On the Approximation Ratio of Optimal Fixed-Price Mechanisms for Single and Multi-Unit Bilateral Trade

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Abstract

Multi-unit bilateral trade refers to the setting, where there is a buyer and a seller, who holds a finite number of units of an indivisible item. An automated mechanism has to decide how many units are transferred from the seller to the buyer and the corresponding payment from the buyer to the seller. The buyer and the seller have both either increasing or increasing submodular valuation functions in the number of units in possession. The (single-unit) bilateral trade problem arises as a particular case.

We study the problem of social welfare maximisation by establishing the fraction (*approximation ratio*) of the optimal social welfare that a fixed-price mechanism can recover. Fixed-price mechanisms, understood as per-unit price in the multi-unit setting, have been characterised as the only truthful, individually rational and strongly budget balanced mechanisms. We narrow the gap on the approximation ratio of optimal fixed-price mechanisms for bilateral trade, which has been shown to lie between 0.72 and 0.7381. We show that it must lie between 0.7292 and 0.73805, which leads to improved bounds on the approximation ratio of optimal fixed-price mechanisms for multi-unit bilateral trade. In particular, we show that multi-unit bilateral trade is at least as hard as single-unit bilateral trade, and obtain several hardness results for different numbers of units.

Code —
https://github.com/giambartolomei/SingleMultiUnitBT.

1 Introduction

Bilateral Trade (BT) is a fundamental problem in mechanism design for two-sided markets: there is a buyer and a seller, who holds a single indivisible item. An automated mechanism has to decide whether the item is transferred from the seller to the buyer and the corresponding payment from the buyer to the seller. Both the buyer and the seller have a private (nonnegative) valuation for the item. We focus on the *Bayesian* setting, where the mechanism designer has access to public distributions these otherwise private valuations are drawn from. This is also known as *full prior information* setting, and serves as a model of the mechanism designer's belief about the buyer's and seller's valuations for the item. The goal of the mechanism designer is maximising *social welfare*, which is defined as the total utility of the buyer and the seller combined.

If S is the valuation of the seller and B is the valuation of the buyer for the item, the (expected) optimal social welfare is simply

$$\mathrm{OPT}(S,B) = \mathbf{E}\left[\max\{S,B\}\right] = \mathbf{E}\left[S + (B-S)\mathbb{1}_{\{S\leq B\}}\right].$$

In order to enable trade between the buyer and the seller in a satisfactory fashion, three fundamental properties must hold for the designed market platform: it must be *Dominant Strategy Incentive Compatible (DSIC)*, that is, in game-theoretic terms, it must be a dominant strategy for both the buyer and the seller to report truthfully their valuation for the item; *Individually Rational (IR)*, that is, whenever trade is enacted both the buyer and the seller have positive utility; *(strongly) Budget Balanced (BB)*, that is all monetary transfers are between the buyer and the seller only. From a seminal result due to Myerson and Satterthwaite (1983), it follows that no such mechanism can achieve optimal social welfare. Further, the only DSIC, IR, BB mechanisms for BT are *fixed-price mechanisms* (Hagerty and Rogerson 1987). Thus in mechanism design for BT we focus on mechanisms of the following form: given the information gained from the public distributions of the valuations S and B, the mechanism designer chooses a price p, and proposes it to the buyer and the seller. The trade happens if $S \leq p \leq B$. The (expected) social welfare obtained by this mechanism is therefore

$$\mathrm{SW}(S,B,p) = \mathbf{E}\left[S + (B-S)\mathbb{1}_{\{S\leq p\leq B\}}\right]$$

and since such mechanisms are suboptimal, it is crucial to know how well the best fixed-price mechanism performs, compared with the optimal benchmark aforementioned. This is investigated through the notion of tight (worst-case) *approximation ratio*, which is defined, if we denote as μ,ν the distribution laws of the seller's and the buyer's valuations respectively, as

$$r^* = \inf_{S\sim\mu, B\sim\nu} \sup_{p} \frac{\mathrm{SW}(S,B,p)}{\mathrm{OPT}(S,B)}.$$

The state of the art regarding bounds on this quantity is $0.72 \leq r^* \leq 0.7381$ (Cai and Wu 2023, Theorem 3.1).

In this work we consider a natural generalisation of the single-item setting just introduced, where the seller has multiple units of an indivisible item for sale. In this case both the seller and the buyer have valuations depending on the number of units in possession, which are naturally modelled as *valuation functions*, again, drawn from public distributions. Since it is naturally desirable to have more units, valuations functions are assumed *increasing*. On the other hand, a standard phenomenon in several economic settings is that the increase in valuation decreases as the amount in possession increases, a property known as *submodularity*. In this work we will focus exclusively on increasing submodular valuations. This generalisation of the single-item setting is known as multi-unit BT, and again, the only DSIC, IR, BB mechanisms are *(sequential) fixed-price mechanisms*: the mechanism designer chooses a fixed per-unit price p, and proposes it to the buyer and the seller repeatedly, a single-unit at a time, until either the buyer or the seller declines to trade because it no longer increases social welfare (Gerstgrasser et al. 2019, Theorem 3.1).

If k is the number of units and $w = (w(q))_{q\in[k]}$ and $v = (v(q))_{q\in[k]}$ are the valuations functions of the seller and the buyer,[1] drawn from public distributions μ and ν respectively, we call $\mathcal{I} = (\mu, \nu)$ an instance. We always impose that $w(0) = v(0) = 0$. The increasing submodularity of, say, v, states that for all $q < q'$, $v(q) < v(q')$ and $v(q) - v(q-1) \geq v(q') - v(q'-1)$. Useful quantities in the study of this model are $\hat{v}(q) := v(q) - v(q-1)$, the *marginal increase function* of the buyer, and $\check{w}(q) := w(k-q+1) - w(k-q)$, the *marginal decrease function* for the seller. The increase in social welfare as a result of trading q units as opposed to $q-1$ units is $\hat{v}(q) - \check{w}(q)$. Then the social welfare is maximised by trading the maximum number of units q such that $\hat{v}(q) > \check{w}(q)$, and this leads to a useful expression of the optimal welfare:

$$\text{OPT}(w, v) = \sum_{q=1}^{k} \mathbf{E}\left[\max\{\check{w}(q), \hat{v}(q)\}\right] \tag{1.1}$$

Under a fixed-price mechanism, the highest quantity that the buyer would trade at unit-price p is $\max\{q : \hat{v}(q) \geq p\}$, and this leads to the following expression of the social welfare of a fixed-price mechanism,

$$\text{SW}(w, v, p) = \sum_{q=1}^{k} \mathbf{E}\left[\check{w}(q) + (\hat{v}(q) - \check{w}(q))\mathbb{1}_{\{\hat{v}(q)\geq p\geq \check{w}(q)\}}\right] \tag{1.2}$$

For $k = 1$, we recover the single-unit BT by setting $S = \check{w}(1) = w(1)$ and $B = \hat{v}(1) = v(1)$. Note that (1.1) and (1.2) express OPT(w, v) and SW(w, v, p) through the marginal decrements and increments of w and v respectively. In terms of total utilities of the buyer and the seller combined, for OPT(w, v), this formulation is equivalent to the expected value of $\max\{v(q) + w(k-q) \ : \ q \in [k] \cup \{0\}\}$, and for SW$(w, v, p)$ this formulation is equivalent to the expected value of $v(q) + w(k-q)$, where q is the number of units that the seller and buyer "agree to trade" under the fixed per-unit price p. For further details on the derivation of (1.1) and (1.2) see Gerstgrasser et al. (2019, §4), where the state-of-the-art $1 - 1/e$-approximation is also obtained. To the best of our knowledge, no nontrivial upper bounds are known for multi-unit BT.

Our contributions. Our results focus on improving the bounds on r^* for both single and multi-unit BT, by refining and generalising the mathematical programming approach of Cai and Wu (2023).[2] We start with our improved bounds for single-unit BT.

Theorem 1.1. *There exists a fixed-price mechanism $\mathbb{M}$ with approximation ratio greater or equal to 0.7292. Further, there exists an instance $\mathcal{I}$ such that no fixed-price mechanism can achieve an approximation ratio better than 0.73805.*

The mechanism $\mathbb{M}$ and the hard instance $\mathcal{I}$ are determined explicitly.

The general form of the mechanism above naturally extends to multi-unit BT and leads to the following improvement on the state-of-the-art $1 - 1/e$-approximation (Gerstgrasser et al. 2019).

Theorem 1.2. *For $k = 2$ units, there exists a fixed-price mechanism $\mathbb{M}$ with approximation ratio greater or equal to 0.65.*

As a corollary, the methods deployed in proving Theorem 1.2 yield a $0.749, 0.76$-appproximation for symmetric (i.e. *iid*) $k = 2, 3$-units BT respectively. We conjecture that the mechanism $\mathbb{M}$ of Theorem 1.2 is also a 0.65-approximation for $k = 3$ units.

An upper bound on the approximation ratio is known as a *hardness* result. When the hardness of a model is strictly less than the approximation ratio possible for another model, we say that the two models are separated. For ease of the reader, our next result is given here in a slightly informal fashion (the formal statement can be found in Section 6).

Theorem 1.3 (Informal version)**.** *Multi-unit BT is at least as hard as single-unit BT. Further, for $2 \leq k \leq 8$ multi-unit BT is respectively 0.7291, 0.7277, 0.7284, 0.7276, 0.7272, 0.7286, 0.7278-hard.*

To be more precise, the first half of the claim means that the hardness of single-unit BT is greater or equal to that of multi-unit BT, for every number of units k. The second part of the claim provides several hardness results for multi-unit BT as the number of units k varies. The corresponding hard instances are determined explicitly, and may not be tight. As a corollary, the methods deployed in proving Theorem 1.3 yield a $0.8372, 0.8483, 0.8482$-hardness for symmetric $k = 2, 3, 4$-units BT respectively. Finally, note that all hardnesses in Theorem 1.3 are strictly less than the approximation ratio of $\mathbb{M}$ for single-unit BT in Theorem 1.1. We conjecture that multi-unit BT is separated from single-unit BT, with the latter beating the former.

[1] $[k] := \{1, \ldots, k\}$.

[2] Our bounds have been found through solving the stated optimisation problems using Gurobi, via the code that we made available through the URL provided on this article's title page. Our computations were partially executed on CREATE Systems of King's College London (King's College London 2022).

Outline. In Section 3 we give formal details about our setup; in Section 4 about Theorem 1.1; in Section 5 about Theorem 1.2; in Section 6 about Theorem 1.3. In Section 7 we discuss promising future directions. Full proofs are omitted due to space constraints, and will be made available in an extended version of this manuscript.

2 Related Literature

Single-unit BT has been studied extensively in algorithmic mechanism design and economics. The basic bilateral trade scenario was first considered by Chatterjee and Samuelson (1983) in the context of bargaining. Myerson and Satterthwaite (1983) proved the celebrated impossibility theorem that states that no Bayesian incentive compatible, weakly budget balanced, and interim individually rational mechanism can be ex-post Pareto efficient, and can thus not always maximise social welfare. Hagerty and Rogerson (1987) were the first to characterise the class of mechanisms for this setting that satisfy dominant strategy incentive compatibility (DSIC), interim individual rationality (IR), and (strong) budget balance (BB): they are the *fixed-price* mechanisms.

Bounds on the social welfare approximation ratio of single-unit bilateral trade. Those results motivate the study of the first-best to second-best approximation ratio for the class of DSIC, IR, BB mechanisms: what is the best fixed price that one can set, and how good is the resulting social welfare of such an optimal fixed-price mechanism (i.e., the "*second-best*") compared to the social welfare attained by the ideal but unachievable scenario, where trade always occurs whenever the buyer's valuation exceeds the sellers (i.e., the "*first-best*"). The precise value r^* of this ratio has been iteratively refined over the course of many, relatively recent, studies: McAfee (2008) proposed setting the price to any value in between the buyer and seller distribution's median, for the case where the buyer's median exceeds the seller's, and showed this to be a $1/2$-approximation to the gain from trade. Blumrosen and Dobzinski (2014) showed that this is also a $1/2$-approximation to the social welfare if one specialises this pricing scheme to pick the seller's median as the price (yielding the *Median Mechanism*), and in the same paper they proposed a better mechanism that achieves an approximation factor of $28/55 \approx 0.5091$, and proved that the best achievable approximation factor r^* is at most 0.8904. Subsequently, the lower and upper bounds on r^* were improved to $13/25 \approx 0.52$ and 0.7485 by Colini-Baldeschi et al. (2016). The lower bound was then significantly improved by Blumrosen and Dobzinski (2021) through the surprisingly simple *Random Quantile Mechanism* which was shown to satisfy an approximation ratio of $1 - 1/e \approx 0.6321$. Through an involved analysis of the problem, Kang, Pernice, and Vondrák (2022) showed that this could be improved to $r^* > 0.6322$, where additionally Kang and Vondrák (2018) proved that $r^* < 0.7385$. This was followed by a significant improvement to both the lower and upper bounds by Liu, Ren, and Wang (2023), and independently Cai and Wu (2023): a lower bound of 0.71 was established by Liu, Ren, and Wang (2023) through a dynamic programming approach, whereas Cai and Wu (2023) establish a lower bound of 0.72 through the analysis of a quadratic programming formulation of the problem. Furthermore, both works prove an upper bound of 0.7381 on r^*. This sequence of improvements thus yields altogether that $r^* \in [0.72, 0.7381]$.

The multi-unit extension. The multi-unit extension of the basic single-unit bilateral trade setting was introduced by Gerstgrasser et al. (2019), who showed that the aforementioned Median Mechanism and the Random Quantile Mechanism can be generalised to yield mechanisms that achieve approximation ratios of $1/2$ and $1 - 1/e$ respectively, when the valuation functions are increasing submodular. They furthermore characterise the class of DSIC, IR, and SBB mechanisms for multi-unit bilateral trade under both general increasing and increasing submodular valuation functions. A variant of multi-unit bilateral trade that strongly resembles the model of Gerstgrasser et al. (2019) has been developed in the economics literature by Loertscher and Marx (2023), where the authors consider exponential valuation functions and are interested in the behaviour of the approximation ratio r^*_{GFT} that is defined in terms of *gain from trade* (as opposed to social welfare, which is what we focus on), as a function of a base parameter by which the valuations are defined.

Results on variants of the approximation ratio. For single-unit bilateral trade, the gain from trade approximation ratio was also considered by Blumrosen and Mizrahi (2016) who proved for the basic single-unit case that there exists a mechanism that achieves a gain-from-trade approximation ratio of $1/e$ under the assumption that the buyer's distribution satisfies a *monotone hazard rate* condition, although this result requires one to relax the incentive compatibility condition imposed on the mechanism to the more permissive notion of *Bayes-Nash incentive compatibility (BNIC)*. Simultaneously, the authors provide $2/e$ as an upper bound to $r^*_{\text{GFT,BNIC}}$, and furthermore show that $r^*_{\text{SW,BNIC}} \leq 0.93$, i.e., no BNIC mechanism can achieve a ratio better than 0.93 with respect to the social welfare. The gain-from-trade for the basic single-unit case approximation ratio was furthermore studied by Colini-Baldeschi et al. (2017), where the authors asymptotically characterised the gain-from-trade approximation ratio $r^*_{\text{GFT,DSIC}}$ as $\Theta(\log(\rho))$, where ρ is the reciprocal of the probability that the buyer's valuation exceeds the seller's valuation. This result was generalised considerably by Cai et al. (2021), who show that the same asymptotic bound holds for a multi-unit setting with multiple sellers, among other results for this generalisation.

The *random offerer mechanism* is a specific choice of BNIC, IR, SBB mechanism, for which the gain-from-trade approximation factor has been analysed in recent literature. Its definition stems from Brustle et al. (2017), who proved that this mechanism achieves a gain from trade approximation of at least $1/2 r^*_{\text{GFT,BNIC}}$, after which it was shown by Deng et al. (2022) that its gain from trade approximation ratio is at least 0.1215. The analysis was subsequently improved by Fei (2022) to 0.3175, which yields a lower bound on $r^*_{\text{GFT,BNIC}}$ that is not conditional on the aforementioned monotone hazard rate property. It was shown by Babaioff, Dobzinski, and Kupfer (2021) that the random offerer mechanisms gain from trade approximation is not better than 0.495.

Generalisations with more than two agents. The bilateral trade setting naturally generalises to a two-sided market setting with multiple buyers and sellers. A relatively simple extension to two buyers was considered by Babaioff, Frey, and N. Nisan (2024), where the authors provide an impossibility result for correlated valuations and a learning algorithm for independent valuations. The *two-sided market* or *double auction* setting refers to the generalisation with an arbitrary number of buyers and sellers, and mechanism design for two-sided markets has been studied extensively under varying levels of restrictions (Colini-Baldeschi et al. 2020, 2016; Brustle et al. 2017; Babaioff et al. 2018; Babaioff, Goldner, and Gonczarowski 2020; Dütting, Roughgarden, and Talgam-Cohen 2014). *Combinatorial exchanges* form a further generalisation of two-sided markets, in which each agent can act as a buyer and seller simultaneously. Such exchanges have been modeled under various contexts throughout the economics and computer science literature. Blumrosen and Dobzinski (2023) provide a result in the spirit of our principal framework, which is a DSIC, IR, BB combinatorial exchange mechanism that achieves a $O(\log t)$-approximation when all agents valuations are sub-additive, where t is the maximum number of items held by any agent.

The approximation ratio under limited information. Certain work in the area focuses on variants of the problem where there is limited information available on the valuation distributions. Deng et al. (2025) consider a setting where pricing is delegated to the buyer and seller, who have only sample access to the distributions. Dütting et al. (2021) show that a $1/2$-approximation to the optimal social welfare can be achieved through taking a single sample from the seller's distribution, and Kang, Pernice, and Vondrák (2022) show that this ratio is $3/4$ when the two distributions are equal. Cai and Wu (2023) characterise the achievable approximation ratio when there is more than one sample, and furthermore show that the approximation ratio is at least $2/3$ if only the mean of one of the agents is known.

3 Preliminaries

Notation. We will use the notation $\mathbb{R}_+$ for the positive reals $(0,\infty)$, and we use $\mathbb{R}_0$ for the non-negative reals, $[0,\infty)$. For an integer $a \in \mathbb{N}$ we write $[a]$ to denote the set $\{i \in \mathbb{N} : i \leq a\}$. Recalling the characterisation of Gerstgrasser et al. (2019) described in Section 1, the tight (worst-case) approximation ratio for the best fixed-price mechanism in the k-unit bilateral trade problem, which we always assume with increasing submodular valuations on the nonnegative reals, can be expressed as

$$r_k^* := \inf_{\mathcal{I}} \sup_{p\in\mathbb{R}_0} \frac{\text{SW}(\mathcal{I},p)}{\text{OPT}(\mathcal{I})} \tag{3.1}$$

$$\text{SW}(\mathcal{I},p) := \sum_{q=1}^{k} \mathbf{E}\left[\check{w}(q) + (\hat{v}(q) - \check{w}(q))\mathbb{1}_{\{\hat{v}(q)\geq p\geq \check{w}(q)\}}\right] \tag{3.2}$$

$$\text{OPT}(\mathcal{I}) := \sum_{q=1}^{k} \mathbf{E}\left[\max\{\check{w}(q),\hat{v}(q)\}\right]. \tag{3.3}$$

The expectations run jointly over the randomness of $w \sim \mu$ for the seller and $v \sim \nu$ for the buyer, which together make up the instance $\mathcal{I} = (\mu,\nu)$, where μ,ν are integrable independent probability laws on the increasing submodular functions. Occasionally, we also write more informally $\mathcal{I} = (w, v)$, as we did in Section 1. Recall that we always assume $v(0) = w(0) = 0$, the valuations v, w both have the set of tradable quantities $[k] \cup \{0\}$ as their domain, and are positive (strictly) increasing in their argument. The *marginal increase function* $\hat{v} : [k] \to \mathbb{R}_+$ of a submodular valuation v for the buyer is defined as $\hat{v}(q) = v(q) - v(q-1)$ and is nonincreasing; the *marginal decrease function* $\check{w} : [k] \to \mathbb{R}_+$ of a submodular valuation w for the seller is defined as $\check{w}(q) = w(k-q+1) - w(k-q)$ and is nondecreasing. Through defining a submodular function via its marginal increases, we obtain that the class of increasing submodular functions are isomorphic to the set of componentwise nonincreasing vectors of $\mathbb{R}_+^k$. We will thus make use of marginal decrements for the seller and marginal increments for the buyer, denoted correspondingly $\check{w}$ and $\hat{v}$, where we use $\check{\mu}$ and $\hat{\nu}$ to denote probability laws over marginal decrease functions and marginal increase functions respectively. To stress the role of the marginal increments in our expressions, we sometimes switch to the notation $\tilde{\mathcal{I}} = (\check{\mu},\hat{\nu})$. When $\tilde{\mathcal{I}}$ (or $\mathcal{I}$) appear without additional qualification, as in (3.1), it is understood that we refer to all instances with increasing submodular valuations. In more explicit formulations we denote this class as $\mathcal{S}_+^k$. We stress that the overloaded notation $\mathcal{S}_+^k$ refers to all distribution pairs over increasing submodular functions, the public knowledge, not to the realised functions themselves, which are private knowledge.

Mathematical programs. The tight approximation ratio for multi-unit BT is easily expressed as a mathematical program. Consider the following semi-infinite program:

$$\begin{aligned} \inf \quad & r && \\ \text{s.t.} \quad & \text{SW}(\tilde{\mathcal{I}},p) \leq r && \forall p \in \mathbb{R}_0 \\ & \text{OPT}(\tilde{\mathcal{I}}) \geq 1 && \\ & 0 \leq r \leq 1, \quad \tilde{\mathcal{I}} \in \mathcal{S}_+^k. && \end{aligned} \tag{3.4}$$

Then the following holds.

Proposition 3.1. *The optimal value of* (3.4) *equals* r_k^*.

There is no loss of generality in considering increasing valuations alone, due to the following.

Proposition 3.2. *The closure, with respect to convergence in distribution, of the space of probability laws over a Borel-measurable subset* $\Omega \subseteq \mathbb{R}_+^k$ *is the space of probability laws over the closure of* Ω. *Formally,* $\Delta(\overline{\Omega}) = \overline{\Delta(\Omega)}$.

Taking Ω to be the the set of componentwise nonincreasing vectors of $\mathbb{R}_+^k$ yields that probability laws on nondecreasing submodular valuations are also captured by the *infimum* in our mathematical programs. In particular, since all hardness results known for the single-unit setting (see Section 2) have considerable mass at 0, they are only apparently excluded by the restriction to $k = 1$ of the multi-unit model as described above. By a standard limiting argument, it is easy to see that r_1^* still captures them. Therefore, for a more unified approach within the multi-unit framework, we equivalently study single-unit BT on $\mathcal{S}_+^1$, the space of distribution pairs $\mathcal{I} = (w(1), v(1))$, rather than in the more conventional framework initially introduced in Section 1.

4 New Bounds for Single-Unit Bilateral Trade

In this section we focus on single-unit BT and give the idea of the proof of Theorem 1.1, which improves on the bounds on r_1^* obtained by Cai and Wu (2023, § 3). In particular, we show that $r_1^* \in [0.7292, 0.73805]$. We leverage the mathematical programming framework adopted by Cai and Wu (2023), which we will further generalise in later sections.

Let $2 \leq n \in \mathbb{N}$ and fixed values $0 = p_1 < p_2 < \ldots < p_n$, which we call *basic prices*.[3] Let $\underline{r}_1$ be the optimal solution to the following program:

$$\begin{aligned} &\inf \quad r && (4.1)\\ &\text{s.t.} \quad \sum_{i=1}^{n} s_i p_i + \sum_{i=1}^{t-1}\sum_{j=t+1}^{n} s_i b_j (p_j - p_i) \leq r \quad \forall t \in [n]\\ &\sum_{i=1}^{n}\sum_{j=1}^{n} s_i b_j \max\{p_i, p_j\} \geq 1\\ &1 \leq \sum_{i=1}^{n} s_i, \sum_{j=1}^{n} b_j \leq 1 + \frac{1}{p_n}\\ &0 \leq r \leq 1, \quad s_i, b_j \geq 0 \qquad \forall i, j \in [n]. \end{aligned}$$

Definition 4.1. *Let $\mathbb{M}$ be the mechanism that uses, on instance $\mathcal{I}$, a fixed price*

$$\underline{p} \in \underset{p \in \{OPT(\mathcal{I}) p_t :\, t \in [n]\}}{\arg\max} SW(\mathcal{I}, p).$$

The social welfare obtained by $\mathbb{M}$ on instance $\mathcal{I}$ is

$$\mathrm{SW}(\mathcal{I}, \underline{p}) = \max\{\mathrm{SW}(\mathcal{I}, p) : \, p \in \{\mathrm{OPT}(\mathcal{I}) p_t : \, t \in [n]\}\}.$$

Clearly, since $\mathbb{M}$ is a fixed-price mechanism, it is DSIC, IR, and BB. Cai and Wu (2023, Lemma 3.2) establish that for every instance $\mathcal{I}$, it holds that

$$\mathrm{SW}(\mathcal{I}, \underline{p}) \geq \underline{r}_1 \, \mathrm{OPT}(\mathcal{I}),$$

and heuristically find a list of 16 basic prices such that the constraints region of (4.1) with $0 \leq r \leq 0.72$ is infeasible, which yields that $\underline{r}_1 \geq 0.72$ (Cai and Wu 2023, Theorem 3.1).

We provide a new list of basic prices similarly ensuring that $\underline{r}_1 \geq 0.7292$. This is similarly obtained via a Gurobi optimisation routine. Our new list of 32 basic prices, also obtained through heuristics,[4] is given in the formal statement below.

Proposition 4.1. *Given basic prices*

$$\{0, 0.12, 0.22, 0.28, 0.3, 0.32, 0.34, 0.36, 0.38, 0.4, 0.42, 0.44, 0.46, 0.48, 0.5, 0.52, 0.54, 0.56, 0.58, 0.6, 0.62, 0.64, 0.66, 0.68, 0.7, 0.72, 0.76, 0.78, 0.84, 0.98, 1.25, 1000\}$$

it holds that $\underline{r}_1 \geq 0.7292$. Therefore, defining $\mathbb{M}$ through the above basic prices, for every instance $\mathcal{I}$ it holds that $SW(\mathcal{I}, \underline{p}) \geq 0.7292\, OPT(\mathcal{I})$.

Let $2 \leq n \in \mathbb{N}$ and fixed values $0 = p_1 < p_2 < \ldots < p_n$, which we call *support points*.[5] Let $\overline{r}_1$ be the optimal solution to the following program:

$$\begin{aligned} &\inf \quad r && (4.2)\\ &\text{s.t.} \quad \sum_{i=1}^{n} s_i p_i + \sum_{i=1}^{t}\sum_{j=t+1}^{n} s_i b_j (p_j - p_i) \leq r \quad \forall t \in [n]\\ &\sum_{i=1}^{n}\sum_{j=1}^{n} s_i b_j \max\{p_i, p_j\} \geq 1\\ &\sum_{i=1}^{n} s_i = \sum_{j=1}^{n} b_j = 1\\ &0 \leq r \leq 1, \quad s_i, b_j \geq 0 \qquad \forall i, j \in [n]. \end{aligned}$$

Cai and Wu (2023, Lemma 3.3) establish that for any valid solution $((s_i), (b_j), r)$ of (4.2) and $\varepsilon > 0$ small enough to have

$$0 = p_1 < p_1 + \varepsilon < p_2 < p_2 + \varepsilon < \ldots < p_n < p_n + \varepsilon,$$

there exists an instance $\mathcal{I} = (w(1), v(1))$(with $w(1)$ supported on $\{p_i + \varepsilon\}$ with corresponding masses (s_i) and $v(1)$ on $\{p_j\}$ with corresponding masses (b_j)) such that for any price p,

$$\mathrm{SW}(\mathcal{I}, p) \leq (r + \varepsilon)\, \mathrm{OPT}(\mathcal{I}).$$

Finally, they heuristically find a list of 80 support points $\{p_t\}$ such that a valid solution to (4.2) $((s_i), (b_j), r)$ is obtained, with $r \approx 0.73810$. Hence $\overline{r}_1 \leq 0.3781$ and $\mathrm{SW}(\mathcal{I}, p) \leq 0.7381\, \mathrm{OPT}(\mathcal{I})$ for any price p. It should be noted that minor amendments are needed to make the hard instance numerically sound.

We provide a new list of support points ensuring the existence of a valid solution $((s_i), (b_j), r)$, with $r \approx 0.738049$, so that $\overline{r}_1 < 0.73805$. This is similarly obtained via a Gurobi optimisation routine. Our list of support points, also heuristically obtained, is given in the formal statement below.

Proposition 4.2. *Given support points*

$$p_i = \frac{29 + i}{100}, \quad 2 \leq i \leq 49, \quad p_{50} = 4000,$$

it holds that $\overline{r}_1 < 0.73805$. Hence there exists an instance $\mathcal{I}$ such that $SW(\mathcal{I}, p) \leq 0.73805\, OPT(\mathcal{I})$ for any price p.

Proof of Theorem 1.1. Propositions 4.1 and 4.2 provide respectively the first and second half of Theorem 1.1. □

[3] The terminology *basic* refers to the fact that they are precisely the prices used by our mechanism on instances having unitary optimal welfare. Equivalently, the prices for a normalised instance, so as to have unitary optimal welfare. *Basic* in the sense of *normal*.

[4] These heuristics are nontrivial, due to the long running times of QCQP solvers on large nonconvex programs.

[5] The terminology *support points* refers to these values providing the support for the hard instance of the buyer.

5 Fixed-Price Mechanisms for Multi-Unit Bilateral Trade

From now on, we fix $k \geq 2$ and focus solely on multi-unit BT. In this section we give the idea of the proof of Theorem 1.2, which improves on the bounds on r_k^*, obtained by Gerstgrasser et al. (2019), for $k = 2$, by generalising the framework introduced in Section 4. Due to the length of some expressions in the remaining mathematical programs, we omit "s.t." in the constraints region.

Let $2 \leq n \in \mathbb{N}$ and fix n basic prices $\{p_i\}$. For all $k \geq 2$ let $\underline{r}_k$ be the optimal solution to the following program:

$$\begin{aligned}
&\inf \quad r && (5.1)\\
&\sum_{q=1}^{k}\sum_{i=1}^{n} s_i(q)p_i + \sum_{i=1}^{t-1}\sum_{j=t+1}^{n} s_i(q)b_j(q)(p_j - p_i) \leq r && \forall t \in [n]\\
&\sum_{q=1}^{k}\sum_{i=1}^{n}\sum_{j=1}^{n} s_i(q)b_j(q)\max\{p_i, p_j\} \geq 1 &&\\
&\sum_{i=1}^{n} s_i(q)p_i \leq \sum_{i=1}^{n} s_i(q+1)p_i && \forall q \in [k-1]\\
&\sum_{j=1}^{n} b_j(q)p_j \geq \sum_{j=1}^{n} b_j(q+1)p_j && \forall q \in [k-1]\\
&\sum_{i=1}^{t} s_i(q) \geq \sum_{i=1}^{t-1} s_i(q+1) && \forall 2 \leq t \leq n-1, q \in [k-1]\\
&\sum_{j=1}^{t-1} b_j(q) \leq \sum_{j=1}^{t} b_j(q+1) && \forall 2 \leq t \leq n-1, q \in [k-1]\\
&1 \leq \sum_{i=1}^{n} s_i(q), \sum_{j=1}^{n} b_j(q) \leq 1 + \frac{1}{pn} && \forall q \in [k]\\
&\sum_{i=1}^{n-1} s_i(q), \sum_{j=1}^{n-1} b_j(q) \leq 1 && \forall q \in [k]\\
&0 \leq r \leq 1, \quad s_i(q), b_j(q) \geq 0 && \forall i, j \in [n], q \in [k].
\end{aligned}$$

We refer to the last three lines of the constraints region of (5.1) as the *relaxed probability mass function (pmf) constraints*, and to the third to sixth line as the *relaxed submodular constraints*. The idea is to think of $(s_i(q))_{i\in[n]}$ (and $(b_j(q))_{j\in[n]}$) not as probability mass functions, but almost: more precisely they can be thought of as arising from the marginal decrement (increment) of the respective submodular laws via mean-preserving discretisations. They would consequently have increasing (decreasing) "expectations" (third and fourth line). Further, since the discretisation used also keeps the resulting "cumulative distribution functions (cdf)" essentially in between the corresponding consecutive samples of the original cdf of the marginal decrements (increments), this provides a relaxed form of stochastic dominance for the resulting "cdf". Altogether, this provides a relaxed form of submodularity (see also Section 6 for further details on the relationship between submodularity and stochastic dominance).

Consider the same mechanism $\mathbb{M}$ of Definition 4.1. We establish the following generalisation of (Cai and Wu 2023, Lemma 3.2), by exploiting the fact that the relaxed submodular constraints arise from mean-preserving discretisations.

Proposition 5.1. *For every $k, n \geq 2$ and instance $\mathcal{I}$, it holds that SW$(\mathcal{I}, \underline{p}) \geq \underline{r}_k$ OPT$(\mathcal{I})$.*

For $k = 2$, we heuristically find a list of 39 basic prices such that the constraints region of (5.1) with $0 \leq r \leq 0.65$ is infeasible, thus obtaining the following.

Proposition 5.2. *Given basic prices*

$$\{0, 0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18, 0.2, 0.22, 0.24, 0.26, 0.28, 0.3, 0.32, 0.34, 0.36, 0.38, 0.4, 0.42, 0.46, 0.48, 0.5, 0.52, 0.54, 0.56, 0.58, 0.6, 0.64, 0.66, 0.7, 0.72, 0.82, 0.92, 0.98, 1.25, 1000\}$$

it holds that $\underline{r}_2 \geq 0.65$.

Proof of Theorem 1.2. Defining $\mathbb{M}$ as per Definition 4.1 through the basic prices of Proposition 5.2 and then exploiting Proposition 5.1 yields that for every instance $\mathcal{I}$ it holds that SW$(\mathcal{I}, \underline{p}) \geq 0.65\,OPT(\mathcal{I})$. ☐

We conclude by noting that restricting (5.1) to the "diagonal" $b_j(q) = s_j(k - q + 1)$ for all $j \in [n]$ and $q \in [k]$ yields lower bounds on symmetric multi-unit BT.

Corollary 5.1. *In symmetric multi-unit BT there exists a fixed-price mechanism $\mathbb{M}$ with approximation ratio greater or equal to $0.749, 0.76$ for $k = 2, 3$ respectively.*

The basic prices for $\mathbb{M}$ in Corollary 5.1 are defined as

$$p_i = \frac{i-1}{50}, \quad 2 \leq i \leq 50, \quad p_{51} = 1.25, \quad p_{52} = 1000.$$

6 Hardness of Multi-Unit Bilateral Trade

In this section we give formal statements and describe the key the idea of the proof of Theorem 1.3, which is obtained by generalising the framework introduced in Section 4. Recall that DSIC, IR and BB mechanisms for multi-unit BT have been characterised as sequential fixed-price mechanisms: a fixed price is set in advance of the trades, and then, one unit at a time, each trade takes place until, for some number of traded units $q-1$, either the marginal decrease function of the seller $\check{w}(q)$ climbs above the price or the marginal increase function of the buyer $\hat{v}(q)$ falls below the price. The iterated structural similarity with single-unit BT and the fact that the price cannot change from one iteration to the next, lead us to expect that r_k^* decreases as k increases. This is the intuition motivating Theorem 1.3.

Let $2 \leq n \in \mathbb{N}$ and fix n support points $\{p_i\}$. Let $\overline{r}_k$ be the optimal solution to the following program:

$$\begin{aligned}
&\inf \quad r && (6.1)\\
&\sum_{q=1}^{k}\sum_{i=1}^{n} s_i(q)p_i + \sum_{i=1}^{t}\sum_{j=t+1}^{n} s_i(q)b_j(q)(p_j - p_i) \leq r && \forall t \in [n]\\
&\sum_{q=1}^{k}\sum_{i=1}^{n}\sum_{j=1}^{n} s_i(q)b_j(q)\max\{p_i, p_j\} \geq 1 &&\\
&\sum_{i=1}^{t} s_i(q) \geq \sum_{i=1}^{t} s_i(q+1) && \forall t \in [n-1], q \in [k-1]
\end{aligned}$$

$$\sum_{j=1}^{t} b_j(q) \leq \sum_{j=1}^{t} b_j(q+1) \qquad \forall t \in [n-1], q \in [k-1]$$

$$\sum_{i=1}^{n} s_i(q) = \sum_{j=1}^{n} b_j(q) = 1 \qquad \forall q \in [k]$$

$$0 \leq r \leq 1, \quad s_i(q), b_j(q) \geq 0 \qquad \forall i, j \in [n], q \in [k].$$

We refer to the last two lines of the constraints region of (6.1), as *pmf constraints*, because they ensure that $(s_i(q))_{i\in[n]}$ and $(b_j(q))_{j\in[n]}$ form probability distributions for every $q \in [k]$. We refer to the third and fourth line as *dominance constraints*, since leveraging them through a *stochastic dominance* argument, we can construct from $(s_i(q))_{i\in[n],q\in[k]}$ and $(b_j(q))_{j\in[n],q\in[k]}$ appropriate probability distributions μ and ν over increasing submodular functions for the seller and the buyer respectively. Further, $\tilde{\mathcal{I}} = (\check{\mu}, \hat{\nu})$ is such that the best possible expected social welfare $\sup_p \mathrm{SW}(\tilde{\mathcal{I}}, p)$ and $\mathrm{OPT}(\tilde{\mathcal{I}})$ are related to the left-hand side of the first constraint and second constraint of (6.1) respectively, in such a way that the following holds.

Lemma 6.1. *For any valid solution* $((s_i(q)), (b_j(q)), r)$ *of* (6.1) *and* $\varepsilon > 0$ *small enough such that*

$$0 = p_1 < p_1 + \varepsilon/k < p_2 < p_2 + \varepsilon/k < \ldots < p_n < p_n + \varepsilon/k,$$

there exists an instance $\tilde{\mathcal{I}} = (\check{w}, \hat{v})$ *(with* $\check{w}(q)$ *supported on* $\{p_i + \varepsilon/k\}$ *with corresponding masses* $(s_i(q))$ *and* $\hat{v}(q)$ *on* $\{p_j\}$ *with corresponding masses* $(b_j(q))$ *for every* $q \in [k]$*) such that for any price* p*,* $SW(\tilde{\mathcal{I}}, p) \leq (r + \varepsilon)\, OPT(\tilde{\mathcal{I}})$*.*

For each number of units $2 \leq k \leq 8$ we found a list of support points ensuring the existence of a valid solution $((s_i(q)), (b_j(q)), r_k)$, with approximate values of r_k reported in Table 1, so that $\bar{r}_k < r_k$. These are obtained heuristically via a Gurobi optimisation routine. The hard instances are given in the formal statement below, which is a direct consequence of Lemma 6.1.

Proposition 6.1. *Given support points of generic form*

$$p_i = \frac{i + \alpha}{\beta}, \quad 2 \leq i \leq n-1, \quad p_n \gg p_{n-1},$$

with parameters set as in Table 1 for each $2 \leq k \leq 8$*, it holds that* $\bar{r}_k < r_k$*. Hence for every such* k *there exists an instance* $\tilde{\mathcal{I}}_k$ *such that* $SW(\tilde{\mathcal{I}}_k, p) \leq r_k\, OPT(\tilde{\mathcal{I}}_k)$ *for any price* p*.*

Further, consider identically distributed marginals for the seller and the buyer. Take the two distributions (each providing a list of identical marginals, one for the seller, one for

k	n	α	β	p_n	r_k
2	30	5	98	2500	0.7291
3	17	5	100	1000	0.7277
4	11	7	98.5	1000	0.7284
5	8	4	82.5	1000	0.7276
6	8	4	95	1000	0.7272
7	8	3.5	95	1000	0.7286
8	6	2.85	96	1000	0.7278

Table 1: Approximate values of r_k for each hard instance $\tilde{\mathcal{I}}_k$.

the buyer) to be the distribution of the seller and the buyer giving a certain hardness for single-unit BT. This set-up allows us to show that said hardness applies to multi-unit BT as well. This is the key idea behind the following result.

Proposition 6.2. *For every* $k \geq 2$*,* $r_k^* \leq r_1^*$*.*

Theorem 6.1 (Formal version of Theorem 1.3)**.** *For all* $k \geq 2$ *it holds that* $r_k^* \leq r_1^*$*. Further,* $r_k^* \leq r_k$ *for all* $2 \leq k \leq 8$*, with values of* r_k *given in Table 1.*

Proof. Propositions 6.1 and 6.2 provide respectively the first and second half of the claim. ☐

We conclude by noting that restricting (6.1) to the "diagonal" $b_j(q) = s_j(k - q + 1)$ for all $j \in [n]$ and $q \in [k]$ and replacing its first quadratic constraint with the constraint $\sum_{q=1}^{k} \sum_{i=1}^{n} s_i(q) p_i + \sum_{i=1}^{t} \sum_{j=t}^{n} s_i(q) b_j(q)(p_j - p_i) \leq r$ for all $t \in [n]$, yields upper bounds on symmetric multi-unit BT. This follows from reasoning as in Lemma 6.1 and Propositions 6.1 and 6.2, with the exception that symmetry does not allow exploiting a perturbed support for $\check{w}(q)$ as in Lemma 6.1, giving rise to the additional t-th term in the aforementioned expression's inner sum indexed in j. Setting the parameters as in Table 1 we have the following upper bounds.

Corollary 6.1. *Symmetric multi-unit BT is* $0.8372, 0.8483, 0.8482$*-hard for* $k = 2, 3, 4$ *respectively.*

For comparison, recall that the optimal approximation ratio of symmetric single-unit BT has been shown to be $(2 + \sqrt{2})/4 \approx 0.8536$ (Kang, Pernice, and Vondrák 2022), which also provides a uniform upper bound for all $k \geq 2$ by a similar argument as that of Proposition 6.2.

7 Discussion

This work improved upper and lower bounds for single and multi-unit BT. Bounds for symmetric multi-unit BT and upper bounds for multi-unit BT had not been investigated yet; for $k = 2$ our bounds show that the state of the art (see Section 1) is not optimal. We recall that (Blumrosen and Dobzinski 2021, Theorem 5.3) converts our 0.7292-approximation for single-unit BT into an improved 0.4217-approximation for multi-unit BT with the general class of *increasing valuations*. Further, our hardness results trivially carry over to this not yet well-understood model.

Our arguments show that mathematical programming techniques can still yield significant improvements for single-unit BT, and that combined with additional probabilistic analysis, they generalise to multi-unit BT. Future work should focus not only on narrowing remaining gaps for every k, but on furthering the understanding of our mathematical programs, so as to obtain uniform bounds.

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