Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence

On Computing Optimal Strategies in Open List Proportional Representation: the Two Parties Case

Ning Ding and **Fangzhen Lin**
Department of Computer Science
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Abstract

Open list proportional representation is an election mechanism used in many elections, including the 2012 Hong Kong Legislative Council Geographical Constituencies election. In this paper, we assume that there are just two parties in the election, and that the number of votes that a list would get is the sum of the numbers of votes that the candidates in the list would get if each of them would go alone in the election. Under these assumptions, we formulate the election as a mostly zero-sum game, and show that while the game always has a pure Nash equilibrium, it is NP-hard to compute it.

Introduction

Proportional representation (PR) is a general concept in voting systems. It includes single transferable vote and party list proportional representation, and is used for the elections in many countries, both at the national and local levels. According to Wikipedia,[1] "PR is used by more nations than the single winner system, and it dominates Europe, including Germany, most of northern and eastern Europe, and is used for European Parliament elections (as enforced by EU law). France adopted PR at the end of World War II, but discarded it in 1958. In 1986 it was used for parliament elections." It was also used in the 2012 Hong Kong Legislative Council Geographical Constituencies (HKLCGC) election [2].

In this paper we consider party list proportional representation. It also has several variants. We consider the open list one using Hare quota and the largest remainder method, also known as the Hamilton method (Hare 1861; Gallagher 1992; Balinski and Young 1975; 2001; Tideman 1995). This is the method used in the HKLCGC elections. In this election, each geographical constituency is given a certain number of seats, and the voters in the constituency are presented with a set of lists consisting of candidates in this constituency. After the election, if the number of votes that a list received reaches k multiple of the quota, which is the ratio between the number of the voters and the number of seats in the constituency, then the top k candidates of the list are elected. If there are still any seats unassigned after this step, they are distributed using a mechanism that ranks lists by the remainders of the numbers of votes divided by the quota, commonly known as the largest remainder method.

Notice that there is no requirement that the candidates in a list must belong to the same political party. Parties are free to trade horses within the same constituency or even across constituencies. In the 2012 HKLCGC elections, there were a total of nine different political parties, some were pro-establishment and others were pro-democracy. After the election, there were reports that the pro-democracy parties did not do as well as they should have done because they did not coordinate well among themselves compared with the pro-establishment parties. This motivated our work in this paper: given this two rounds open party list proportional representation election rule, how political parties should form lists to enter into election to maximize the number of seats that they would win. As we shall see, this is a very hard computational problem, even for the case when there is just one constituency, two parties, and two seats, and under the simplified assumption that the number of votes that a given list would get is the sum of the numbers of votes that all the candidates in the list would get if each of them goes alone in the election.

The rest of this paper is organized as follows. We first formulate the election with two parties as a mostly zero-sum (constant-sum) game. We then show that this game always has a pure Nash equilibrium. We then consider the complexity of computing such a Nash equilibrium, discuss some related work in proportional representation, and conclude the paper.

Two parties case: basic definitions

The open list proportional representation election rule used in the 2012 HKLCGC election goes as follows. Each constituency is allocated a certain number of seats in the Legislative Council. In each constituency, voters are given a set of lists to vote, and each voter is asked to cast his/her vote to exactly one list. The set of lists need to satisfy the following conditions:

- Each list is a set of candidates with a total order, and
- no two lists can share candidates.

After the election, the winners are computed in two rounds using the Hamilton method (Balinski and Young 1975;

[1] http://en.wikipedia.org/wiki/Proportional_representation

[2] http://www.eac.gov.hk/en/legco/2012lc_guide.htm

2001) as follows. Suppose the constituency has K seats, and V voters. Then $Q = V/K$ is called the quota of the constituency. In social choice literature, this quota is often called the Hare quota (see, e.g. (Hare 1861)), in contrast to other notions of quotas such as the Droop quota (Droop 1881; Lundell and Hill 2007). In the first round, if a list L receives at least $M * Q$ votes, then the top M candidates from L are elected, assuming that L has at least M candidates. If L has less than M candidates, then all candidates in L are elected, and the list is eliminated. After this, any remaining seats are distributed in the second round using the largest remainder method as follows: the remaining K_1 seats are given to the K_1 lists that still have any candidates left and have the largest remainders of the vote counts divided by the quota, with ties broken by lottery.

For example, suppose $K = 5$ (5 seats) and $V = 100$ (100 voters). Then the quota is $Q = 100/5 = 20$. Suppose there are four lists: $l_1 = [a, b, c]$, $l_2 = [d]$, $l_3 = [e, f]$, and $l_4 = [g]$, and they received 50, 30, 10, and 10 votes, respectively. Then in the first round, the top two candidates from l_1, a and b, and the candidate from l_2 are declared winners. This leaves 2 seats to be filled in the second round. The list l_2 is discarded now as it has no more candidates left. For the remaining lists l_1, l_3, and l_4, the remainders of their vote counts over the quota are all 10. A lottery is then held to give the two seats to two of the lists.

Notice that it is possible for a list that receives no vote to be a winner. For instance, suppose there are two seats, and two lists: one that has a single candidate and gets all the votes, and the other that receives no vote. Then in the first round, the candidate in the first list is elected. The remaining one seat will be given to the top candidate in the second list in the second round. It is also possible that not all seats can be allocated. For instance, suppose there are three seats, and again two lists: one that has one candidate and gets all the votes, and the other that includes all of the other candidates and receives no vote. Then again, the candidate in the first list is elected, and the top candidate in the second list is also elected in the second round. These two are the only winners. However, these are all extreme cases that are unlikely to occur in practice.

Notice also that according to the election rule, there is no requirement as who can form a list. However, it is clear that the system is designed with political parties in mind, and the candidates in a list should come from either the same party or the parties that are willing to form an alliance. Thus to make the problem interesting, we assume that each candidate is affiliated with a certain party, and the problem is how a party should form lists to enter into election so as to maximize the number of seats that will be won by the party as a whole. In other words, the game is not about which candidate would win, but which party would win the most number of seats.

In general, when there are multiple parties participating in elections in multiple constituencies, there are issues such as whether any group of parties should form an alliance, and once an alliance is formed, how to put forward candidate lists. In this paper, we consider only when there are two parties. In this case, the two parties are competing against each other very much like in a zero-sum game. The only issue is then how each party should form lists so as to maximize the number of seats that it would win. Furthermore, maximizing the number of seats won in all constituencies amounts to maximizing the number of seats won in each constituency. Thus for the two parties case, we can assume there is just one constituency.

Obviously, to decide how to form lists from its pool of candidates, a party needs to have information about how voters are going to vote. This information can be modeled in many ways. For example, it can be modeled by a probability distribution of how each voter is going to vote in all possible situations. In this paper, we make the simple assumption that each voter has exactly one candidate in mind, and she casts her vote on a list iff her candidate is in the list. At first glance, it may seem that this simple assumption would make the problem trivial. However, as we shall see, even under this simple assumption, how to form lists to maximize the number of seats won is computationally hard.

To summarize, we have the following definitions.

Definition 1 *A simple two parties open list proportional representation election game, called shortly an election game below, is a tuple (C_1, C_2, S, π), where C_i, $i = 1, 2$, is a non-empty set standing for the set of candidates for party i, S a positive integer standing for the number of seats up for election, and π a function from $C_1 \cup C_2$ to non-negative integers representing the votes that each candidate gets.*

Notice that the number of voters in an election game, V, can be computed from the vote count function π as follows:

$$V = \sum_{x \in C_1 \cup C_2} \pi(x).$$

The quota Q is then $\lfloor V/S \rfloor$, the largest integer that is smaller or equal to V/S.

Example 1 Table 1 describes an election game with 2 seats up for election. Notice that the number of voters is simply

Candidates in C_1	π	Candidates in C_2	π
c_{11}	3	c_{21}	2
c_{12}	2	c_{22}	1
c_{13}	1	-	-

Table 1: An election game with 2 parties, 2 seats, 5 candidates, and 9 voters.

the sum of the numbers in the two columns under π. Thus the quota Q is $\lfloor 9/2 \rfloor = 4$.

In an election, parties put forward non-overlapping lists of candidates. Given that we consider only two parties that will not join candidates in any lists, and that the utility of a party is the number of seats that it will win in the election, we can simplify our presentation by representing a list as a set of candidates from a single party.

Definition 2 *Given an election game (C_1, C_2, S, π), a strategy L for party i is a partition of C_i: $C_i = \cup_{l \in L} l$, and $l \cap l' = \emptyset$ for any two distinct members l and l' in L. Each*

$l \in L$ represents a list in an election. Below, we use A_i to stand for the set of strategies of party i. Thus a strategy profile (L_1, L_2) is a member of $A_1 \times A_2$.

We now define the payoff (utility) function for each party. Intuitively, the payoff of a party in an election game is the number of seats it expects to win in the election. The formal definition below is tedious as the election rule is procedural.

Definition 3 *Let (C_1, C_2, S, π) be an election game, Q the quota, and A_1 and A_2 the sets of strategies for parties 1 and 2, respectively. The payoff function u_i for party i is defined as follows: for any $(L_1, L_2) \in A_1 \times A_2$,*

$$u_i(L_1, L_2) = \sum_{l \in L_i} Seats(l, L_1, L_2),$$

where $Seats(l, L_1, L_2)$ is the expected number of seats won by list l, and is defined as the sum of the seats won in the two rounds:

$$Seats(l, L_1, L_2) = Seat_1(l) + Seat_2(l, L_1, L_2),$$

where

- *$Seat_1(l)$ is the number of seats won in the first round:*

  $$Seat_1(l) = \min\{|l|, \lfloor \pi(l)/Q) \rfloor\},$$

  where $|l|$ is the number of candidates in l, and $\pi(l)$ the number of votes that the list gets, $\pi(l) = \sum_{x \in l} \pi(x)$.
- *$Seat_2(l, L_1, L_2)$ is the expected number of seats won by l in the second round, given that there may be ties, and is defined as follows. Let S' be the number of seats left after the first round:*

  $$S' = S - \sum_{l \in L_1 \cup L_2} Seat_1(l).$$

  Let L_i' be the set of lists from L_i that are going to the second round:

  $$L_i' = \{l \mid l \in L_i, |l| > Seat_1(l)\}.$$

  Let π' be the new vote count (remainder) function:

  $$\pi'(l) = \pi(l) - Seat_1(l) \times Q.$$

  Then $Seat_2(l, L_1, L_2)$ is defined as follows:

  1. *if*
     $$|\{l' \mid l' \in L_1' \cup L_2', \pi'(l') \geq \pi'(l)\}| \leq S',$$
     then $Seat_2(l, L_1, L_2) = 1$;
  2. *if*
     $$|\{l' \mid l' \in L_1' \cup L_2', \pi'(l') \geq \pi'(l)\}| > S'$$
     but
     $$|\{l' \mid l' \in L_1' \cup L_2', \pi'(l') > \pi'(l)\}| < S',$$
     then $Seat_2(l, L_1, L_2)$ is
     $$\frac{S' - |\{l' \mid l' \in L_1' \cup L_2', \pi'(l') > \pi'(l)\}|}{|\{l' \mid l' \in L_1' \cup L_2', \pi'(l') = \pi'(l)\}|};$$
  3. *otherwise $Seat_2(l, L_1, L_2) = 0$.*

Example 2 Consider again the election game in Example 1 (Table 1). Recall that the quota Q is 4. Consider the following two strategies L_1 and L_2 for parties 1 and 2, respectively: $L_1 = \{\{c_{11}\}, \{c_{12}, c_{13}\}\}$ and $L_2 = \{\{c_{21}, c_{22}\}\}$. Since $\pi(\{c_{11}\}) = \pi(\{c_{12}, c_{13}\}) = \pi(\{c_{21}, c_{22}\}) = 3 < Q$, we have that $Seat_1(\{c_{11}\}) = Seat_1(\{c_{12}, c_{13}\}) = Seat_1(\{c_{21}, c_{22}\}) = 0$. Thus $\pi'(\{c_{11}\}) = \pi'(\{c_{12}, c_{13}\}) = \pi'(\{c_{21}, c_{22}\}) = 3$ and $S' = 2$. So we have $Seat_2(\{c_{11}\}) = Seat_2(\{c_{12}, c_{13}\}) = Seat_2(\{c_{21}, c_{22}\}) = \frac{2}{3}$. Summing them up, we get that the payoff vector is $(1\frac{1}{3}, \frac{2}{3})$, meaning that party 1 can expect to win $1\frac{1}{3}$ seats, and for party 2, $\frac{2}{3}$ seats. As can be easily verified, this strategy profile is the only pure Nash equilibrium.

Notice that it may not always be the case that $u_1(L_1, L_2) + u_2(L_1, L_2) = S$ (all seats are assigned). But if the number of candidates in $C_1 \cup C_2$ is greater or equal to S, then $u_1(L_1, L_2) + u_2(L_1, L_2) = S$ holds for most profiles (as well as on all pure Nash equilibria). So in essence, an election game defined above is a constant-sum (zero-sum) game.

One of our main results is that every election game has a pure Nash equilibrium. In the following, given a set C, we denote by $\hat{C}$ the singleton partition of C: $\hat{C} = \{\{c\} | c \in C\}$.

Theorem 1 *An election game (C_1, C_2, S, π) always has a pure Nash equilibrium. Furthermore, if $|C_1| + |C_2| < S$, then $(\hat{C_1}, \hat{C_2})$ is a Nash equilibrium. If $|C_1| + |C_2| \geq S$, and (L_1, L_2) is a Nash equilibrium, then $u_1(L_1, L_2) + u_2(L_1, L_2) = S$.*

We prove this theorem by reducing an election game to a number game. The proof can be found in the appendix.

The complexity of computing Nash equilibria

We have shown that an election game always has a pure Nash equilibrium. Assuming that each party has the required information about how voters would cast their votes, all a party needs to do is to compute a Nash equilibrium, and play the corresponding strategy. But can a Nash equilibrium be computed efficiently? The short answer is that there is no tractable algorithm for doing this, assuming that P $\neq$ NP.

Formally, we formulate the problem as the decision problem of checking whether there is a Nash equilibrium (L_1, L_2) such that party 1 (or symmetrically, party 2) can win at least N seats, i.e. $u_1(L_1, L_2) \geq N$, for a given $N \leq S$.

Definition 4 SEATS(S,N) Decision Problem
INSTANCE: *An election game $EG = (C_1, C_2, S, \pi)$.*
QUESTION: *Does EG have a pure Nash equilibrium (L_1, L_2) such that $u_1(L_1, L_2) \geq N$?*

We show that this is in general an intractable problem. Our first result is that SEATS(2,2), which asks whether a party can win both of the given 2 seats, is NP-complete.

Theorem 2 *SEATS(2,2) is NP-complete.*

Proof: We show this by giving a reduction from the NP-complete partition problem (Garey and Johnson 1979) to SEATS(2,2). Recall that a partition problem asks that given a finite set D and a "size" function $s : D \to Z^+$, whether there is a subset $D' \subseteq D$ such that $\sum_{d \in D'} s(d) = \sum_{d \in D-D'} s(d)$.

Suppose that we are given such a partition problem, and that $D = \{d_1, ..., d_n\}$, $n > 2$. We construct an election with $S = 2$, $V = 3\sum_{d \in D} s(d) - 1$, $C_1 = \{c_{11}, c_{12}, ..., c_{1n}\}$ (n candidates for party 1), and $C_2 = \{c_{21}\}$ (just one candidate for party 2). Now consider the election game $EG = (C_1, C_2, 2, \pi)$ with π defined as follows: $\pi(c_{1i}) = 2s(d_i)$, $1 \le i \le n$, and $\pi(c_{21}) = \sum_{d \in D} s(d) - 1$.

We prove that for this game, there is a Nash equilibrium (L_1, L_2) such that $u_1(L_1, L_2) = 2$ if and only if the partition has a solution, i.e. D can be partitioned into two subsets with equal size.

"$\Leftarrow$": Given a partition $\{D', D - D'\}$ of D, we construct the strategy $L_1 = \{l, C_1 - l\}$, where $l = \{c_{1i} \mid d_i \in D'\}$, for party 1. Party 2 has only one strategy, that is, $L_2 = \{C_2\}$. We have $\pi(l) = \pi(C_1 - l) = \sum_{d \in D} s(d) > \pi(C_2)$. In this profile, we have that $Seat_1(l) = Seat_1(C_1 - l) = 0$, and $Seat_2(l, L_1, L_2) = Seat_2(C_1 - l, L_1, L_2) = 1$. thus $u_1(L_1, L_2) = 2$ and $u_2(L_1, L_2) = 0$, and (L_1, L_2) is a Nash equilibrium.

"$\Rightarrow$": On the other hand, suppose (L_1, L_2) is a Nash equilibrium such that $u_1(L_1, L_2) = 2$. First of all, $L_2 = \{C_2\}$ as this is the only strategy for party 2. Suppose $L_1 = \{l_1, l_2, ..., l_k\}$.

We claim that $k \neq 1$. Suppose otherwise, then $l_1 = C_1$, and $\pi(l_1) = 2\sum_{d \in D} s(d)$. The quota $Q = \lfloor (3\sum_{d \in D} s(d) - 1)/2 \rfloor$. Thus $Seat_1(l_1) = 1$. It is easy to see that $Seat_1(C_2) = 0$. To compute $Seat_2$, notice that $\pi'(l_1) = \lceil \frac{1}{2}\sum_{d \in D} s(d) + \frac{1}{2} \rceil$, and $\pi'(C_2) = \pi(C_2) = \sum_{d \in D} s(d) - 1$. Thus $\pi'(l_1) \le \pi'(C_2)$. So $Seat_2(l_1, L_1, L_2) < 1$, and $u_1(L_1, L_2) < 2$, a contradiction.

Similarly, it is easy to see that k cannot be more than 2, because with more than two lists, in second round, party 2's list C_2 will either win a seat outright or some fraction of a seat for being in a "tie" with some of party 1's lists.

So $k = 2$. It's easy to see that it's impossible for $\pi(l_i)$ to be greater or equal to Q for both $i = 1, 2$. Thus party 1 cannot claim the 2 seats in the first round. If one of the lists receives at least Q number of votes, and get a seat in the first round, then neither list can get any seat in the second round. This means that party 1 gets the two seats in the second round, which means that $\pi'(l_i) = \pi(l_i) > \sum_{d \in D} s(d) - 1$ for $i = 1, 2$. But $\pi(l_1) + \pi(l_2) = 2\sum_{d \in D} s(d)$. So it must be that $\pi(l_1) = \pi(l_2) = \sum_{d \in D} s(d)$. Thus $D' = \{d_i \mid c_{1i} \in l_1\}$ is the evidence that D has a partition.

Since the partition problem is NP-hard, so SEATS(2,2) is NP-hard as well.

To see that SEATS(2,2) is in NP, notice that for an election game with $S = 2$, there is a Nash equilibrium (L_1, L_2) such that $u_1(L_1, L_2) = 2$ iff there is a strategy L_1 for party 1 such that $u_1(L_1, \{C_2\}) = 2$. Given a specific L_1, checking if $u_1(L_1, \{C_2\}) = 2$ can be done in polynomial time. Thus SEATS(2,2) is in NP. ■

Our second complexity result is that given 2 seats, checking whether the Nash equilibrium is a profile where a party gets at least 1 seat is coNP-complete:

Theorem 3 *SEATS(2,1) is coNP-complete.*

Proof: Party 1 can get 1 seat in an equilibrium iff for all $L_2 \in A_2$, $u_1(\{C_1\}, L_2) \ge 1$. Since checking whether $u_1(\{C_1\}, L_2) \ge 1$ can be done in polynomial time, thus SEATS(2,1) is in coNP.

To show that it is coNP-hard, notice that in our proof of Theorem 2, a partition problem is mapped to an election game whose payoff vectors are either $(2, 0)$ or $(1, 1)$, and that the partition problem has a solution iff in the equilibria, the payoff vector is $(2, 0)$ (notice that all pure Nash equilibria in this game have the same payoff vectors). Thus the partition problem has no solution iff in an equilibrium, party 2 wins one seat. Thus the problem of deciding if party 2 can win one or more seats is coNP-hard. Given that party 1 and party 2 are symmetric, thus the problem of deciding if party 1 can win one or more seats is coNP-hard as well, i.e. SEATS(2,1) is coNP-hard. ■

Theorem 4 *In general, SEATS(S,N) is in Σ_2^{P}.*

Proof: SEATS(S,N) is true if and only if there is a strategy $L_1 \in A_1$ such that for every $L_2 \in A_2$, $u_1(L_1, L_2) \ge N$. Since checking if $u_1(L_1, L_2) \ge N$ can be done in polynomial time, we conclude that SEATS(S,N) is in Σ_2^{P}. ■

Related work

Proportional representation includes a variety of voting models. Perhaps because the winner determination problems for the so-called fully proportional representation under Monroe's rule (1995), and that under Chamberlin and Counrant's rule (1983) are NP-hard, there has been much work recently on these two voting rules, ranging from some detailed complexity analysis of the winner determination problem to approximation algorithms for computing the winners (e.g. (Procaccia, Rosenschein, and Zohar 2008; Betzler, Slinko, and Uhlmann 2013; Skowron et al. 2013; Skowron, Faliszewski, and Slinko 2013a; 2013b; Cornaz, Galand, and Spanjaard 2012; Lu and Boutilier 2011; Yu, Chan, and Elkind 2013)).

In comparison, for the open list proportional representation studied in this paper, the winner determination problem is easy. Instead, what is computationally hard is deciding the best way to form lists so as to maximize the number of seats that a party would win. As a way to jump start this line of work, we have made some simplified assumptions in this paper. We have assumed that there are just two parties participating in the election, and that it is a common knowledge how many votes each candidate will get, and that the number of votes that a list will get is the sum of the numbers of the votes that the candidates in the list will get. Perhaps a good news for the election rule is that even in this simple case, and even assuming that there are just two seats up for election, it is computationally intractable for a party to compute the optimal way to partition its set of candidates to form lists to enter into the election. One may compare this with the single transferrable vote, for which the standard manipulation problem has been proved to be NP-hard (Bartholdi and Orlin 1991; Walsh 2010).

Concluding remarks

We have considered a form of open list proportional representation used for the 2012 Hong Kong Legislative Council Geographical Constituencies election. From a computational point of view, the interesting problem here is how political parties should form alliances so as to maximize the number of seats that they would win. As a starting point, we assume in this paper that there are just two parties in the election, and that the number of votes that a list would get is the sum of the numbers of votes that the candidates in the list would get. Under these assumptions, we formulate the election as a normal game that is mostly zero-sum (constant-sum), and show that this game always has a pure Nash equilibrium. However, computing a Nash equilibrium is in general hard. It is even NP-hard when there are just two seats.

We are currently working on extending this work to elections that can have more than two parties and multiple constituencies. The problem becomes much more complicated here. First of all, we cannot consider a list to be a set of candidates anymore because in general, it may contain candidates from multiple parties. More importantly, there are issues about whether some parties are going to form an alliance, and once an alliance is formed, how they are going to agree on the order of the candidates in a list. We hope to have some results on this extension to report in the near future.

Appendix: Proof of Theorem 1

Let $EG = (C_1, C_2, S, \pi)$ be the given election game. Let Q be the quota, u_i, $i = 1, 2$, the payoff functions computed from EG.

We have treated lists as sets of candidates as only the number of seats that a party has won matters. We can go one step further: it does not matter who the candidates are in a list, all it matters is how many votes the list would get.

First of all, each of the payoff function u_i is the sum of two functions, $Seat_1$ and $Seat_2$, that correspond to round one and round two, respectively, of the election rule. The first function, $Seat_1$, is independent of the other party's strategy. It computes the quotient of the list's vote count over the quota, and checks if the list has enough number of candidates to fill the quotient. For the second function, $Seat_2$, what matters is the remainder of the list's vote count over the quota, provided that the list has enough candidates to go to the second round. Thus we can use a list x of numbers to represent a list l of candidates: if l wins n seats in the first round, then we put n copies of the quota Q into x, and if l also goes to the second round, then we put the remainder, $\pi(l) - n \times Q$ into x. Given that the number of seats that a party has won is the sum of the seats that each of its lists has won, once we have a list of numbers for each list of candidates, we can merge the lists of numbers into a single list of numbers. This leads to the following definition of a reduced game, which makes use of the following list notation and functions:

- $\texttt{merge}(x_1, ..., x_n)$ denotes the result of merging lists x_1 through x_n into a new list. For instance, $\texttt{merge}([1, 2], [2, 3], [3, 1]) = [1, 2, 2, 3, 3, 1]$.
- $\texttt{sort}(x)$ denotes the result of sorting a list of numbers in descending order. For instance, $\texttt{sort}([1, 2, 2, 3, 3, 1]) = [3, 3, 2, 2, 1, 1]$.

Definition 5 *The reduced game of EG is $RG = (R_1, R_2, v_1, v_2)$, where*

- *for each $i = 1, 2$, R_i is a set of lists of integers defined from A_i as follows: if $L = \{l_1, ..., l_m\} \in A_i$, then $f(L) = \texttt{sort}(\texttt{merge}(g(l_1), ..., g(l_m))) \in R_i$, where $g(l)$ is the list consisting of $Seat_1(l)$ copies of quota Q, plus the remainder if any: let $k = Seat_1(l)$,*

$$g(l) = \begin{cases} [N_1, N_2, ..., N_k, N_{k+1}], & \text{if } |l| > k \text{, where} \\ \quad N_i = Q,\ 1 \le i \le k,\ N_{k+1} = \pi(l) - kQ & \\ [N_1, N_2, ..., N_k], & \text{if } |l| = k \text{, where} \\ \quad N_i = Q,\ 1 \le i \le k & \end{cases}$$

  Notice that when $Seat_1(l) = 0$, then $g(l) = [\pi(l)]$. Notice also that if $Seat_1(l) > 0$, then $N_{k+1} < Q$.
- *the payoff function v_i is defined as follows: for $i = 1, 2$, $X_1 \in R_1, X_2 \in R_2$,*

$$v_i(X_1, X_2) = \sum_{x \in X_i} v(x, X_1, X_2),$$

  where $x \in X_i$ means that x is an element of the list X_i, and $v(x, X_1, X_2)$ is defined as follows: let $X = \texttt{sort}(\texttt{merge}(X_1, X_2))$,
  - *if the number of elements in X is less than S, then $v(x, X_1, X_2) = 1$ for all x,*
  - *otherwise, let x^* be the Sth element in X, t the number of occurrences of x^* in X, and w the number of the elements in X that are greater than x^*,*

$$v(x, X_1, X_2) = \begin{cases} 1, & \text{if } x > x^* \\ \dfrac{S - w}{t}, & \text{if } x = x^* \\ 0, & \text{if } x < x^* \end{cases}$$

Example 3 Consider the election game given by Table 1, Example 1. Recall that the quota is 4. Consider the strategy profile $L_1 = \{\{c_{11}, c_{12}, c_{13}\}\}$ and $L_2 = \{\{c_{21}, c_{22}\}\})$. Since $\pi(\{c_{11}, c_{12}, c_{13}\}) = 6 > 4$, $Seat_1(\{c_{11}, c_{12}, c_{13}\}) = 1$ and $\pi'(\{c_{11}, c_{12}, c_{13}\}) = 2$, we have that $g(\{c_{11}, c_{12}, c_{13}\}) = [4, 2]$. Thus $f(L_1) = [4, 2]$. For L_2, $\pi(\{c_{21}, c_{22}\}) = 3 < 4$, thus $g(\{c_{21}, c_{22}\}) = [3]$ and $f(L_2) = [3]$. It is easy to see that the payoff vector of the strategy profile $(f(L_1), f(L_2))$ in the reduced game is $(1, 1)$. Now consider $L'_1 = \{\{c_{11}\}, \{c_{12}, c_{13}\}\}$. Since $\pi(\{c_{11}\}) = 3$ and $\pi(\{c_{12}, c_{13}\}) = 3$, we have that $g(\{c_{11}\}) = g(\{c_{12}, c_{13}\}) = [3]$. Thus $f(L'_1) = [3, 3]$, and $(f(L'_1), f(L_2)) = ([3, 3], [3])$. For this strategy profile in the reduced game, the payoff vector is $(1\frac{1}{3}, \frac{2}{3})$.

Lemma 1 *An election game $EG = (C_1, C_2, S, \pi)$ has a Nash equilibrium iff its reduced game $RG = (R_1, R_2, v_1, v_2)$ defined above has a Nash equilibrium.*

Proof: This is proved by showing that for any profile (L_1, L_2) in EG, $u_i(L_1, L_2) = v_i(f(L_1), f(L_2))$, $i = 1, 2$. Thus if (L_1, L_2) is a Nash equilibrium of EG, then $(f(L_1), f(L_2))$ is a Nash equilibrium of RG, and conversely, if (X_1, X_2) is a Nash equilibrium of RG, then for any L_i such that $X_i = f(L_i)$, (L_1, L_2) is a Nash equilibrium of EG. ■

The following lemma says that RG, like EG, is essentially a zero-sum game.

Lemma 2 *For every profile $(X_1, X_2) \in R_1 \times R_2$, we have that $v_1(X_1, X_2) + v_2(X_1, X_2) \leq S$. Furthermore, if $|X_1| + |X_2| \geq S$, then $v_1(X_1, X_2) + v_2(X_1, X_2) = S$.*

We now proceed to show that RG always has a Nash equilibrium.

Theorem 5 *A reduced game RG always has a pure Nash equilibrium.*

Proof: Suppose the given election game is (C_1, C_2, S, π). Suppose the quota is Q, and the reduced game is $RG = (R_1, R_2, v_1, v_2)$.

First of all, we define the "size" of R_i, written V_i below, to be the size of the longest list among all lists in R_i: for each i, V_i is the largest $|X|$ among all $X \in R_i$.

If $V_1 + V_2 < S$, then it is easy to see that for any X_i such that $V_i = |X_i|$, (X_1, X_2) is a Nash equilibrium of RG.

So we assume that $V_1 + V_2 \geq S$. First we define the jth largest element in each player's strategy set: for each $i = 1, 2$, and each $j > 0$, let R_i^j be the set obtained from R_i by taking the jth element, if it exists, in X for each $X \in R_i$:

$$R_i^j = \{x \mid X \in R_i, x \text{ is the } j\text{th element in } X\},$$

and let $m_{i,j}$ be the largest number in R_i^j. Notice that $m_{i,j}$ is well-defined only when $j \leq V_i$, and for any $0 < j_1 < j_2 \leq V_i$, $m_{i,j_2} \leq m_{i,j_1}$.

Now let

$$M = \texttt{sort}([m_{1,1}, m_{1,2}, ..., m_{1,V_1}, m_{2,1}, m_{2,2}, ..., m_{2,V_2}]).$$

Notice that as $V_1 + V_2 \geq S$, we have at least S elements in M. The significant number of RG is then the Sth element in M. We denote this number by θ, and evaluate players' strategies according to it: for any $X \in R_1 \cup R_2$, let $W(X)$ be the number of elements in it that are greater than θ, and $T(X)$ the number of elements in it that are equal to θ. Intuitively, $W(X)$ is the number of "seats" that the strategy X will "win" for sure, and $T(X)$ the number of "seats" that may be in a "tie". Since there is at least one seat in tie position, for any profile $(X_1, X_2) \in R_1 \times R_2$, $W(X_1) + W(X_2) < S$ must hold.

It can be shown that any profile $(X_1, X_2) \in R_1 \times R_2$ that minimizes

$$\frac{S - W(X_1) - W(X_2)}{T(X_1) + T(X_2)}$$

is a pure Nash equilibrium. ■

The first part of Theorem 1, the existence of pure Nash equilibria of every election game, now follows from Lemma 1 and Theorem 5. The second part of Theorem 1 is straightfroward.

Acknowledgements

This work was supported in part by HK RGC under GRF 616013. We thank the anonymous reviwers for their helpful comments on an earlier version of this paper.

References

Balinski, M. L., and Young, H. P. 1975. The quota method of apportionment. *American Mathematical Monthly* 82(7):701–730.

Balinski, M. L., and Young, H. P. 2001. *Fair representation: meeting the ideal of one man, one vote*. Brookings Institution Press.

Bartholdi, J. J., and Orlin, J. B. 1991. Single transferable vote resists strategic voting. *Social Choice and Welfare* 8(4):341–354.

Betzler, N.; Slinko, A.; and Uhlmann, J. 2013. On the computation of fully proportional representation. *J. Artif. Intell. Res. (JAIR)* 47:475–519.

Chamberlin, J. R., and Courant, P. N. 1983. Representative deliberations and representative decisions: Proportional representation and the Borda rule. *The American Political Science Review* 718–733.

Cornaz, D.; Galand, L.; and Spanjaard, O. 2012. Bounded single-peaked width and proportional representation. In *ECAI*, 270–275.

Droop, H. R. 1881. On methods of electing representatives. *Journal of the Statistical Society of London* 141–202.

Gallagher, M. 1992. Comparing proportional representation electoral systems: Quotas, thresholds, paradoxes and majorities. *British Journal of Political Science* 22(4):469–496.

Garey, M. R., and Johnson, D. S. 1979. *Computers and Intractability: A Guide to the Theory of NP-completeness*. WH Freeman and Company, New York.

Hare, T. 1861. *A treatise on the election of representatives, parliamentary and municipal*. Longman, Green, Longman, & Roberts.

Lu, T., and Boutilier, C. 2011. Budgeted social choice: From consensus to personalized decision making. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence*, 280–286. AAAI Press.

Lundell, J., and Hill, I. 2007. Notes on the Droop quota. *Voting matters* 24:3–6.

Monroe, B. L. 1995. Fully proportional representation. *American Political Science Review* 925–940.

Procaccia, A. D.; Rosenschein, J. S.; and Zohar, A. 2008. On the complexity of achieving proportional representation. *Social Choice and Welfare* 30(3):353–362.

Skowron, P.; Yu, L.; Faliszewski, P.; and Elkind, E. 2013. The complexity of fully proportional representation for single-crossing electorates. In *Algorithmic Game Theory*. Springer. 1–12.

Skowron, P.; Faliszewski, P.; and Slinko, A. 2013a. Achieving fully proportional representation is easy in practice. In *Proceedings of the 2013 international conference on Autonomous Agents and Multi-Agent Systems*, 399–406. IFAAMAS.

Skowron, P.; Faliszewski, P.; and Slinko, A. 2013b. Fully proportional representation as resource allocation: Approximability results. In *Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence*, 353–359. AAAI Press.

Tideman, N. 1995. The single transferable vote. *The Journal of Economic Perspectives* 27–38.

Walsh, T. 2010. An empirical study of the manipulability of single transferable voting. In *ECAI*, 257–262.

Yu, L.; Chan, H.; and Elkind, E. 2013. Multiwinner elections under preferences that are single-peaked on a tree. In *Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence*, 425–431. AAAI Press.