θ_l and all $\vartheta_i, \vartheta_j \in \Theta_l$ we have that $|N_{\#k}^{-1}(\vartheta_i)| = |N_{\#k}^{-1}(\vartheta_j)|$ and $|R_{\#k}^{-1}(\vartheta_i)| = |R_{\#k}^{-1}(\vartheta_j)|$, satisfying $(M3)$ and $(M4)$. Hence, $\psi \in \mathscr{M}$.

Regarding $\mathscr{M}^*$, property $(M1^*)$ holds trivially, since clearly $h(G) \leq |V|$ and from the definition of $\psi^D(G)(C)$ property $(M2^*)$ holds trivially as well. For each $v \in V$ and $1 \leq l \leq h(G)$ it is easy to compute $|R_{\Theta_l}(\{v\})|$ and $|\widetilde{R}_{\Theta_l}(\{v\}|$, which satisfies $(M3^*)$. Finally, $|N_{\#k}^{-1}(\vartheta)|$ and $|R_{\#k}^{-1}(\vartheta)|$ can be computed in polynomial time simply from their definitions shown above, satisfying property $(M3^*)$.

Due to Theorem 1, we can compute semivalue-based centralities based on v_G^D in polynomial time. Due to Corollary 1, we can compute $\phi_v(\nu_G^D)$ in time $O(|V|^2 + |V|^{m_1+1} + |V|^{m_2+1})$. □

Parameterised betweenness centrality

In this subsection, we show that it is possible to express in our framework the class of cooperative games, where a node's value is based on its betweenness centrality. To this end, we will denote the set of paths between s and t by $\Pi_{s,t}$ and the set of all shortest paths in G by SP(G). Also, we will denote by σ_{st} the number of shortest paths between s and t (if $s = t$ then $\sigma_{st} = 1$) and by $\sigma_{st}(C)$ the number of shortest paths between s and t that pass through some node $v \in C$.

The general parameterised group betweenness centrality of a coalition C in graph G is defined by:

$$\psi^B(G)(C) = \nu_G^B(C) = g(|C|) \sum_{s,t \notin C} \sigma_{st}(C) f(s,t),$$

where f and g polynomially computable parameters.[2] Using this characteristic function, we build game-theoretic extensions of stress, betweenness, distance-scaled betweenness and normalized betweenness centralities (Table 2).

Proposition 2 *The parameterised representation function ψ^B (for polynomially computable parameters f and g) belongs to $\mathscr{M}^*$. Consequently,* SEMIVALUE(ψ^B, ϕ) *can be solved in polynomial time and the corresponding semivalue-based centrality can be computed in time $O(|V|^3 + |V|^{m_1+2} + |V|^{m_2+1})$.*

[2] If $\sigma_{xy}(C') = 0$, we postulate $\frac{0}{0} = 0$.

Centrality measure	$f(s,t)$	$g(\|C\|)$	Complexity
stress$^{(*)}$ (Szczepański et al. 2012)	1	1	$O(\|V\|^3)$
betweenness$^{(*)}$ (Everett and Borgatti 1999)	$\frac{1}{\sigma_{st}}$	1	$O(\|V\|^3)$
distance-scaled betw. (Brandes 2008)	$\frac{1}{\sigma_{st} dist(s,t)}$	1	$O(\|V\|^3)$
normalized betw. (Everett and Borgatti 1999)	$\frac{1}{\sigma_{st}}$	$\frac{2}{(\|V\|-\|C\|)(\|V\|-\|C\|-1)}$	$O(\|V\|^3)$

$^{(*)}$ Szczepański et al. 2012 presented the algorithms for the Shapley Value-based betweenness and stress centralities. Our algorithm applies to all semivalues.

Table 2: f and g for various betweenness centralities

Centrality measure	$f(d)$	$g(\|C\|)$	Algorithm
closeness (Everett and Borgatti 1999)	1	1	$O(\|V\|^4)$
harmonic$^{(*)}$ (Boldi and Vigna 2013)	$\frac{1}{d}$	1	$O(\|V\|^4)$
influence game$^{(*)}$ (Michalak et al. 2013b)	positive value decreasing	1	$O(\|V\|^4)$
normalized closeness (Everett and Borgatti 1999)	$\frac{1}{\sigma_{st}}$	$\frac{1}{(\|V\|-\|C\|)}$	$O(\|V\|^4)$

$^{(*)}$ Michalak et al. 2013b presented for these functions an algorithm for the Shapley Value. Our algorithm applies to all semivalues.

Table 3: f and g for various closeness centralities

Sketch of proof: We will show how to define for every ν_G^B the model M_G satisfying the conditions $(M1)$ and $(M2)$ for class $\mathscr{M}$. To this end, we define $\Theta_G = SP(G)$. Also, we define the two relations, $R, \widetilde{R} \subseteq V \times SP(G)$, as follows:

$$R(v, \pi_{st}) \quad \text{if and only if} \quad v \in \pi_{st} \text{ and } v \neq s \text{ and } v \neq t$$
$$\widetilde{R}(v, \pi_{st}) \quad \text{if and only if} \quad v = s \text{ or } v = t.$$

and we define the of partitions of Θ_G as $\Theta_{st} = \{\pi_{st} \in SP(G) : \pi_{st} \in \Pi_{st}\}$. Thus $\nu_G^D(C) = g(|C|)\sum_{\vartheta \in R(C)} f(\vartheta)$, satisfying $(M1)$ and $(M2)$. □

Parameterised closeness centrality

The last class of characteristic functions that we will define in our framework assigns value to a node based on its distance to other nodes. The parameterised closeness centrality is defined as follows:

$$\psi^{CL}(G)(C) = \nu_G^{CL}(C) = g(|C|) \sum_{v \in V} f(dist(C, v)),$$

where f and g are polynomially computable parameters. Using this characteristic function, we build game-theoretic extensions of closeness, harmonic, influence game and normalized closeness centralities (Table 3).

Proposition 3 *The parameterised representation function ψ^{CL} (for polynomially computable parameters f and g) belongs to $\mathscr{M}^*$. Consequently,* SEMIVALUE(ψ^{CL}, ϕ) *can be solved in polynomial time and the corresponding semivalue-based centrality can be computed in time $O(|V|^4 + |V|^{m_1+2} + |V|^{m_2+1})$.*

Sketch of proof: We will show how to define a model M_G for every ν_G^{CL}. To this end, we will define items as pairs of the form $\langle u, l\rangle \subseteq V \times \{0, .., |V|-1\}$. They will represent nodes and all possible distances between them. We define $\Theta_G = \{\langle u, l\rangle \in V \times \mathbb{N} : \exists_{v \in V}\, dist(u, v) = l\}$ and the partition $\{\{\vartheta\} | \vartheta \in \Theta_G\}$ of singletons. Next, we define the two relations, $R, \widetilde{R} \subseteq V \times V \times \{0, \ldots, |V|-1\}$, as follows:

$$R(v, \langle u, l\rangle) \text{ if and only if } dist(v, u) = l$$
$$\widetilde{R}(v, \langle u, l\rangle) \text{ if and only if } dist(v, u) < l.$$

Thus $\nu_G^{CL}(C) = g(|C|) \sum_{\vartheta \in R(C)} f(\vartheta)$, satisfying $(M1)$.[3] $(M2)$ holds for all entries in Table 3. $(M3)$ and $(M4)$ hold since item groups are singletons. Hence $\psi^{CL} \in \mathscr{M}$. □

Related Work

Grofman and Owen (1982) were the first to apply a game-theoretic solution concept—the Banzhaf index—as a centrality measure. Gómez et al. (2003) proposed a centrality measure based on the Shapley value and Myerson value for graph-restricted games Myerson (1977). Amer and Giménez (2004) proposed to use semivalues as a measure of the importance of nodes, and Amer, Giménez, and Magaña (2012) followed for directed networks. Recently, Szczepański, Michalak, and Wooldridge (2014) developed the first game-theoretic measure of centrality based on the Owen value (Owen 1977) that takes into account the *community structure* of the underlying network. We also mention some works on the computational aspects of cooperative games on graphs (Bachrach and Rosenschein 2009; Greco et al. 2011).

Works on the computational characteristics of game-theoretic centrality include Michalak et al. (2013b), who proposed, inter alia, polynomial algorithms for Shapley value-based degree and closeness centralities, and Szczepański, Michalak, and Rahwan (2012), who proposed polynomial algorithms for Shapley value-based betweenness and stress centralities. Michalak et al. (2013a) proposed fast algorithms for connectivity games.

Game-theoretic centralities were first used by Suri and Narahari (2010) in the interesting application of influence propagation in networks. In particular, the authors used the Shapley value to approximate the solution to the top k-node problem, i.e., the problem of identifying the k most influential nodes in a network. Solution concepts from cooperative game theory have been applied to many other graph-related problems, such as studies of the synergies between agents (Procaccia, Shah, and Tucker 2014), wire-tapping communication networks (Aziz et al. 2009).

Our paper also contributes to the line of research on modelling characteristic functions using specific combinatorial structures such as graphs. This includes works by Deng and Papadimitriou (1994), Greco et al. (2009), and Wooldridge and Dunne (2006). Such representations are guaranteed to be concise, however they are not fully expressive.

[3] In our examples f ignores the first parameter of ϑ and is simply a function of distance.

Future Work

A few future research directions stem from this work. First, it may be possible to broaden the scope of the class of game-theoretic network centralities presented in this paper or to find new classes altogether. For example, perhaps it would be possible to extend our model to *coalitional semivalues* (Szczepański, Michalak, and Wooldridge 2014): the family of solution concepts for cooperative games with *a priori*-given unions. Second, we have presented generalised degree, betweenness and closeness centralities, but many more cooperative games and their semivalues can be used to analyse networks. Finally, it would be interesting to perform an experimental analysis of the suitability of different semivalues for particular applications. The algorithms presented in this paper aim to facilitate the tractable computations that are necessary for such studies.

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