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Preventing Unraveling in Social Networks Gets Harder*

Rajesh Chitnis† and **Fedor V. Fomin**‡ and **Petr A. Golovach**‡

Abstract

The behavior of users in social networks is often observed to be affected by the actions of their friends. Bhawalkar et al. (Bhawalkar et al. 2012) introduced a formal mathematical model for user engagement in social networks where each individual derives a benefit proportional to the number of its friends which are engaged. Given a threshold degree k the equilibrium for this model is a maximal subgraph whose minimum degree is $\geq k$. However the dropping out of individuals with degrees less than k might lead to a cascading effect of iterated withdrawals such that the size of equilibrium subgraph becomes very small. To overcome this some special vertices called "anchors" are introduced: these vertices need not have large degree. Bhawalkar et al. (Bhawalkar et al. 2012) considered the ANCHORED k-CORE problem: Given a graph G and integers b, k and p do there exist a set of vertices $B \subseteq H \subseteq V(G)$ such that $|B| \leq b, |H| \geq p$ and every vertex $v \in H \setminus B$ has degree at least k is the induced subgraph $G[H]$. They showed that the problem is NP-hard for $k \geq 2$ and gave some inapproximability and fixed-parameter intractability results. In this paper we give improved hardness results for this problem. In particular we show that the ANCHORED k-CORE problem is W[1]-hard parameterized by p, even for $k = 3$. This improves the result of Bhawalkar et al. (Bhawalkar et al. 2012) (who show W[2]-hardness parameterized by b) as our parameter is always bigger since $p \geq b$. Then we answer a question of Bhawalkar et al. (Bhawalkar et al. 2012) by showing that the ANCHORED k-CORE problem remains NP-hard on planar graphs for all $k \geq 3$, even if the maximum degree of the graph is $k + 2$. Finally we show that the problem is FPT on planar graphs parameterized by b for all $k \geq 7$.

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Introduction

A social network can be thought as the graph of relationships and interactions within a group of individuals. Social networks play a leading role in various fields such as social sciences (Mikolajczyk and Kretzschmar 2008; Milgram 1967), life sciences (Dezső and Barabási 2002; Wilson 1989) and medicine (Dezső and Barabási 2002; Pastor-Satorras and Vespignani 2001). Social networks today perform a fundamental role as a medium for the spread of information, ideas, and influence among its members. As an example, Facebook reported a figure of one billion active users as of October 2012 (BBC 2012). An important characteristic of social networks is that the behavior of an individual is often influenced by the actions of their friends. New events occur quite often in social networks: some examples are usage of a particular cell phone brand, adoption of a new drug within the medical profession, or the rise of a political movement in an unstable society. To estimate whether these events or ideas spread extensively or die out soon, we need to model and study the dynamics of *influence propagation* in social networks. We consider the following model of *user engagement* defined by Bhawalkar et al. (Bhawalkar et al. 2012): there is a single product and each individual has two options of "engaged" or "drop out". Initially we assume that all individuals are engaged. There is a given threshold parameter k such that a person finds it worthwhile to remain engaged if and only if at least k of her friends are still engaged. For example engagement could represent active participation in a social network, and individuals might drop out and switch to a new social network if less than k of his friends are active on the current social network. Indeed such a phase transition has been observed in the popularity of social networks: in India the leading social network was Orkut until Facebook surpassed it in August 2010 (Facebook 2005).

In our model of *user engagement* all individuals with less than k friends will clearly drop out. Unfortunately this can be contagious and may affect even those individuals who initially had more than k friends in the social network. An extreme example of this is given in page 17 of (Schelling 2006): consider a path on n vertices and let $k = 2$. Note that $n - 2$ vertices have degree two in the network. However there will be a *cascade of iterated withdrawls*. An endpoint has degree one, it drops out and now its neighbor in the path has only one friend in the social network and it drops out as well. It is not hard to see that the whole network eventually drops out. In general at the end of all the iterated withdrawls the remaining engaged individuals form a unique maximal induced subgraph whose minimum degree is at least k. This is called as the *k-core* and is a well-known concept in the theory of social networks. It was introduced by Seidman (Seidman 1983) and also been studied in various social sciences literature (Chwe 1999; 2000).

A Game-Theoretic Model: Consider the following game-theoretical model (Bhawalkar et al. 2012): each user in a social network pays a cost of k to remain engaged. On the other hand it receives a profit of 1 from every neighbor who is engaged. The "network effects" come into play, and an individual decides to remain engaged if has non-negative payoff, i.e., it has at least k neighbors who are engaged. The k-core can be viewed as the unique maximal equilibrium in this model. Assuming that all the players make decisions simultaneously the model can be viewed as a simultaneous-move game where each individual has two strategies viz. remaining engaged or dropping out. Consider a graph G defined on the set of players by adding an edge between two players if and only if they are friends in the network. For every strategy profile δ let S_δ denote the set of players who remain engaged. The payoff for a person v is 0 if she is not engaged, otherwise it is the number of her friends among engaged players minus k. We can easily characterize the set of pure Nash equilibria for this game: a strategy profile δ is a Nash equilibrium if and only if the following two conditions hold:

- No engaged player wants to drop out, i.e., minimum degree of the induced graph $G[S_\delta]$ is $\geq k$
- No player who has dropped out wants to become engaged, i.e., no $v \in V(G) \setminus S_\delta$ has $\geq k$ neighbors in S_δ

In general there can be many Nash equilibria. For example, if G itself has minimum degree $\geq k$ then $S_\delta = \emptyset$ and $S_\delta = V(G)$ are two equilibria (and there may be more). Recall that the goal of the product company is to attract as many people as possible. Owing to the fact that it is a maximal equilibrium, the k-core has the special property that it is beneficial to both parties: it maximizes the payoff of every user, while also maximizing the payoff of the product company. Chwe (Chwe 1999; 2000) and Sääskilahti (Sääskilahti 2007) claim that one can reasonably expect this maximal equilibrium even in real-life implementations of this game.

Preventing Unraveling: The unraveling described above in Schelling's example of a path is highly undesirable since the goal is to keep as many people engaged as possible. How can we attempt to prevent this unraveling? In Schelling's example it is easy to see: if we "buy" the two end-point players into being engaged then the whole path becomes engaged.

In general we overcome the issue of unraveling by allowing some "anchors": these are vertices that remain engaged irrespective of their payoffs. This can be achieved by giving them extra incentives or discounts. The hope is that with a few *anchors* we can now ensure a large subgraph remains engaged. This subgraph is now called as the *anchored k-core*: each non-anchor vertex in this induced subgraph must have degree at least k while the anchored vertices can have arbitrary degrees. We use the notation $\deg_S(v)$ to denote the degree of v in the graph S. Bhawalkar et al. (Bhawalkar et al. 2012) formally defined the ANCHORED k-CORE problem :

The ANCHORED k-CORE Problem (AKC)
Input : An undirected graph $G = (V, E)$ and integers b, k
Question: Find a set of vertices $H \subseteq V$ of maximum size such that

- There is a set $B \subseteq H$ and $|B| \leq b$
- Every $v \in H \setminus B$ satisfies $\deg_{G[H]}(v) \geq k$

The AKC problem deals with finding a small group of individuals whose engagement is essential for the health of the social network. We call the set B as *anchors*, the set H as the *anchored k-core* and the set $H \setminus B$ as the *anchored k-supercore*. The decision version of the ANCHORED k-CORE problem deals with anchoring a given number of vertices to maximize the number of engaged vertices. More formally:

p-AKC
Input : An undirected graph $G = (V, E)$ and integers b, k, p
Question: Do there exist sets $B \subseteq H \subseteq V$ such that

- $|B| \leq b$ and $|H| \geq p$
- Every $v \in H \setminus B$ satisfies $\deg_{G[H]}(v) \geq k$

Previous Work: Bhawalkar et al. (Bhawalkar et al. 2012) introduced the ANCHORED k-CORE problem and gave some positive and negative results for this problem. Noting that the problem is trivial for $k = 1$, they showed that AKC is polynomial time solvable for $k = 2$ but NP-hard for all $k \geq 3$. They also gave a strong inapproximability result: it is NP-hard to approximate the AKC problem to within an $O(n^{1-\epsilon})$ factor for any $\epsilon > 0$. From the viewpoint of parameterized complexity they showed that for every $k \geq 3$ the p-AKC problem is W[2]-hard with respect to b. Finally on the positive side they give a polynomial time algorithm on graphs of bounded treewidth. On graphs with treewidth at most w their algorithm runs in $O(3^w (k+1)^{2w} b^2) \cdot \text{poly}(n)$ time.

Our Results: It is easy to see that ANCHORED k-CORE can be solved in time $n^{b+O(1)}$, as we can try all subsets B of size b of the set of vertices of the input graph, and for each B, find the unique k-core H of maximum size such that $\deg_{G[H]}(v) \geq k$ if $v \in H \setminus B$ by the consecutive deletions of small degree vertices. We show that this result is optimal in some sense by proving that p-AKC problem is W[1]-hard parameterized by $b + k + p$. We also show that the p-AKC problem is W[1]-hard parameterized by p even for $k = 3$. This improves the result of Bhawalkar et al. (Bhawalkar et al. 2012) (who show W[2]-hardness parameterized by b), because our parameter is always bigger since $p \geq b$. Bhawalkar et al. raised the question of resolving the complexity of the

ANCHORED k-CORE problem on special graph classes. In this paper we consider the complexity of the AKC problem on the class of planar graphs. We show that the ANCHORED k-CORE problem is NP-hard on planar graphs for all $k \geq 3$, even if the maximum degree of the graph is $k+2$. Finally on the positive side we show that the p-AKC problem on planar graphs is FPT parameterized by b for all $k \geq 7$.

Fixed-Parameter Intractability Results

In this section we give two parameterized intractability results. Before that we give a brief introduction to parameterized complexity.

Parameterized Complexity: *Parameterized Complexity* is basically a two-dimensional generalization of "P vs. NP" where in addition to the overall input size n, one studies the effects on computational complexity of a secondary measurement that captures additional relevant information. This additional information can be, for example, a structural restriction on the input distribution considered, such as a bound on the treewidth of an input graph or the size of solution set. For general background on the theory see (Downey and Fellows 1999; Flum and Grohe 2006; Niedermeier 2006). For decision problems with input size n, and a parameter k, the two dimensional analogue (or generalization) of P, is solvability within a time bound of $O(f(k)n^{O(1)})$, where f is a computable function of k alone. Problems having such an algorithm are said to be *fixed parameter tractable* (FPT). Such algorithms are practical when small parameters cover practical ranges. The W-hierarchy is a collection of computational complexity classes: we omit the technical definitions here. The following relation is known amongst the classes in the W-hierarchy: $FPT = W[0] \subseteq W[1] \subseteq W[2] \subseteq \ldots$. It is widely believed that $FPT \neq W[1]$, and hence if a problem is hard for the class $W[i]$ (for any $i \geq 1$) then it is considered to be fixed-parameter intractable.

W[1]-hardness parameterized by b+k+p: In this section we show that the p-AKC problem is W[1]-hard even when parameterized by $b+k+p$. We reduce from the well-known W[1]-hard problem CLIQUE

CLIQUE
Input : An undirected graph $G = (V, E)$ and an integer ℓ
Question: Does G have a clique of size at least ℓ ?

Theorem 1 $[\star]$[1] *The p-AKC problem is W[1]-hard parameterized by $b + k + p$ for $k \geq 3$.*

W[1]-hardness parameterized by p

Bhawalkar et al. (Theorem 3 in (Bhawalkar et al. 2012)) showed that the p-AKC problem is W[2]-hard parameterized by b for every $k \geq 3$. In this section we prove that it is in fact W[1]-hard parameterized by p for $k = 3$.

[1]The proofs marked with $[\star]$ have been deferred to a full version of this paper due to space constraints.

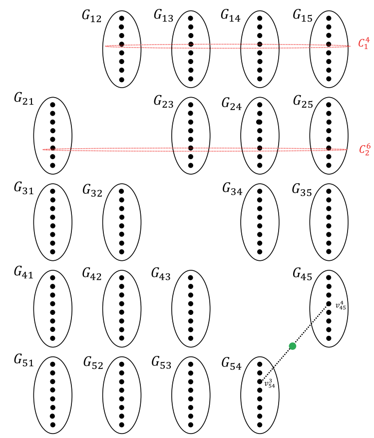

Figure 1: The graph G' constructed in Theorem 2 for the special case when $n = 8$ and $\ell = 5$.

Theorem 2 *The p-AKC problem is W[1]-hard parameterized by p for $k = 3$.*

Proof. We again reduce from the CLIQUE problem. Consider an instance $(G = (V, E), \ell)$ of CLIQUE where $V = (v^1, v^2, \ldots, v^n)$. Construct a new graph $G' = (V', E')$ as follows. For each $1 \leq i \neq j \leq \ell$ make a copy G_{ij} of the vertex set V (do not add any edges). Let the vertex v^r in the copy G_{ij} be labeled v^r_{ij}. Add the following edges to G':

- For each $1 \leq i \neq j \leq \ell$ and $r, s \in [\ell]$ we add an edge between v^r_{ij} and v^s_{ji} if and only if $v^r v^s \in E$. Subdivide each such edge by adding a new vertex called *green*.
- For each $1 \leq i, j \leq \ell$ add the cycle $v^j_{i1} - v^j_{i2} - \ldots v^j_{i,i-1} - v^j_{i,i+1} - \ldots - v^j_{i\ell} - v^j_{i1}$. Let us denote this cycle by C^j_i.

This completes the construction of G'. Let $k = 3,\ b = \binom{\ell}{2}$ and $p = 3b$. The claim is that the instance (G, ℓ) of CLIQUE answers YES if and only if the instance (G', b, k, p) of p-AKC answers YES.

Suppose G has a clique of size ℓ, say $C = \{v_1, v_2, \ldots, v_\ell\}$. For $1 \leq i \neq j \leq \ell$ pick the vertex v^i_{ij} from G_{ij}. This gives a set S of $2b$ vertices. It is easy to see that $G'[S]$ consists of disjoint cycles, and hence is regular of degree two. Now for every v^i_{ij} there is an unique vertex v^j_{ji} which is connected to it by a subdivided edge in G'. The green vertices of this subdivided edges become the anchors. Note that we pick exactly $\frac{|S|}{2} = b$ anchors. It is easy to see that each vertex of S now has degree three in the resulting induced subgraph, and hence S becomes the k-supercore. Therefore the instance (G', b, k, p) of p-AKC answers YES.

Now suppose that the instance (G', b, k, p) of p-AKC answers YES. Let S be the k-supercore and B be the set of anchors. Then we know that $|S| \geq p - b = 2b$. Each green vertex has degree two in G', and hence cannot be in S. Each vertex in S needs at least one green vertex to achieve degree three in $G'[S \cup B]$. But any green vertex can be used by at most two vertices in S. Therefore we have $2b \geq 2|B| \geq |S| \geq 2b$ which implies $|B| = b$ and $|S| = 2b$.

Hence the budget must come from the green vertices only, and that the vertices of S form a *matching* under the relation of sharing a common green vertex. Without loss of generality let $v_{12}^{i_1}$ and $v_{21}^{i_2}$ be two vertices in S such that they share a green vertex from B. Now we know that $v_{12}^{i_1}$ has degree at least three in $G'[B \cup S]$ but cannot be incident to any other green vertex. So we need to include $v_{13}^{i_1}$ and $v_{1\ell}^{i_1}$ in S. Again each of these two vertices can be incident to at most one green vertex in $G'[B \cup S]$ and ultimately this means that we must have $C_1^{i_1} \subseteq G'[B \cup S]$. For $2 \leq j \leq \ell$ we know that the vertex $v_{1j}^{i_1}$ needs one more edge to achieve degree at least three. This edge must be towards a green vertex which is adjacent to some vertex in G_{j1}, say $v_{j1}^{i_j}$. By reasoning similar to above we must have $C_j^{i_j} \subseteq G'[B \cup S]$ for every $2 \leq j \leq \ell$. So we have chosen $2b$ vertices in S, which is the maximum allowed budget. Therefore $S \cap G_{jj'} = \{v_{jj'}^{i_j}\}$ for every $1 \leq j \neq j' \leq \ell$.

The claim is that the set $\{v^{i_1}, v^{i_2}, \ldots, v^{i_\ell}\}$ forms a clique in G. Consider any two indices $1 \leq q \neq r \leq \ell$. We know that the vertex $v_{qr}^{i_q}$ is in S and has degree two in $G'[B \cup S]$ as it is in the cycle $C_q^{i_q}$. To achieve degree three it must be incident to some green vertex. Also it must share this green vertex with some other vertex from $G_{rq} \cap S$. But we know that $G_{rq} \cap S = \{v_{rq}^{i_r}\}$. Therefore $v_{qr}^{i_q}$ and $v_{rq}^{i_r}$ share a green vertex, i.e., v^{i_q} and v^{i_r} are adjacent in G, i.e, the vertices $\{v^{i_1}, v^{i_2}, \ldots, v^{i_\ell}\}$ form a clique in G. $\square$

NP-hardness Results on Planar Graphs

Bhawalkar et al. (Bhawalkar et al. 2012) raised the question of investigating the complexity of the ANCHORED k-CORE problem on special cases of graphs such as planar graphs. In this section we provide some answers by showing NP-hardness results for planar graphs for $k \geq 3$. The case $k \geq 4$ can be handled by a single reduction, but $k = 3$ is more complicated and requires a separate reduction. We reduce from the following problem which was shown to be NP-hard by Dahlhaus et al. (Dahlhaus et al. 1994):

RESTRICTED-PLANAR-3-SAT
Input : A Boolean CNF formula ϕ such that

- Each clause has at most 3 literals
- Each variable is used in at most 3 clauses
- Each variable is used at least once in positive and at least once in negation
- The graph G_ϕ (described below) is planar

Question: Is the formula ϕ satisfiable ?

Consider an instance ϕ of RESTRICTED-PLANAR-3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$. We associate the following graph G_ϕ with ϕ:

- For each $1 \leq i \leq n$ introduce the vertices r_i, x_i and $\overline{x_i}$. Add the edges $r_i x_i$ and $r_i \overline{x_i}$.
- For each $1 \leq j \leq m$ introduce the vertex c_j.
- For each $1 \leq i \leq n$ and $1 \leq j \leq m$ add an edge between x_i (or $\overline{x_i}$) and c_j iff x_i (or $\overline{x_i}$) belongs to the clause C_j.

NP-hardness on Planar Graphs for $k = 3$

In this section we show that the ANCHORED k-CORE problem is NP-hard on planar graphs for all $k = 3$, even in graphs of maximum degree 5.

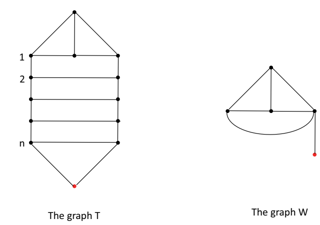

Figure 2: The graphs T and W used in construction of G in Theorem 3. Note that T has $2n + 3$ vertices, and exactly one vertex has degree two. The graph W has exactly one vertex of degree one.

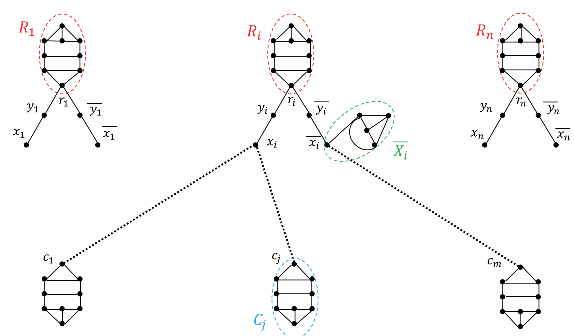

Figure 3: The graph G constructed in Theorem 3.

Theorem 3 *For $k = 3$ the* ANCHORED k-CORE *problem is NP-hard even on planar graphs of maximum degree* 5.

Proof. We reduce from the RESTRICTED-PLANAR-3-SAT problem. For an instance ϕ of RESTRICTED-PLANAR-3-SAT let G_ϕ be the associated planar graph. We define two special graphs T and W (see Figure 2) and build the graph G as follows (see Figure 3):

- For each $1 \leq i \leq n$ subdivide the edge $r_i x_i$ and let the newly introduced vertex be y_i
- For each $1 \leq i \leq n$ subdivide the edge $r_i \overline{x_i}$ and let the newly introduced vertex be $\overline{y_i}$.
- For each $1 \leq i \leq [n$ attach a copy of T by identifying its degree two vertex with vertex r_i. Call this gadget as R_i
- For each $1 \leq i \leq n$ if x_i appears in exactly one clause then attach a copy of W by identifying its degree one vertex with vertex x_i. Call this gadget as X_i
- For each $1 \leq i \leq n$ if $\overline{x_i}$ appears in exactly one clause then attach a copy of W by identifying its degree one vertex with vertex $\overline{x_i}$. Call this gadget as $\overline{X_i}$

- For each $1 \leq j \leq m$ attach a copy of T by identifying its degree two vertex with vertex c_j. Call this gadget as C_j

This completes the construction of the graph $G = (V, E)$. All the gadgets we added are planar, and it is easy to verify that the planarity is preserved when we construct G from G_ϕ. Let $k = 3,\ b = n$ and $p = |V| - 2n$. The claim is that ϕ is satisfiable if and only if the instance (G, b, k, p) of p-AKC answers YES.

Suppose ϕ is satisfiable. For each $1 \leq i \leq n$, if $x_i = 1$ in the satisfying assignment for ϕ then pick y_i in B and $\overline{x_i},\ \overline{y_i}$ in B'. Otherwise pick $\overline{y_i}$ in B and $x_i,\ y_i$ in B'. Clearly $|B| = n$ and $|B'| = 2n$. Let B be the set of anchors and set $H = V \setminus B'$. Now the claim is that every vertex $w \in H \setminus B$ has degree at least three in the induced subgraph $G[H]$. For each $1 \leq i \leq n$, exactly one of y_i or $\overline{y_i}$ is in B. Hence r_i (and also each vertex of R_i) has degree exactly three in H. Consider a literal x_i. We have the following two cases:

- $\underline{y_i \in B}$: x_i gets one edge from y_i. If x_i appears in exactly one clause then it gets one edge from that clause vertex and one edge from its neighbor in X_i (and each vertex in X_i has degree at least three in H). Otherwise x_i gets two edges from the two clause vertices which it appears in.
- $\underline{\overline{y_i} \in B}$: $\overline{x_i}$ gets one edge from $\overline{y_i}$. If $\overline{x_i}$ appears in exactly one clause then it gets one edge from that clause vertex and one edge from its neighbor in $\overline{X_i}$ (and each vertex in $\overline{X_i}$ has degree at least three in H). Otherwise $\overline{x_i}$ gets two edges from the two clause vertices which it appears in.

Finally consider a clause vertex c_j. It has at least one true literal say x_i in it. In addition c_j has two neighbors in C_j, and hence the degree of c_j is at least three in H. Consequently each vertex in C_j has degree at least three in H. Therefore with $b = |B| = n$ anchors we can cover a 3-core of size at least $|V \setminus B'| = |V| - 2n = p$, and hence (G, b, k, p) answers YES.

Suppose that the instance (G, b, k, p) of p-AKC answers YES. Let us denote the 3-core by H. Note that we can afford to not have at most $2n$ vertices in the 3-core. Each y_i and $\overline{y_i}$ have degree two in G: so either we cannot have them in the 3-core or we need to pick them as anchors. Also for $i \in [n]$ if we do not pick at least one of y_i or $\overline{y_i}$ then the vertex r_i also cannot be in the 3-core. This will lead to a cascade effect and the whole gadget R_i cannot be the in the 3-core, which is a contradiction since it has $2n + 3$ vertices and we could have left out at most n vertices from the core. If for some $i \in [n]$ we pick both y_i and $\overline{y_i}$ as anchors then for some $j \neq i$ we cannot pick either of y_j and $\overline{y_j}$ as anchors since the total budget for anchors is at most n. Therefore we must anchor exactly one of $y_i, \overline{y_i}$ for each $1 \leq i \leq n$. Let B be the set of anchors. Consider the assignment $f : \{1, 2, \dots, n\} \to \{0, 1\}$ given by $f(x_i) = 1$ if $y_i \in B$ or $f(x_i) = 0$ otherwise. We claim that f is indeed a satisfying assignment for ϕ. Consider a clause vertex c_j. We know that c_j must lie in the 3-core: otherwise we lose the entire gadget C_j which has $2n+3$ vertices which is more than our budget. Therefore c_j has an edge in $G[H]$ to some vertex say x_i. If $y_i \notin B$ then x_i can have degree at most two in $G[H]$: either it appears in exactly one clause and has a copy of W attached to it, or it is adjacent to two clause vertices. Therefore $y_i \in B$ which implies $f(x_i) = 1$, and so the clause c_j is satisfied.

Finally note that the maximum degree of G is five, which can occur if c_j has three literals. $\square$

NP-hardness on Planar Graphs for $k \geq 4$

In this section we show that the ANCHORED k-CORE problem is NP-hard on planar graphs for all $k \geq 4$, even in graphs of maximum degree $k + 2$.

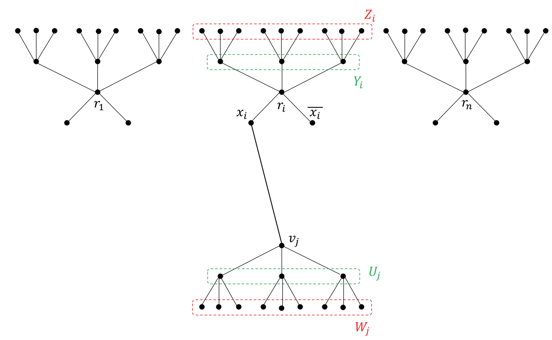

Figure 4: The graph G constructed in Theorem 4 for $k = 4$.

Theorem 4 *For any $k \geq 4$ the* ANCHORED k-CORE *problem is NP-hard even on planar graphs of maximum degree $k + 2$.*

Proof. Fix any $k \geq 4$. We reduce from the RESTRICTED-PLANAR-3-SAT problem. For an instance ϕ of RESTRICTED-PLANAR-3-SAT let G_ϕ be the associated planar graph. We build a graph $G = (V, E)$ from G_ϕ as follows (see Figure 4):

1. For each $1 \leq i \leq n$
   - Add a set Y_i of $k - 1$ vertices and make all of them adjacent to r_i.
   - For each vertex $y \in Y_i$ add $k - 1$ vertices and make all of them adjacent to y. Let Z_i be the set of all these $(k - 1)^2$ vertices.
2. For each $1 \leq j \leq m$
   - Add a set U_j of $k - 1$ vertices and make all of them adjacent to v_j.
   - For each vertex $u \in U_j$ add $k - 1$ vertices and make all of them adjacent to u. Let W_j be the set of all these $(k - 1)^2$ vertices.

Set $b = n((k-1)^2 + 1) + m(k-1)^2$ and $p = n(k(k-1) + 2) + m(k(k-1) + 1) = b + nk + mk$. Note that degree of each x_i and $\overline{x_i}$ is at most three in G. We claim that ϕ is satisfiable if and only if the instance (G, b, k, p) of p-AKC answers YES.

Suppose ϕ is satisfiable. For each $1 \leq i \leq n$, if $x_i = 1$ in the satisfying assignment then select x_i in B' and $\overline{x_i}$ in B''. Else select $\overline{x_i}$ in B' and x_i in B''. Let $B = B' \bigcup (\cup_{i=1}^{n} Z_i) \bigcup (\cup_{j=1}^{m} W_j)$. Let us place the anchors at vertices of B. Then $|B| = n + n(k-1)^2 + m(k-1)^2 = b$.

Now the claim is that $V \setminus B''$ forms a k-core. This would conclude the proof since $|V \setminus B''| = |G| - n = p$. For each $1 \le i \le n$ the vertex r_i gets one neighbor from either x_i or $\overline{x_i}$ and it has $k-1$ neighbors in Y_i. Each vertex in Y_i has one neighbor in r_i and $k-1$ neighbors in Z_i. Each vertex in U_j has one neighbor in v_j and $k-1$ neighbors in W_j. Since there is a satisfying assignment we know that v_j has at least one neighbor in B', and of course has $k-1$ neighbors in U_j. So $V \setminus B''$ forms a k-core with B as the anchor set, and the instance (G, b, k, p) of p-AKC answers YES.

Now suppose that the instance (G, b, k, p) of p-AKC answers YES. Let us denote the anchors by B and the k-core by H. Consider $S = (\cup_{i=1}^{n}\{x_i, \overline{x_i}\}) \bigcup (\cup_{i=1}^{n} Z_i) \bigcup (\cup_{j=1}^{m} W_j)$. Note that $|S| = b + n$. Any vertex in S has degree at most $\max\{k-1, 3\}$ in G: so if it is present in the k-core then it must be an anchor. Since $p = |V| - n$ at least $|S| - n$ vertices from S must be anchors. Since $|S| - n = b$ these vertices must be the anchor set say B and the k-core is $H = B \cup (V \setminus S)$. Let $z \in Z_i$ for some $1 \le i \le n$ be adjacent to $y \in Y_i$ in G. If $z \notin B$ then y has at most $k-1$ neighbors in H, which contradicts the fact that $Y_i \subseteq H \setminus B$. Therefore $Z_i \subseteq B$ for every $1 \le i \le [n]$. Similarly we have $W_j \subseteq B$ for every $1 \le j \le m$. So now we can only choose n more anchors from the set $\cup_{i=1}^{n}\{x_i, \overline{x_i}\}$. Suppose for some $1 \le i \le n$ we have both $x_i \notin B$ and $\overline{x_i} \notin B$. Then the vertex r_i has degree at most $k-1$ in $G[H]$, contradicting the fact that $r_i \in H \setminus B$. Therefore for every $1 \le i \le n$ at least one of x_i or $\overline{x_i}$ must be in B. As the budget for the anchors is n we know that for every $1 \le i \le n$ exactly one of x_i or $\overline{x_i}$ is in B. Consider the assignment $f : \{1, 2, \ldots, n\} \to \{0, 1\}$ for ϕ given by $f(i) = 1$ if $x_i \in B$ and 0 otherwise. The claim is that f is a satisfying assignment for ϕ. Consider any clause C_j of ϕ. The vertex v_j has exactly $k-1$ neighbors in U_j, and hence must have at least one neighbor in B (which appears in the clause C_j). If this neighbor is some $x_i \in B$ then the assignment would set $x_i = 1$. Else the neighbor is of the type $\overline{x_i} \in B$ and then our assignment would have set $x_i = 0$. Hence f is a satisfying assignment for ϕ. Finally note that the maximum degree of G is $k+2$, which can occur if v_j has three literals. $\square$

FPT on Planar Graphs Parameterized by b

In this section we show that the p-AKC problem is FPT parameterized by b on planar graphs when $k \ge 7$.

Lemma 1 *The problem of checking whether there is an anchored k-core such that $q \ge |H| \ge p$ can be expressed in first-order logic.*

Proof. Consider the following formula in first-order logic: $\phi_q = \bigvee_{p \le i \le q} \big(\exists h_1, h_2, \ldots h_i : H_i \wedge \bigvee_{1 \le j \le b} \big(\exists b_1, b_2, \ldots b_j : B_j \wedge B_j H_i \wedge \forall y : \big(\bigvee_{1 \le i_1 \le i}(y = h_{i_1}) \wedge Y B_j\big) \to \exists v_1 \ldots v_k : V_k \wedge V_k H_i \wedge Y V_k\big)\big)$
where
$H_i = \bigwedge_{1 \le i_1 \ne i_2 \le i}(h_{i_1} \ne h_{i_2})$
$B_j = \bigwedge_{1 \le j_1 \ne j_2 \le j}(b_{j_1} \ne b_{j_2})$
$B_j H_i = \bigwedge_{1 \le j_1 \le j}(\bigvee_{1 \le i_1 \le i}(b_{j_1} = h_{i_1})$
$Y B_j = \bigwedge_{1 \le j_1 \le j}(y \ne b_{j_1})$
$V_k = \bigwedge_{1 \le k_1 \ne k_2 \le k}(v_{k_1} \ne v_{k_2})$
$V_k H_i = \bigwedge_{1 \le k_1 \le k}(\bigvee_{1 \le i_1 \le i}(v_{k_1} = h_{i_1}))$
$Y V_k = \bigwedge_{1 \le k_1 \le k}(y v_{k g_1} \in E))$

We claim that the formula ϕ_q correctly expresses the problem of checking whether there is an anchored k-core such that $q \ge |H| \ge p$. The formulae H_i, B_j and B_k just check that all the variables in the respective formula are pairwise distinct. The formula $B_j H_i$ checks every anchor is present in the anchored k-core H. Finally for every $y \in H \setminus B$ we enforce that that there are at least k elements $v_1, v_2, \ldots, v_k$ which are pairwise distinct, present in H and adjacent to y. It is now easy to see that any solution H such that $q \ge |H| \ge p$ gives a solution to the formula ϕ_q and vice versa, i.e., the formula ϕ_q exactly expresses this problem. Note that the length of ϕ_q is $\mathrm{poly}(q)$ since $q \ge p \ge b$ and $q \ge k-1$. $\square$

Seese (Seese 1996) showed that any graph problem expressible in first-order logic can be solved in linear FPT time on graphs of bounded degree. More formally, let $\mathcal{X}$ be a graph problem and $\phi_{\mathcal{X}}$ be a first-order formula for $\mathcal{X}$. For a constant $c > 0$ consider the graph class $\mathcal{G}_c = \{\, G \mid \Delta(G) \le c\}$. Then for every $G \in \mathcal{G}$ we can solve $\mathcal{X}$ in $O(f(|\phi_{\mathcal{X}}|) \cdot |G|)$ where f is some function. This was later extended by Dvorak et al. (Dvorak, Král, and Thomas 2010) to a much richer graph class known as graphs with *bounded expansion*. We refer to (Dvorak, Král, and Thomas 2010; Nesetril and de Mendez 2006) for the exact definitions. However we remark that examples of such graph classes are graphs of bounded degree, graphs of bounded genus (including planar graphs), graphs that exclude a fixed (topological) minor, etc. Using these results we can give FPT algorithm parameterized by b on some classes of sparse graphs when k is sufficiently large. However for the sake of simplicity we just state the result for planar graphs.

Lemma 2 [$\star$] *Let G be a planar graph on n vertices. Let $k \ge 7$ and m be the set of vertices of degree at least k. Then $\frac{m}{n} < \frac{6}{7}$.*

Theorem 5 [$\star$] *Let $k \ge 7$. Then for the class of planar graphs the p-AKC problem can be solved in linear FPT time parameterized by the number of anchors b.*

Conclusions and Open Problems

We studied the complexity of the AKC problem on the class of planar graphs, thus answering the question raised in (Bhawalkar et al. 2012). We showed that the AKC problem is NP-hard on planar graphs, even if the graph has maximum degree $k+2$. We also improve some fixed-parameter intractability results for the p-AKC problem. Finally on the positive side we show that for all $k \ge 7$ the p-AKC problem on planar graphs is FPT parameterized by b.

There are still several interesting questions remaining. We mention some of them here: what is the parameterized complexity status of the problem parameterized by b on planar graphs for $3 \le k \le 6$? What happens when we consider the problem on random graphs? Can we get reasonable approximation algorithms on some restricted graph classes?

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