Topological Relations between Convex Regions *

Sanjiang Li[1,2] and **Weiming Liu**[1]
[1] Centre for Quantum Computation and Intelligent Systems (QCIS)
Faculty of Engineering and Information Technology, University of Technology, Sydney, Australia
sanjiang.li@uts.edu.au, weiming.liu-1@student.uts.edu.au
[2] State Key Laboratory of Intelligent Technology and Systems, TNLIST
Department of Computer Science and Technology,Tsinghua University, Beijing, China

Abstract

Topological relations between spatial objects are the most important kind of qualitative spatial information. Dozens of relation models have been proposed in the past two decades. These models usually make a small number of distinctions and therefore can only cope with spatial information at a fixed granularity of spatial knowledge. In this paper, we propose a topological relation model in which *the* topological relation between two convex plane regions can be uniquely represented as a circular string over the alphabet $\{\mathsf{u},\mathsf{v},\mathsf{x},\mathsf{y}\}$. A linear algorithm is given to compute the topological relation between two convex polygons. The infinite relation calculus could be used in hierarchical spatial reasoning as well as in qualitative shape description.

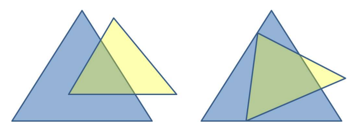
Keywords: qualitative spatial reasoning; topological relation; convex region; intersection; computational geometry

Introduction

Human beings are very good at making qualitative distinctions for spatial configurations. The challenge of the AI approach to spatial reasoning — Qualitative Spatial Reasoning (QSR) — is to "provide calculi which allow a machine to represent and reason with spatial entities without resort to the traditional quantitative techniques prevalent in, for e.g. the computer graphics or computer vision communities." (Cohn and Renz 2007)

It is evident that a single calculus is insufficient to represent all aspects of space. In the past two decades, we have seen dozens of spatial calculi, each of which introduces a finite number of basic distinctions to the spatial relations.

As for topological relations, the region connection calculus (RCC) (Randell, Cui, and Cohn 1992) is perhaps the most well-known topological formalism. Based on one primitive connectedness relation, many different topological relations can be defined. In particular, the RCC supports the definition of two spatial relation algebras, i.e. the RCC5 and the RCC8. These two algebras make a small number (5 and 8, respectively) of topological distinctions. It is of no surprise that many topologically different configurations are classified as the same. The following figure illustrates two topologically different configurations of the same RCC8 relation **PO** (partially overlap).



The RCC8 basic relations are the only atomic topological relations between closed disks. There are, however, more (actually 32, see §4 of this paper) atomic topological relations between triangles. *Is it possible to make a complete classification for topological relations between spatial objects? How to represent them? And, how to tell if two configurations have the same topological relation?*

Following the tradition of spatial database research, in this paper *we use semi-algebraic sets to model spatial regions* (cf. e.g. (Benedikt et al. 2006)). A planar set is called *semi-algebraic* if it can be defined by a Boolean combination of polynomial inequalities.

We give answers to all above problems for convex regions. Although not a topological property, convexity is preserved under projections such as translation, rotation, and scale, and hence is still qualitative in nature. Convexity plays a central role in computational geometry, geographical information science, and several other disciplines. An arbitrarily shaped object is often approximated by its convex hull in practical applications. This is clearly more precise than its minimum bounding rectangle (MBR).

This paper proposes a topological relation model for convex regions. We uniformly represent the topological relation between any pair of convex regions by a finite string over $\{\mathsf{u},\mathsf{v},\mathsf{x},\mathsf{y}\}$. This means, two configurations (each consists of two convex regions) are topologically equivalent iff they have the same string representation. This provides a complete classification for topological relations between convex regions. Moreover, a string (of length greater than 1) represents the topological relation between some convex regions iff characters in $\{\mathsf{u},\mathsf{v}\}$ and characters in $\{\mathsf{x},\mathsf{y}\}$ appear in turn. We also give a linear (in the number of the vertices of the two polygons) algorithm to compute the topological relation between two convex polygons.

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The remainder of this work is structured as follows. Section 2 prepares the paper with some preliminaries. Section 3 describes the model and Section 4 applies the model to triangles and gives illustrations of topological relations between triangles. We then introduce the linear algorithm in Section 5. Further discussions and related work are given in Section 6, which is followed by a concluding section.

Preliminaries

In this paper, the usual topology on $\mathbb{R}^2$ is assumed. A set a in the plane is called a region if it is nonempty and regular closed, i.e. $a = \overline{a^\circ} \neq \varnothing$, where x° and $\overline{x}$ denote the interior and, respectively, the closure of a set x.

RCC8 Basic Relations

For two regions a, b, we have
$(a, b) \in \mathbf{DC}$ if $a \cap b = \varnothing$;
$(a, b) \in \mathbf{EC}$ if $a^\circ \cap b^\circ = \varnothing$ but $a \cap b \neq \varnothing$;
$(a, b) \in \mathbf{PO}$ if $a^\circ \cap b^\circ \neq \varnothing$ and $a \not\subseteq b$ and $b \not\subseteq a$;
$(a, b) \in \mathbf{TPP}$ if $a \subseteq b$ but $a \not\subseteq b^\circ$;
$(a, b) \in \mathbf{NTPP}$ if $a \subseteq b^\circ$;
$(a, b) \in \mathbf{EQ}$ if $a = b$.

The above relations, together with the converses of **TPP** and **NTPP**, are jointly exhaustive and pairwise disjoint (JEPD). This means, any two regions are related by exactly one basic relation. This classification is simple and general, but not precise when fine grained information is important.

Homeomorphism and Isotopy

A *homeomorphism* of the plane is a mapping f from $\mathbb{R}^2$ to itself which is a bijection and both f and f^{-1} are continuous. Write Hom for the set of all homeomorphisms of $\mathbb{R}^2$. An *isotopy* is a homeomorphism of the plane that is isotopic to the identity id, where two functions are *isotopic* if one can be changed into another continuously. More formally, we say two homeomorphisms f and g in Hom are isotopic, if there exists a function $F(x, t) : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$, such that

- For any $t \in [0, 1]$, $F_t(x) = F(x, t)$ is a homeomorphism.
- For any $x \in \mathbb{R}^2$, $F(x, t)$ is continuous at t.
- $F_0 = f$ and $F_1 = g$.

A homeomorphism of $\mathbb{R}^2$ is either isotopic to identity id, or isotopic to the reflection mr, which maps each (x, y) to $(-x, y)$ (Moise 1977).

Atomic Topological Relations

Write $\mathcal{U}$ for the set of plane regions. A binary relation α on $\mathcal{U}$ is called a *topological relation* if for any instance (a, b) of α and any homeomorphism $f \in \mathsf{Hom}$, $(f(a), f(b))$ is also an instance of α.

A topological relation α is called *atomic* if, for any two instances (a, b) and (a', b') of α, there exists a homeomorphism $f \in \mathsf{Hom}$ such that $a' = f(a)$ and $b' = f(b)$.

Proposition 1. *Let a, b be two plane regions. The relation*

$$\alpha_{a,b} = \{(f(a), f(b)) : f \in \mathsf{Hom}\} \tag{1}$$

is an atomic topological relation. Moreover, $\alpha_{a,b}$ is the smallest topological relation which contains (a, b).

This means, $\alpha_{a,b}$ is *the* topological relation of a to b. Since each pair of regions is contained in a unique atomic topological relation, the set of atomic topological relation is a complete classification of the topological relations between plane regions. It is also clear that a relation on $\mathcal{U}$ is a topological relation iff it is the union of a set of atomic topological relations. In particular, the RCC8 relations are all topological relations. On the other hand, no RCC8 basic relation is an atomic topological relation. This is because there exist topologically different regions.

So, how many atomic topological relations are there?

Proposition 2. *There are uncountably many atomic topological relations.*

Proof. This is because there are uncountably many topologically different regions in the plane. Note that a plane region may contain infinite holes and connected components. □

Restriction to Convex Regions

Quite often, we need to restrict the discussion to a set of special regions, e.g. simple regions (i.e. regions homeomorphic to a closed disk), convex regions, rectangles, or disks.

Suppose $\mathcal{U}'$ is a subset of $\mathcal{U}$. For an (atomic) topological relation α on $\mathcal{U}$, $\alpha|_{\mathcal{U}'}$, the restriction of α to $\mathcal{U}'$, could be empty. The number of atomic topological relations will decrease significantly if we restrict regions to special ones. For example, suppose $\mathcal{U}'$ is the set of closed plane disks. Then there are only eight atomic topological relations, viz. the RCC8 basic relations (restricted to disks). If the choice of $\mathcal{U}'$ is understood, we also write α for its restriction.

In the remainder of this paper, we always assume a convex region is semi-algebraic closed.

Because all convex regions are homeomorphic, we have

Proposition 3. *The RCC8 basic relations* **DC**, **NTPP**, **EQ**, *and* **NTPP**$^\sim$, *the converse of* **NTPP**, *are all atomic topological relations on convex regions.*

We show there are infinite but still countable atomic topological relations on convex regions, each of which can be represented by a finite string over a finite alphabet. To this end, we need some preliminary results.

Proposition 4. *Suppose a, b are two convex regions that are externally connected, i.e. $(a, b) \in$* **EC**. *Then $a \cap b$ is either a singleton or a line segment.*

By the above result, it is easy to show the following

Proposition 5. *The* **EC** *relation on convex regions contains exactly two atomic topological relations (cf. Table 3).*

For convex regions a and b, it is clear that $a^\circ \cap \partial b$, $\partial a \cap b^\circ$, and $\partial a \cap \partial b$ form a partition of $\partial(a \cap b)$, where ∂x is the boundary of x. Note that $a^\circ \cap \partial b$ or $\partial a \cap b^\circ$ or $\partial a \cap \partial b$ may be empty. For example, if $a \subseteq b^\circ$, i.e. $(a, b) \in \mathbf{NTPP}$, then we have $a^\circ \cap \partial b = \varnothing$, $\partial a \cap \partial b = \varnothing$, and $\partial a \cap b^\circ = \partial(a \cap b)$.

The following proposition is easy to prove.

Proposition 6. *Suppose $a \neq b$ are two convex regions and a is not contained in the interior of b. Each mcc (maximally connected component) of $a^\circ \cap \partial b$ or $\partial a \cap b^\circ$ is homeomorphic to the open interval $(0, 1)$; and each mcc of $\partial a \cap \partial b$ is a single point or homeomorphic to $[0, 1]$.*

Definition 1. Suppose $a \neq b$ are two convex regions. We say (cf. Fig. 1) a subset X of $\partial(a \cap b)$ is a

- type u component if X is a mcc of $a^\circ \cap \partial b$;
- type v component if X is a mcc of $\partial a \cap b^\circ$;
- type x component if X is a 0-dimensional mcc of $\partial a \cap \partial b$;
- type y component if X is a 1-dimensional mcc of $\partial a \cap \partial b$.

For two semi-algebraic closed convex regions $a \neq b$, $a \cap b$ has finite typed components in the above sense.

For two strings s_1 and s_2 of same length, we say s_2 is a *circular rotation* of s_1 if it consists of a suffix of s_1 followed by a prefix of s_1. For example, $s' = $ (uxuxvy) is a circular rotation of $s = $ (vyuxux). A *circular string* of length n is a string in which the last character is considered to precede the first character (Gusfield 1997). We say $s_2 = (\gamma_0\gamma_1 \cdots \gamma_{n-1})$ is the *inverse* of $s_1 = (\delta_0\delta_1 \cdots \delta_{n-1})$, written s_1^{-1}, if $\gamma_i = \delta_{n-1-i}$ for each $i = 0, 1, \cdots, n-1$.

A String Representation

For each pair of non-equal convex regions (a, b), we show the atomic topological relation $\alpha_{a,b}$ can be represented by a circular string over $\{$u, v, x, y$\}$.

Starting from one typed component, we travel clockwise along $\partial(a \cap b)$ until arriving at the starting component. Recording the type of each component in order, we get a string s over $\{$u, v, x, y$\}$. We say s represents (a, b).

Take the two convex regions in Figure 1 as example. Starting from the type v component, we get a string $s = $ (vyuxux). That is, s represents (a, b).

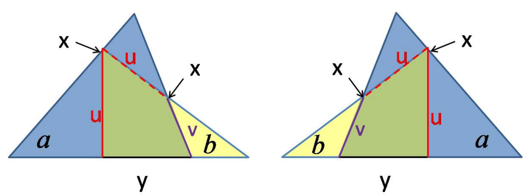

Figure 1: A configuration (left) and its mirror image (right)

We examine some simple cases.

Proposition 7. *The circular strings*

$$\varepsilon,\ (v),\ (u),\ (x),\ (y)$$

represent the atomic topological relations **DC**, **NTPP**, **NTPP**$^\sim$, *and the two sub-relations of* **EC** *(cf. Table 3), respectively.*

If s represents (a, b), so does any of its circular rotation.

Proposition 8. *Let $a \neq b$ be two convex regions and let s be a string over $\{$u, v, x, y$\}$. Suppose s represents (a, b). If s' is a circular rotation of s, then s' also represents (a, b).*

We henceforth regard a representation string s of a configuration as a circular string.

Proposition 9. *Let $a \neq b$ be two convex regions. Suppose s is a string over $\{$u, v, x, y$\}$ that represents (a, b). Then the inverse string s^{-1} represents $(mr(a), mr(b))$, where mr is the homeomorphism defined as $mr(x, y) = (-x, y)$.*

Proof. Let $c_0, c_1, \cdots, c_{n-1}$ be the components of $\partial(a \cap b)$, arranged clockwise. Since mr is a homeomorphism, $mr(c_i)$ is also a component of $\partial(mr(a) \cap mr(b))$ with the same type of c_i. Note that the order is opposite, i.e., $mr(c_0), mr(c_1), \cdots, mr(c_{n-1})$ are arranged counterclockwise. So if we start from component $mr(c_{n-1})$ and travel clockwise along $\partial(mr(a)) \cap (mr(b))$, we get string s^{-1}. This shows that s^{-1} represents $(mr(a), mr(b))$. □

In Figure 1, the inverse of s is (xuxuyv), which represents $(mr(a), mr(b))$, the mirror image of (a, b).

The next proposition shows that if a circular string s represents two pairs of convex regions (a, b) and (a', b'), then they are topologically equivalent.

Proposition 10. *Let (a, b) and (a', b') be two pairs of non-equal convex regions. Suppose s is a circular string that represents both (a, b) and (a', b'). Then there exists a homeomorphism f such that $f(a) = a'$ and $f(b) = b'$.*

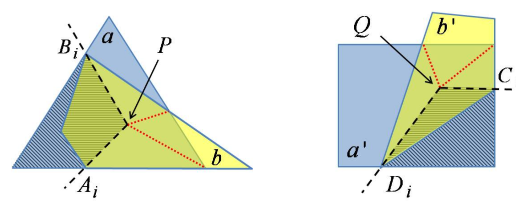

Figure 2: Construction of a homeomorphism

Proof. Without loss of generality, we assume $(a \cap b)^\circ \neq \varnothing$. Suppose $s = (\delta_0\delta_1 \cdots \delta_{n-1})$. Let c_i and c_i' be, respectively, the components of $\partial(a \cap b)$ and $\partial(a' \cap b')$ corresponding to δ_i. We select a point P from $(a \cap b)^\circ$ and a point Q from $(a' \cap b')^\circ$. Suppose $\delta_i \neq$ x. Consider the components c_i and c_i'. Let A_i, B_i and C_i, D_i be the endpoints of c_i and c_i', respectively. Suppose $\angle A_iPB_i = \phi_i$ and $\angle C_iQD_i = \psi_i$. Write r_i (r_i', resp.) for the area obtained by rotating ray PA_i (QC_i, resp.) clockwise to PB_i (QD_i, resp.).

Each point X in r_i is uniquely represented by a pair (θ_X, d_X), where $\theta_X = \angle A_iPX$, and $d_X = |PX|$, the distance from P to X. Similarly, each point Y in r_i' is represented by (θ_Y', d_Y').

For any $0 \leq \theta \leq \phi_i$, there exists a unique point X in the boundary of a (or b) such that $\angle A_iPX = \theta$. This is because a and b are convex and P is in the interior of $a \cap b$. Suppose $P_{1,\theta}$ and $P_{2,\theta}$ are two points in the boundaries of a and b such that $\angle A_iPP_{1,\theta} = \angle A_iPP_{2,\theta} = \theta$ and $\mu_{1,\theta} \equiv |PP_{1,\theta}| \geq |PP_{2,\theta}| \equiv \mu_{2,\theta}$. For any $0 \leq \theta' \leq \psi_i$, we similarly define two points $Q_{1,\theta'}$ and $Q_{2,\theta'}$ in the boundaries of a' and b' such that $\angle C_iQQ_{1,\theta'} = \angle C_iQQ_{2,\theta'} = \theta'$ and $\mu_{1,\theta'}' \equiv |QQ_{1,\theta'}| \geq |QQ_{2,\theta'}| \equiv \mu_{2,\theta'}'$.

For each point $X = (\theta, d)$ in r_i, let $\theta' = \theta \times \frac{\psi_i}{\phi_i}$ and define

$$d' = \begin{cases} d \times \frac{\mu_2'(\theta')}{\mu_2(\theta)} & \text{if } 0 \leq d \leq \mu_2(\theta); \\ \mu_2'(\theta') + (d - \mu_2(\theta)) \times \frac{\nu'(\theta')}{\nu(\theta)} & \text{if } \mu_2(\theta) < d < \mu_1(\theta); \\ d \times \frac{\mu_1'(\theta')}{\mu_1(\theta)} & \text{if } d \geq \mu_1(\theta); \end{cases}$$

where $\nu(\theta) = \mu_1(\theta) - \mu_2(\theta)$ and $\nu'(\theta') = \mu_1'(\theta') - \mu_2'(\theta')$.

Clearly, $Y = (\theta_i', d')$ is a point in r_i'. Define $f_i(X) = Y$. It is straightforward to prove that f_i is a homeomorphism from r_i to r_i', and $f_i(a \cap r_i) = a' \cap r_i'$ and $f_i(b \cap r_i) = b' \cap r_i'$.

These functions $\{f_i\}$ are compatible (i.e. $f_i(X) = f_j(X)$ for any $X \in \mathsf{dom}(f_i) \cap \mathsf{dom}(f_j)$). We amalgamate these f_i into one function f. It's not hard to prove that f is a homeomorphism of $\mathbb{R}^2$, which maps a to a', b to b'. □

On the other hand, if $(f(a), f(b))$ is the image of (a, b) under a homeomorphism f, then either s or its inverse s^{-1} represents $(f(a), f(b))$.

Proposition 11. *Let $a \neq b$ be two convex regions. Assume f is a homeomorphism on the plane such that $f(a)$ and $f(b)$ are convex regions. Suppose s is a circular string that represents (a, b). Then either s or s^{-1} represents $(f(a), f(b))$.*

Proof. Suppose $s = (\delta_0\delta_1 \cdots \delta_{n-1})$, and the components of $\partial(a \cap b)$ are $c_0, c_1, \cdots, c_{n-1}$, where the type of c_i is δ_i. As f is a homeomorphism, $f(c_0), f(c_1), \cdots, f(c_{n-1})$ are components of $\partial(f(a) \cap f(b))$. If f is isotopic to the identity, then the orientation of $c_0, c_1, \cdots, c_{n-1}$ are preserved, i.e., the orientation of $f(c_0), f(c_1), \cdots, f(c_{n-1})$ is still clockwise. That is, s represents $(f(a), f(b))$. If f is isotopic to the reflection mr, then s^{-1} represents $(mr(a), mr(b))$ by Prop. 9. Therefore, it also represents $(f(a), f(b))$. □

From the above results, we know

Theorem 1. *For each atomic topological relation $\alpha \neq$ **EQ** on convex regions, there exists a circular string s such that, for any two convex regions a, b, $(a, b) \in \alpha$ iff either s or s^{-1} represents (a, b).*

Clearly, if s and s' are two circular strings that satisfy the above property, then s' is either s or s^{-1}. In this sense, we say each atomic topological relation has a unique representation circular string.

We next characterize when a string is valid. By Prop. 7, the empty string and all strings with length 1 are valid.

Proposition 12. *Suppose $s = (s_1 s_2 \cdots s_k)$ $(k \geq 2)$ is a circular string over $\{\mathsf{u}, \mathsf{v}, \mathsf{x}, \mathsf{y}\}$. Then s represents some atomic topological relation iff characters from $\{\mathsf{u}, \mathsf{v}\}$ and characters from $\{\mathsf{x}, \mathsf{y}\}$ appear in turn in s.*

Proof. The 'only if' part lies in that the boundary of $a \cap b$ is connected and that type u components and type v components are open sets, but type x components and type y components are all closed sets. For the 'if' part, we construct

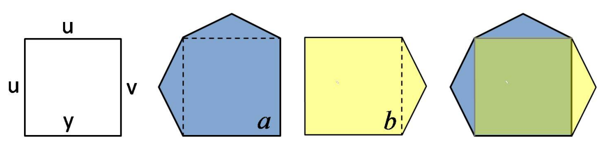

Figure 3: Constructing a configuration for $s =$ (uxuxvy)

polygons a and b such that s represents (a, b). Suppose n is the total number of characters u, v, and y in s. We construct a regular n-polygon circumscribed in the unit circle. The regular polygon represents the intersection of a, b, while

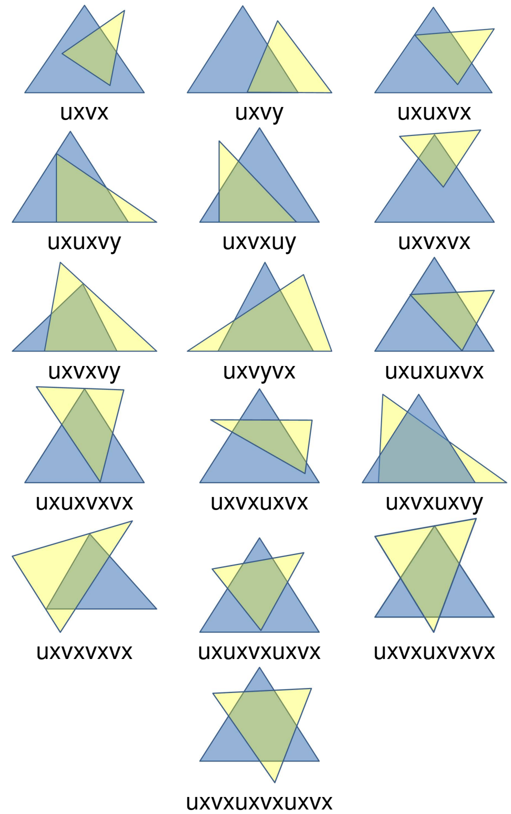

Table 1: Triangle Topological Relations: **PO**

its n edges correspond to the n non-x characters in s in order. Suppose AB is an edge that corresponds to u (or v). Note AB is a chord in the unit circle. Let C be the middle point of the arc AB. We extend the regular polygon to a (or b) by adding the equilateral triangle ABC. These triangles are still circumscribed in the unit circle, and both a and b are convex polygons (see Figure 3). It is straightforward to check that s represents (a, b). □

Note that the last character and the first character in a circular string are regarded as consecutive. This implies the length of a valid string is 1 or $2l$ for $l \geq 0$.

Topological Relations between Triangles

Applying our method to triangles, we obtain a complete classification of topological relations over triangles. These include 1 **DC** relation, 1 **EQ** relation, 1 **NTPP** relation, 2 **EC** relations, 5 **TPP** relations, and 16 **PO** relations. Illustrations for **PO**, **TPP**, and **EC** relations are given in Tables 1, 2, and 3, respectively.

Algorithm for Convex Polygons

We first give an estimation of the length of the representation string.

Proposition 13. *If $a \neq b$ are convex polygons with m and n vertices, respectively, then the circular string s that represents (a, b) has at most $2(m + n)$ characters.*

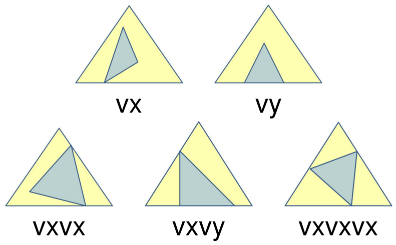

Table 2: Triangle Topological Relations: **TPP**

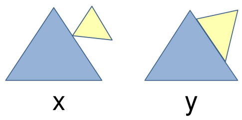

Table 3: Triangle Topological Relations: **EC**

Proof. If $a \cap b$ is not a polygon, then by Prop. 7 s is a string of length 0 or 1. Suppose $a \cap b$ is a polygon. Note that each character u (v, resp.) in s corresponds to exclusively one or several consecutive edges of a (b, resp.), and different u's (v's, resp.) can not correspond to the same edge of a (b, resp.). So the number of u's and v's in s is at most $m+n$. By Prop. 12, we know s has at most $2(m+n)$ characters. $\square$

The intersection $a \cap b$ of two convex polygons a, b can be computed in linear time (Chazelle and Dobkin 1987). If $a \cap b$ is empty, we know $(a, b) \in \mathbf{DC}$; if $a \cap b$ is nonempty but contains no interior point, then $(a, b) \in \mathbf{EC}$ and (x) or (y) represents (a, b) according to whether $a \cap b$ is a singleton.

A linear algorithm to compute the topological relation of two convex polygons is given in Algorithm 1, where we *assume the intersection of two convex polygons is also a polygon*. The function $T : \mathbb{R} \to \{\mathsf{u}, \mathsf{v}, \mathsf{x}\}$ used in Algorithm 1 is defined as $T(x) = \mathsf{u}$ if $x < 0$, and $T(x) = \mathsf{v}$ if $x > 0$, and $T(x) = \mathsf{x}$ if $x = 0$.

Remark 1. When calculating i_k and j_k, if $\max \beta_j < \alpha_k < 2\pi$, we add 2π to $\min \beta_j$; if $0 \leq \alpha_k < \min \beta_j$, we subtract 2π from $\max \beta_j$; if $\alpha_m < \max \beta_j < 2\pi$, we add 2π to $\alpha_1 = 0$. When merging α_i and β_j, it may happen that $\alpha_i = \beta_j$ for some i, j. In this case, $\delta_i = \theta_j$. Only one of them is kept. This is because, when $P_i = Q_j$, it will generate xx in the first version of s, and introduce an incorrect y in the final string. When post-processing the string, the last and the first characters of a circular string are regarded as consecutive.

Remark 2. A common interior point O can be found in $O(\log(m+n))$ time (Chazelle and Dobkin 1987). The procedure of computing j_k and i_k can be completed in $O(m+n)$ time. This is because $\alpha_1, \cdots, \alpha_m$ is ordered and $\beta_1, \cdots, \beta_n$ is a circular rotation of some ordered sequence (i.e., $\beta_k < \beta_{k+1} < \cdots < \beta_n < \beta_1 < \cdots < \beta_{k-1}$). It is also clear that δ_k and θ_k can be computed in $O(m+n)$ time.

The length of the first version of s (which is generated after merging α_i and β_j) is no more than $m+n$. So it is not hard to prove that post-processing s needs $O(m+n)$ time.

In conclusion, the time complexity of the algorithm is $O(m+n)$.

Algorithm 1 COMPUTING THE TOPOLOGICAL RELATION OF TWO CONVEX POLYGONS

Require: Vertices of two convex polygons, clockwise, $P_1, P_2, \cdots, P_m$ and $Q_1, Q_2, \cdots, Q_n$.
Ensure: The circular string that represents the topological relation of the two polygons.
  $O \leftarrow$ an interior point of both polygons;
  $\alpha_i \leftarrow \angle P_1 O P_i$;
  $\beta_j \leftarrow \angle P_1 O Q_j$;
  For each α_k, find j_k such that $\beta_{j_k} \leq \alpha_k < \beta_{j_k+1}$;
  For each β_k, find i_k such that $\alpha_{i_k} \leq \beta_k < \alpha_{i_k+1}$;
  for $k = 1, 2, \cdots, m$ **do**
    $P'_k \leftarrow$ the intersection of ray OP_k and $Q_{j_k} Q_{j_k+1}$;
    $\delta_k \leftarrow T(|OP'_k| - |OP_k|)$;
  for $k = 1, 2, \cdots, n$ **do**
    $Q'_k \leftarrow$ the intersection of ray OQ_k and $P_{i_k} P_{i_k+1}$;
    $\theta_k \leftarrow T(|OQ_k| - |OQ'_k|)$;
  Merge sort $\{\alpha_i\}$ and $\{\beta_j\}$, meanwhile compose the corresponding δ_i and θ_j into a circular string s;
  Replace consecutive u's (v's) in s with one u (v);
  Replace consecutive x's in s with one y;
  Insert an x between each pair of neighboring u (or v) in s;
  Output s.

We give an example to illustrate the idea of the algorithm.

Example 1. *Consider the configuration in Fig. 4. Notice P'_2, the intersection of OP_2 and Q_2Q_3, is in polygon $P_1P_2 \cdots P_7$ (i.e., $|OP'_2| < |OP_2|$). Therefore, $\delta_2 =$ u. Similarly, we have $\delta_1 = \delta_2 = \delta_5 = \delta_6 = \delta_7 =$ u, $\delta_3 =$ v, $\delta_4 =$ x, and $\theta_1 = \theta_4 = \theta_5 =$ v, $\theta_2 = \theta_3 = \theta_6 =$ x. As $\alpha_1 < \alpha_2 < \beta_3 < \alpha_3 = \beta_4 < \beta_5 < \alpha_4 < \beta_6 < \alpha_5 < \beta_1 < \alpha_6 < \alpha_7 < \beta_2$, δ_i and θ_j are combined into string (uuxvvxxuvuux). Note that as $\alpha_3 = \beta_4$, only one of δ_3 and θ_4 is adopted in the string . After post-processing, we get the representation string (uxvyuxvxux).*

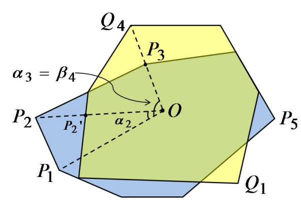

Figure 4: Illustration of Algorithm 1

Compare Two Configurations

Given two pairs of convex polygons (a, b) and (a', b'), we show how to determine if they are topological equivalent. First, we use Algorithm 1 to compute their circular strings s and s'. If the length of s and s' are different, then the two pairs are topologically different. Otherwise, based on a theorem of (Gusfield 1997), we can decide in linear time if s' is the circular rotation of s or s^{-1}, and hence, decide in linear time whether(a, b) and (a', b') are topologically equivalent.

Related Work and Further Discussions

(Egenhofer and Franzosa 1995) refined the well-known 4-Intersection Method with further topological invariants of the boundary-boundary intersection, including "the dimension of the components, their types (touching, crossing, and different refinements of crossings), their relationships with respect to the exterior neighborhoods, and the sequence of the components." These invariants are claimed to completely characterize two simple regions up to homeomorphism (the result is stated without proof).

Because of the arbitrary shape of simple regions, the representation given in (Egenhofer and Franzosa 1995) is quite complicated. When only convex regions are concerned, lots of the arguments in the model are redundant. In our paper, *the boundary of the intersection* of two regions, instead of *the intersection of the boundaries* of two regions, are used for representing topological relations. We give rigorous proof to justify the completeness of our classification. Moreover, a uniform string representation and computational method are presented in our paper.

(Papadimitriou, Suciu, and Vianu 1999) proved that the topological properties of semi-algebraic spatial regions can be completely specified by using, roughly speaking, the embedded planar graph of the region boundaries. This explains why it suffices to characterize the topological relation of two convex regions by encoding the boundary of the intersection of the two convex regions. They also showed that any spatial configuration of semi-algebraic regions can be represented simply as polygonal regions. This is consistent with what we have seen in the proof of Prop. 12, where a polygonal instance is constructed for each atomic topological relation.

(Benedikt et al. 2006) characterized the topological properties of planar datasets expressible in the relational calculus with real polynomial constraints. They used the notion of isotopy to formalize the concept that two datasets A and B are topologically the same. As noted earlier, isotopy is a little finer than the usual notion of topological equivalent. Suppose α is an atomic topological relation determined by a circular string s. It is possible to make isotopic distinctions for instances of α. Actually, write α^+ and, respectively, α^- for the sets of instances of α that are represented by s and s^{-1}. In case s^{-1} is a circular rotation of s, $\alpha^+ = \alpha^- = \alpha$. Otherwise, α^+ and α^- are disjoint and their union is α.

Convexity has been studied by several researchers in QSR (see e.g. (Cohn 1995; Davis, Gotts, and Cohn 1999; Pratt 1999)). But the topological relations between convex objects have not been well studied before. (Galton 1998) develops a system for representing overlap relations by counting components. His system is incomplete in the sense that two topologically different configurations may be classified as the same relation.

Conclusion

In this paper we began with a clear formulation of what is *the* topological relation between two regions, and then gave a uniform string representation for topological relations between convex regions. We associated each atomic topological relation with a (unique in a sense) circular string over $\{\mathsf{u}, \mathsf{v}, \mathsf{x}, \mathsf{y}\}$, and characterized when a circular string is valid. For two convex polygons, we gave a linear algorithm to compute the representation circular string. Based upon this result, we can decide in linear time whether two pairs of convex regions are topologically equivalent.

This computational relation model provides the complete topological information for convex regions. The model could be extended to represent information of more general spatial objects via the convex hull operation. The next important step will be defining a metric to measure the similarity of two arbitrary atomic topological relations, which can be used in clustering spatial relations and image retrieval. Future work will also consider the topological relation between 3-dimensional convex objects.

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