Fair Incentives for Early Arrival in 0-1 Cooperative Games

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Abstract

Incentives for early arrival (I4EA) was recently proposed for studying online cooperative games. In an online cooperative game, players arrive in an unknown order, and the value increase after each player arrived should be distributed immediately among all the arrived players. Although there is only one arriving order in the game, we also hope that the value distribution is equal to their Shapley value in expectation. To achieve these goals, the early solutions ignored the fairness in each single arriving order. More specifically, an important player may receive nothing in a game, which seems unfair in reality. To combat this, we propose refined fairness in this paper and design new solutions in 0-1 value games. Specifically, we compute the distance of the distribution in each order to the Shapley value and aim to minimize it. We propose a new mechanism called Egalitarian Value-Sharing (EVS) to do so. We also show that the mechanism can maximize the egalitarian welfare among all the players who made contributions.

Introduction

Compared to statistical models, online models that allow agents to enter the market at different times are more suitable for many real-world scenarios where instant and irrevocable decisions must be made (Simon 1945). This has attracted a substantial body of literature focusing on online economic markets, including auctions (Ockenfels, Reiley Jr, and Sadrieh 2006; Bergemann and Said 2010), matching (Huang et al. 2020; Huang, Tang, and Wajc 2024), hedonic games (Bullinger and Romen 2023; Flammini et al. 2021), and other related topics (Porter 2004; Borodin and El-Yaniv 2005). Such motivations can also be extended to cooperative games, whose objective is to fairly allocate value among a set of collaborating players. In online scenarios, it is impractical to determine value distribution only after the grand coalition is fully formed (for example, when forming a startup, it is impossible to know in advance when all potential participants will have joined).

Ge et al. (2024) were the first to formalize the model of online cooperative games, identifying two primary concerns within this framework: *incentives for early arrival* and *Shapley-fairness*. The former ensures that players are motivated to join the coalition as early as possible, which is crucial for practical applications to prevent the project from stalling due to potential participants waiting for one another. The latter is a fairness axiom requiring that the expected share of each player, averaged over all possible arrival orders, is equal to their Shapley value—a classic solution concept that satisfies most basic and intuitive fairness axioms (Shapley et al. 1953; Winter 2002). They achieve both objectives by decomposing the game into 0-1 monotone games and applying the *Rewarding the First Critical Player* (RFC) method, which allocates rewards exclusively to the first player whose participation is indispensable for generating the coalition's value.

Although the RFC method satisfies the axiom of Shapley fairness, its sharing scheme still exhibits several notable limitations. The most significant issue is that, for any specific arrival order, the resulting value allocation may deviate substantially from the corresponding Shapley value. This drawback poses practical concerns, since in real-world scenarios only a single arrival order is actually realized. To address this issue, we require a property that emphasizes fairness with respect to a single joining order, and propose a new value allocation mechanism based on this property.

Fortunately, we observe that although the RFC scheme is proven to satisfy both incentives for early arrival and Shapley-fairness, it is not the only such mechanism. This observation suggests that it may not be necessary to sacrifice current properties in order to achieve a more balanced allocation in each possible arrival order. Our work seeks to address the following question: *Can we design a fairer allocation for each possible arrival order without compromising Shapley-fairness?*

The main reason a Shapley-fair scheme does not guarantee balanced allocations in every order is that, although the expected share matches the Shapley value, the shares assigned in specific orders can deviate significantly from it. To address this, we introduce the Euclidean (ℓ_2) distance between the Shapley value and the allocation in each specific order, which we refer to as the Shapley distance. This metric enables us to explore various approaches to improve Shapley-fairness at the level of individual orders. In this work, we focus on minimizing the expected Shapley distance across all possible arrival orders.

One theoretical advantage of adopting the above objective is that, in the context of decomposed 0-1 monotone games, it can be reformulated as maximizing egalitarian welfare—that is, increasing the minimum share among all contributing players—subject to an intuitive constraint: among players with symmetric contributions, those who arrive earlier should receive a larger share. The notion of egalitarian welfare is particularly natural in 0-1 games, as all contributing players are symmetric in the moment when the coalition value increases from 0 to 1.

Finally, our contributions can be summarized as follows:

- We propose a generalized class of allocation policies for online cooperative games that satisfy both incentives for early arrival and Shapley-fairness. Unlike previous approaches, our policies can allocate non-zero shares to all contributing players in each order.
- We introduce new concepts for evaluating fairness in the implementation of specific arrival orders, including Shapley distances and egalitarian welfare. We further show that, in 0-1 games, there exists a single mechanism that simultaneously maximizes both objectives, subject to the constraint that players with symmetric contributions who arrive earlier receive larger shares.

The remainder of the paper is organized as follows. In **Preliminaries**, we introduce the setting and summarize the theoretical results from Ge et al. (2024), including the definition of online cooperative games, the formalization of properties, and the RFC mechanism. In **New Fairness**, we present our new fairness notions for specific orders and further refine them in the context of 0-1 monotone games. In **The Mechanism for New Fairness**, we describe our proposed class of mechanisms, analyze their properties, and discuss the minimization of the expected Shapley distance. We offer formal proofs of the theoretical results in **The Proofs**.

Related Work

Incentives for Early Arrival In addition to Ge et al. (2024), Zhang et al. (2025) investigated incentives for early arrival in the context of cost-sharing problems. Adopting the previous requirements, but with the valuation replaced by a cost function, they designed a mechanism that solves all cost-sharing games. Aziz, Guo, and Sun (2025) abandoned the requirement of Shapley-fairness and instead explored other axioms, such as incentives to stay and incentives to participate which ensure that contributing players receive a positive share. Zhao (2025) provided an in-depth discussion of I4EA and emphasizes its significance across a variety of domains. However, they did not address the question of how to formulate in-order fairness requirements.

Online Coalition Formation There is a body of literature focusing on online coalition formation, where players arrive sequentially and the objective is to partition them in a way that maximizes social welfare (Bullinger and Romen 2023; Flammini et al. 2021). In these settings, agents derive utility from their preferences over coalition members. Cohen and Agmon (2025) further introduce the challenge of maximizing egalitarian welfare, i.e., the minimum utility among all participants. Bullinger and Romen (2025) aim to achieve stable coalition structures in online environments, addressing common stability concepts based on deviations by individual agents or groups of players. In contrast to these works, our setting assumes that all players form a single coalition, and our goal is to determine a fair value allocation that can incentivize the players to join as soon as possible.

Online Mechanism Design We also review previous work on mechanism design in online settings. Bergemann and Välimäki (2019) and Pavan, Segal, and Toikka (2014) focus on designing auction mechanisms for environments where players with private valuations arrive over time and may change their valuations. Another line of research concerns online fair division, as studied in Elkind et al. (2025); Aleksandrov and Walsh (2020); Amanatidis et al. (2023), where typically a fixed set of players participate and the items to be allocated arrive sequentially. In contrast to these studies, our work primarily addresses incentives for early arrival, which are crucial to ensuring player participation.

Preliminaries

We introduce the formal setting of online cooperative games in this section. The main notations follow the initial work by Ge et al. (2024), and we also give a brief and necessary introduction of their *Rewarding the First Critical Player (RFC)* mechanism.

The Model

A transferable-utility game is denoted by a pair (N, v), where N is the set of players and $v : 2^N \to \mathbb{R}_{\geq 0}$ is the *valuation function*. For any coalition $S \subseteq N$, $v(S)$ is interpreted as the value that can be produced by S, and we assume $v(\emptyset) = 0$. For two players $i, j \in N$, we say i, j are *symmetric* if $\forall S \subseteq N \setminus \{i, j\}, v(S \cup \{i\}) = v(S \cup \{j\})$. In this paper, we focus on *0-1 monotone games*[1] where $v(S) \in \{0, 1\}$ for any $S \subseteq N$, and $v(T) \leq v(S)$ for any $T \subseteq S$.

In an online cooperative game, there is an additional element of some order $\pi \in \Pi(N)$, where $\Pi(N)$ is the set of all permutations of N. The players join sequentially following the order π. If player i joins before player j in order π, we denote it as $i \prec_\pi j$. Moreover, we denote $p^\pi(i)$ as the players in π who arrive before i and herself.

We say a set of players S is a *prefix* of π, denoted as $S \sqsubseteq \pi$, if the players in S are the first $|S|$ players in π. Then, the local game on prefix S consists of a local valuation function $v_{|S} : 2^S \to \mathbb{R}_+$ and a local order $\pi_{|S} \in \Pi(S)$, where $v_{|S}(T) = v(T)$ for any $T \subseteq S$ and $i \prec_{\pi_{|S}} j \Leftrightarrow i \prec_\pi j$ for any $i, j \in S$. Overall, (v, π) describes an online game in which a set of players joins following the order π and cooperates following the valuation function v, while before all players have joined, $(v_{|S}, \pi_{|S})$ describes the local state that only the players in S have arrived. Moreover, in each local state, only the information about the local game is accessible. Our goal is to design a value-sharing policy that not only determines a value share for (v, π), but also includes value shares for any $(v_{|S}, \pi_{|S})$[2].

[1] In Ge et al. (2024), the authors have shown that a mechanism on 0-1 monotone games can be generalized through the decomposition of games and linear properties can be maintained.

[2] We omit the player set in this tuple because the subscript indicates the set of players and, when there is no subscript, the set of

Definition 1 (Mechanism)**.** *A* ***value-sharing policy*** ϕ *maps a game* (v, π) *to an* $|N|$*-tuple of allocations, so that* $\phi_i(v, \pi) \geq 0$ *is player* i*'s share of the value, and* $\sum_i \phi_i(v, \pi) = v(N)$*.*

An ***online value-sharing mechanism*** *is given by a policy* ϕ*, so that after the arrival of each prefix* $S \sqsubseteq \pi$*, each player* $i \in S$ *gets a (cumulated) share of* $\phi_i(v_{|S}, \pi_{|S})$*.*

To keep the players staying in the cooperation, we require the players' shares to be non-decreasing through other players' arrival. Moreover, to prevent the players from delaying their arrival strategically, we require a player's share to be non-increasing when choosing to delay, assuming the order of others to be fixed.

Definition 2 (OIR)**.** *An online mechanism is* ***online individually rational (OIR)*** *for value function* v *if for any arrival order* π *and any* $T, S \sqsubseteq \pi$ *with* $T \subseteq S$*, we have* $\phi_i(v_{|T}, \pi_{|T}) \leq \phi_i(v_{|S}, \pi_{|S})$ *for every player* $i \in T$*.*

Definition 3 (I4EA)**.** *An online mechanism is* ***incentivizing for early arrival (I4EA)*** *if for any player* i*,* $\phi_i(v, \pi) \geq \phi_i(v, \pi')$ *for all* π *and* π' *such that* $\pi_{|N \setminus \{i\}} = \pi'_{|N \setminus \{i\}}$ *and* $p^{\pi}(i) \subset p^{\pi'}(i)$*.*

Another desired property is fairness. Here we follow the definition of the *Shapley-fair* proposed in Ge et al. (2024), and in the next section we give our new fairness properties. Shapley-fairness requires that the expected share of a player equals *Shapley value* when π is uniformly selected at random. Intuitively, the Shapley value can be regarded as the expectation of a mechanism which rewards each player with her marginal contribution under uniformly selected π. We give formal definitions of these concepts as follows.

Definition 4. *Given* (v, π)*, the* ***marginal contribution*** *of a player* i *is* $\mathrm{MC}_i(v, \pi) := v(p^{\pi}(i)) - v(p^{\pi}(i) \setminus \{i\})$*.*

Definition 5 (Shapley et al. (1953))**.** *Given* v*, the* ***Shapley value*** *of a player* i *in* (N, v) *is*

$$\mathrm{SV}_i(v) := \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \mathrm{MC}_i(v, \pi).$$

Definition 6 (SF)**.** *Given* v*, an online mechanism is* ***Shapley-Fair (SF)*** *if for any player* i*,*

$$\frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \phi_i(v, \pi) = \mathrm{SV}_i(v).$$

The Early Solutions

In Ge et al. (2024), the authors demonstrate that there exist certain unsolvable games for which no mechanism can simultaneously satisfy SF, OIR, and I4EA. They further characterize the class of solvable games, as detailed in the following, and propose a complete mechanism called *Rewarding the First Critical Player (RFC)*. RFC is formalized in Algorithm 1. Intuitively, in an online 0-1 monotone game, there is at most one player with a marginal contribution of 1 (referred to as the *marginal player*), while all other players have marginal contributions of 0. If the value distribution is based solely on marginal contributions, those players who are essential for value creation—termed *critical players*—may strategically delay their actions to become the marginal player. The RFC mechanism addresses this by awarding the value to the first critical player, thereby ensuring I4EA in most games. However, in unsolvable games, there exists a "super-complementary" player who, despite having an individual value of 0, can become the unique critical player through delayed participation.

players is referred to by default as N.

Order	Critical Players	Marginal Player	RFC Share
A-B-C	A,B	B	A$\leftarrow$1
A-C-B	A,C	C	A$\leftarrow$1
B-A-C	B,A	A	B$\leftarrow$1
B-C-A	A	A	A$\leftarrow$1
C-A-B	C,A	A	C$\leftarrow$1
C-B-A	A	A	A$\leftarrow$1

Table 1: The critical players, marginal player and the share determined by RFC of the game in Example 1. Each row corresponds to a joining order listed in the first column.

Algorithm 1: Rewarding the First Critical Player (RFC)

Input: (v, π)
Output: $\phi \leftarrow \{\phi_j\}_{j \in \pi}$
Initialize $\phi_j \leftarrow 0, \forall j \in \pi$.
if $|\mathrm{CR}(v, \pi)| > 0$ **then**
  $i \leftarrow$ the first player in $\mathrm{CR}(v, \pi)$.
  $\phi_i \leftarrow 1$.
end if

Definition 7. *Given a 0-1 monotone* v *and some order* π*, if there exists* $S \sqsubseteq \pi$ *that* $v(S) = 1$*, then the* ***marginal player*** *is some* $i \in S$ *satisfying*

$$v(p^{\pi}(i)) = 1 \text{ and } v(p^{\pi}(i) \setminus \{i\}) = 0;$$

and the ***critical players*** *are defined as*

$$\mathrm{CR}(v, \pi) = \{j \in \pi \mid v(p^{\pi}(i) \setminus \{j\}) = 0\}.$$

Also note that the marginal player is the last critical player.

Definition 8 (Solvability)**.** *A 0-1 monotone* v *is* ***unsolvable*** *if there exists a player* i *such that* $v(\{i\}) = 0$ *and* $\exists S, \{i\} = \{j \in S \mid v(S) = 1 \text{ and } v(S \setminus \{j\}) = 0\}$*. A 0-1 monotone* v *is solvable in contrast.*

It has been pointed out that only on solvable games, there exists a mechanism that is OIR, SF and I4EA. RFC satisfies these properties on all solvable games. Example 1 illustrates how RFC operates on an unsolvable game.

Example 1. *Consider a 0-1 monotone game with* $N = \{A, B, C\}$*, and* $v(S) = 1$ *only if* $\{A, B\} \subseteq S$ *or* $\{A, C\} \subseteq S$*. The marginal player, critical players, and the share of RFC are shown in Table 1. The Shapley value of* A, B, C *are* $4/6, 1/6, 1/6$ *respectively. Also note that this game is unsolvable. A is the "super-complementary" player in* $\{A, B, C\}$ *who can delay to be the unique critical player, e.g., from order* $[B, A, C]$ *to* $[B, C, A]$*.*

Order	Share to 1st	Share to 2nd	Share to 3rd	Share to 4th
□□□△	$1-2\epsilon$	ϵ	ϵ	0
□□△□	$1-\epsilon$	0	0	ϵ
□△□□	1	0	0	0
△□□□	0	1	0	0

Table 2: A share of the game in Example 2. Each row represents joining orders where □ corresponds to one of A, B, C and △ corresponds to the null player D. The share of the $(k-1)$th player in the order is listed in the kth column.

At the end of this section, we provide some useful observations of the critical players and marginal players as follows. Given (v, π), let i be the marginal player, then

Observation 1. *$\forall j \in p^{\pi}(i) \setminus \{i\}$, if j moves her position in order but still arrives before i, then the critical players are unchanged.*

Observation 2. *If i delays, the set of critical players becomes a subset of the original one and if i arrives earlier, the set of critical players becomes a superset of the original one until i is no longer the marginal player.*

Observation 3. *The critical players are symmetric to each other in local game $v_{|p^{\pi}(i)}$.*

New Fairness

Although the property of SF guarantees fairness in an expectation manner, the exact implementation in a specific order may not feel so fair. To overcome this problem, we propose two new considerations. For symmetric players, we ask the mechanism to determine a weakly higher share for the player arriving earlier.

Definition 9 (MOS)**.** *Given v, a mechanism $\mathcal{M}$ is* ***monotone on symmetric players (MOS)*** *if for any symmetric players i, j and an order π where $i \prec_{\pi} j$*

$$\phi_i^{\mathcal{M}}(v, \pi) \geq \phi_j^{\mathcal{M}}(v, \pi).$$

Another key flaw of SF is that the value shares in each possible order can always be far away from Shapley value, without hurting the expected one. Hence, to evaluate these gaps, we introduce the Euclidean distance between the Shapley value and the share in a specific order.

Definition 10 (SD)**.** *Given (v, π), the* ***Shapley distance (SD)*** *of a mechanism $\mathcal{M}$ on a player i is defined as*

$$\mathrm{SD}^{\mathcal{M}}(v, \pi) := \| \mathrm{SV}_i(v) - \phi_i^{\mathcal{M}}(v, \pi)\|^2.$$

In this paper, our objective is to design mechanisms for 0-1 monotone games that satisfy SF, OIR, I4EA, and MOS, while minimizing expected SD (later we will also show that minimizing SD on every joining order is impossible). We present equivalent conditions for these requirements in the context of 0-1 monotone games, which serve as the foundation for our proposed mechanism.

Implementation in 0-1 Games

For 0-1 monotone games, we first demonstrate that any mechanism satisfying OIR and SF must allocate the value exclusively to the critical players, and this allocation should occur precisely when the marginal player joins.

Theorem 1. *For any mechanism $\mathcal{M}$ satisfying OIR and SF on all solvable games, there is $\sum_{j \in \mathrm{CR}(v,\pi)} \phi_j^{\mathcal{M}}(v, \pi) = 1$ for any solvable v and arbitrary π. Moreover, $\phi_j^{\mathcal{M}}(v, \pi) = \phi_j^{\mathcal{M}}(v_{|p^{\pi}(i)}, \pi_{|p^{\pi}(i)})$ for any j where i is the marginal player.*

As noted in Observation 3, the critical players are symmetric when the marginal player joins. Intuitively, earlier critical players should receive higher rewards. In fact, this forms an equivalent condition for MOS.

Proposition 1. *For an OIR and SF mechanism $\mathcal{M}$, it satisfies MOS for any 0-1 monotone v and any order π iff $\phi_i^{\mathcal{M}}(v, \pi) \geq \phi_j^{\mathcal{M}}(v, \pi), \forall i, j \in \mathrm{CR}(v, \pi)$ satisfying $i \prec_{\pi} j$.*

Notice that RFC satisfies MOS as only the share of the first critical player is 1. However, we justify that MOS is not inherently guaranteed by mechanisms that satisfy SF, OIR, and I4EA by Example 2. In the next Section, we offer a family of mechanisms that are OIR, I4EA, SF and MOS. Moreover, the expected SD over all joining orders is minimized when we maximize the minimal share of the critical players.

Definition 11 (EW)**.** *Given (v, π), the* ***egalitarian welfare of critical players (EW)*** *of a mechanism $\mathcal{M}$ is defined as $\mathrm{EW}^{\mathcal{M}}(v, \pi) = \min_{i \in \mathrm{CR}(v,\pi)} \phi_i^{\mathcal{M}}(v, \pi)$.*

Example 2. *Consider a game containing $N = \{A, B, C, D\}$ where $v(S) = 1$ only if $\{A, B, C\} \subseteq S$ and the share shown in Table 2. To avoid listing all 24 joining orders, we represent A, B, C with □ and D with △. This share is obviously not MOS, but it is SF, I4EA and could be output by an OIR mechanism.*

The Mechanism for New Fairness

Following the proposed properties, an intuitive idea is to share the value equally among the critical players. We claim that this idea fails to satisfy I4EA in all solvable games, since the marginal player may delay to decrease the number of critical players, as shown in the following example.

Example 3. *Consider the game where $N = \{A, B, C, D\}$ and $v(S) = 1$ only if $\{A, B, C\} \subseteq S$ or $\{A, B, D\} \subseteq S$. Consider the orders $\pi = [C, A, B, D]$ and $\pi' = [C, A, D, B]$. Notice that B is always the marginal player in both orders. However, the sets of critical players in two orders are different. As listed in Table 3, C is a critical player in π buts not in π'. Therefore in order π, B can delay to arrive after D so that the number of critical players decreases and the share of B increases from $1/3$ to $1/2$ if we simply make equal shares (abbreviated as ES in the table) among the critical players.*

To fix this problem, we propose the idea of determining the *minimal critical prefix* in (v, π): for the marginal player i, we move i one position forward until the player before i is a critical player, and determine the share of i in that order.

	Joining Orders						
	C-A-B-D			C-A-D-B		C-D-A-B	
CR	C	A	B	A	B	A	B
RFC	1	0	0	1	0	1	0
ES	1/3	1/3	1/3	1/2	1/2	1/2	1/2
MCP	C-A-B			C-A-B		C-D-A-B	
EVS	1/3	1/3	1/3	2/3	1/3	1/2	1/2
SV	A:5/12; B: 5/12; C:1/12; D: 1/12						

Table 3: Three joining orders of the game in Example 3 and 4. The critical players (CR), result of equally sharing among critical players (ES), minimal critical prefix (MCP), result of EVS (WVS with $w(k) = 1$) and the result of RFC are listed respectively. We only show the share of critical players following their joining order. Moreover, we also present the Shapley value (SV) of the game in the last row.

Algorithm 2: Weighted Value-Sharing Mechanism (WVS)

Input: v, π and a weight function w
Output: value share $\phi \leftarrow \{\phi_j\}_{j\in\pi}$

Initialize $\phi_j \leftarrow 0, \forall j \in \pi$.
Let i^* be the marginal player, and let m be the number of critical players in π.
Let $S \leftarrow$ be the minimal critical prefix $\hat{S}(v,\pi)$, and let m' be the number of critical players in S.
if $m' > m = 1$ **then**
  WARNING: "v is not solvable".
  $\phi_i \leftarrow 1$ and RETURN.
end if
Set $\phi_{i^*} \leftarrow \frac{w(m')}{\sum_{k=1}^{m'} w(k)}$.
For each critical player j in $\mathrm{CR}(v,\pi) \setminus \{i\}$, assume j is the t-th arrived player in $\mathrm{CR}(v,\pi) \setminus \{i\}$, set

$$\phi_j \leftarrow \frac{w(t)}{\sum_{k=1}^{m-1} w(k)}(1 - \phi_{i^*}).$$

Definition 12. *Given* (v,π), *the* minimal critical prefix *is*

$$\hat{S}(v,\pi) := \left(\arg\min_{\substack{S\sqsubseteq\pi \\ v(S\cup\{i\})=1}} |S|\right) \cup \{i\}$$

where i is the marginal player in (v,π).

Notice that the process of determining the minial critical prefix is not equivalent to simply take i together with the agents (weakly) before the penultimate critical player. As we have shown in Observation 2, if i is moved forward, the critical players may increase. For instance, we examine the game and two orders in Example 3. In order π, B cannot maintain being the marginal player by moving her to earlier positions, so that the minimal critical prefix just includes all players (weakly) before B. In contrast, in order π', B is still the marginal player if moved one step earlier, which finally induces the same minimal critical prefix as that of π. Intuitively, we should give B the same share in both orders because of the same minimal critical prefix, or B may delay to increase her value share.

Based on the above idea, we now propose our main result, the *Weighted Value-Sharing Mechanism (WVS)*, as summarized in Algorithm 2. We show that, taking use of the minimal critical prefix, we can improve not only the idea of equally sharing among critical players, but also any policy determined by a non-increasing weight function $w : \mathbb{N}_+ \rightarrow \mathbb{R}_{\geq 0}$, to satisfy I4EA. When w is a constant function, we call it a *Egalitarian Value-Sharing Mechanism (EVS)*. Moreover, in the next Section, we claim that EVS optimizes EW and expected SD. Intuitively, different weight functions impose different importance of players who arrive earlier: if the weight function decreases sharply, we finally give more reward for those joining earlier. In the extreme case where $w(1) \neq 0$ and $w(k) = 0$ for any $k > 1$, the corresponding WVS degenerates to the RFC mechanism.

A WVS is parameterized by a weight function w, and the idea of WVS is firstly to determine the share of the marginal players in the minimal critical prefix by weighted average, and then to distribute the rest value to other critical players proportional to their weights. When v is unsolvable, this procedure would produce an inefficient share, where the share of the marginal player is less than 1 and the shares of others are 0. To maintain efficiency, we give all the value to the marginal player in such cases.

Example 4. *Consider the same game as described in Example 3, and here we show the results of WVS with a constant weight function, i.e., the EVS mechanism, which are also listed in Table 3. As we mentioned before, the minimal critical prefix of $[C, A, D, B]$ is $[C, A, B]$ which is the same as the minimal critical prefix of $[C, A, B, D]$. Therefore, the shares of B in both orders are $1/3$ if applying the EVS. For the remaining critical players, they will equally share the rest value since they have equal weights. Now, we can see that the mechanism avoid the strategic delay of player B. Moreover, in each order, compared the share of that in RFC, where player B gets nothing, the value distribution is fairer since it is closer to Shapley value.*

Properties

We now present the theoretical properties of WVS. In Theorem 2, we show that WVS satisfies SF, OIR, I4EA and MOS if the weight function is weakly monotone decreasing.

Theorem 2. *For any solvable v, WVS defined by w is SF, OIR, I4EA and MOS if w is weakly monotone decreasing.*

Also note that not all mechanisms satisfying these properties can be written as a WVS defined by some specific $w(k)$, because there might exist mechanisms that output different value share for orders with same numbers of critical players. Such mechanisms require a weight system defined on different joining orders, which raises the complexity of mechanism definition and computation to exponential level.

Now we move to the question that how to optimize the EW and SD. First, we naturally constrain the space of all possible mechanisms into *anonymous* ones. That is, if we replace a player i with an equivalent player i' in (v, π) to create a game (v', π'), then $\phi_i(v, \pi) = \phi_{i'}(v', \pi')$. Surprisingly, we find that the EVS, i.e, a WVS with the constant weight function optimizes EW and the expected SD.

Definition 13. *The Egalitarian Value-Sharing Mechanism is an instance of the WVS mechanism, where the weight system is given by $w(k) = c$ for some constant $c > 0$.*

Theorem 3. *Given any solvable v and joining order π,*

$$\mathrm{EW}^{\mathrm{EVS}}(v, \pi) \geq \mathrm{EW}^{\mathcal{M}}(v, \pi),$$

where $\mathcal{M}$ is an arbitrary mechanism satisfying SF, OIR, I4EA and MOS.

Theorem 4. *Given any solvable v, there is*

$$\mathbb{E}_{\pi\sim\mathcal{U}(\Pi(N))}[\mathrm{SD}^{\mathrm{EVS}}(v, \pi)] \leq \mathbb{E}_{\pi\sim\mathcal{U}(\Pi(N))}[\mathrm{SD}^{\mathcal{M}}(v, \pi)]$$

where $\mathcal{M}$ is an arbitrary anonymous mechanism satisfying SF, OIR, I4EA and MOS.

We further argue that no mechanism can optimize SD for every single (v, π). Although critical players may initially be symmetric when the marginal player joins, this symmetry can be disrupted as additional players arrive. For example, in Example 4, players A, B, and C are symmetric in the local order $[A, B, C]$, but this symmetry no longer holds in the order $[A, B, C, D]$. Due to such asymmetry, a mechanism that optimizes SD would ideally allocate shares to critical players proportionally to their Shapley values. However, the Shapley value itself may change as new players join, while any value allocation becomes irrevocable once it is made.

The Proofs

Now we offer formal proofs of the theorems and propositions in this paper. For simplicity, we denote $\pi = [..., i, ..., j, ...]$ and $\pi' = [..., j, ..., i, ...]$ as a pair of joining orders where only the positions of i and j are exchanged and the order of others is fixed. When we refer to i, j as adjacent players, we use $\pi = [...i, j...]$ (no ellipsis between i, j).

Sharing among Critical Players

Proof of Theorem 1. For a SF mechanism, the value should be shared to the players entirely at any time. On 0-1 monotone games, the value becomes 1 when the marginal player joins and keeps unchanged. Therefore, the share should be determined at that time and due to OIR, it should not be changed. As for sharing only on critical players, we prove by induction: given solvable v, the equality holds if for any $i \in N$ and any $\pi_{|N\setminus\{i\}} \in \Pi(N \setminus \{i\})$, only players in $\mathrm{CR}(v_{|N\setminus\{i\}}, \pi_{|N\setminus\{i\}})$ receive the value share in the local game $(v_{|N\setminus\{i\}}, \pi_{|N\setminus\{i\}})$. For simplicity, let $S_i = N \setminus \{i\}$.

For $\pi \in \Pi(N)$, let i be the last player in π. We discuss by categories:

- If i is not the marginal player, then $\phi_j(v, \pi) = \phi_j(v_{|S_i}, \pi_{|S_i})$ following OIR, and only critical players have a positive share as $\mathrm{CR}(v, \pi) = \mathrm{CR}(v_{|S_i}, \pi_{|S_i})$.
- If i is the marginal player, then i is also a critical player whenever i is not the last to join. Moreover, following SF, the value share should satisfy

$$\begin{aligned}
&\sum_{\pi\in\Pi(N)} \phi_i(v, \pi) - \sum_{j\in S_i} \sum_{\pi_{|S_j}\in\Pi[S_j]} \phi_i(v_{|S_j}, \pi_{|S_j}) \\
=& \sum_{\pi\in\Pi(N)} \mathrm{MC}_i(v, \pi) - \sum_{j\in S_i} \sum_{\pi_{|S_j}\in\Pi[S_j]} \mathrm{MC}_i(v_{|S_j}, \pi_{|S_j}) \\
=& \sum_{\pi_{|S_i}\in\Pi(S_i)} \mathrm{MC}_i(v, \pi).
\end{aligned}$$

  i.e., over all joining orders, the total value share increase of i equals her total marginal contribution. If any player $j \notin \mathrm{CR}(v, \pi)$ gets a positive share, she can not pay the value back to those critical players as no marginal contribution is created when she is the last to join. Therefore, the value should be shared only among critical players.

□

Proof of Proposition 1. "⇒": Given 0-1 monotone v and an order π, notice that if i, j are symmetric, then $i \in \mathrm{CR}(v, \pi) \Leftrightarrow j \in \mathrm{CR}(v, \pi)$. Suppose $i \prec_\pi j$, if both i, j are critical players, MOS is guaranteed when the share of i is higher than j; if i, j are not critical players, their share should be 0 following SF and OIR.

"⇐": If the share of i is less than j, then the local game on $p^\pi(i^*)$ is a counter example for MOS. □

WVS is SF, OIR, I4EA and MOS

Before the proof of Theorem 2, we emphasize that WVS can verify if a game is solvable properly. WVS firstly determines the share of the marginal player, and shares the rest of the value among other critical players. The share is inefficient only when (1) the share of the marginal player is less than 1 and (2) the marginal player is the unique critical player. Therefore, there exists a joining order where the marginal player is not the unique critical player, and she can delay to be the unique one. Such cases are unsolvable following Definition 8. Now, we give the proof of Theorem 2 formally.

Proof of Theorem 2. **OIR**. WVS is obviously OIR as the value is shared only once when the marginal player joins, and the share is unchanged in the future.

SF. Given N and $i \in N$, we construct a partition $\mathcal{P}(i, \Pi) = \{\Pi_{i0}, \Pi_{i1}, ..., \Pi_{im}\}$ of the joining order set $\Pi(N)$ as follows:

- for any π where $i \notin \mathrm{CR}(v, \pi)$, $\pi \in \Pi_{i0}$;
- for every $1 \leq k \leq m$, there exists only one $\pi \in \Pi_{ik}$, where i is the marginal player of π;
- for Π_{ik} where $1 \leq k \leq m$, if $\pi \in \Pi_{ik}$, then $\pi' \in \Pi_{ik}$. Here π' is the joining order where i exchanges her position with each one previous critical player in π.

Notice that for any π such that $i \in \mathrm{CR}(v, \pi)$ while i is not the marginal player, it corresponds to another π' where i exchanges the position with the marginal player in π to become the marginal player in π' without changing the critical player set. Hence, the above partition finally includes all $\pi \in \Pi(N)$. As we mentioned, the critical players are symmetric with each other in the local game $(v_{|p^\pi(i)}, p^\pi(i))$. Therefore, in an anonymous mechanism for $\pi = [..., j, ..., i, ...]$ and $\pi' = [..., i, ..., j, ...]$ satisfying $\{i, j\} \subseteq \mathrm{CR}(v, \pi) = \mathrm{CR}(v, \pi')$, we have $\phi_i^{\mathcal{M}}(v, \pi) = \phi_j^{\mathcal{M}}(v, \pi')$. Notice that $\sum_{j \in \mathrm{CR}(v,\pi)} \phi_i^{\mathrm{WVS}}(v, \pi) = 1$ according to the efficiency, we have $\sum_{\pi \in \Pi_{ik}} \phi_i^{\mathrm{WVS}}(v, \pi) = 1$. Now consider the total marginal contribution produced by i in the joining orders in Π_{ik}. For $1 \le k \le m$, as there is only one order that i is the marginal player, we have $\sum_{\pi \in \Pi_{ik}} \mathrm{MC}_i(v, \pi) = 1$. For $k = 0$, i is never the marginal player so the total marginal contribution is 0. Hence, $\sum_{\pi \in \Pi_{ik}} \phi_i^{\mathrm{WVS}}(v, \pi) = \sum_{\pi \in \Pi_{ik}} \mathrm{MC}_i(v, \pi)$ Overall,

$$\begin{aligned}
&\frac{1}{n!} \sum_{\Pi_{ik} \in \mathcal{P}(i,\Pi)} \sum_{\pi \in \Pi_{ik}} \phi_i^{\mathrm{WVS}}(v, \pi) \\
=&\frac{1}{n!} \sum_{\Pi_{ik} \in \mathcal{P}(i,\Pi)} \sum_{\pi \in \Pi_{ik}} \mathrm{MC}_i(v, \pi) \\
=&\frac{1}{n!} \sum_{\pi \in \Pi(N)} \mathrm{MC}_i(v, \pi) = \mathrm{SV}_i(v).
\end{aligned}$$

I4EA. For any solvable game v, and $\pi = [..., j, i...]$,$\pi' = [...i, j...]$, we discuss the share of i by categories:

- If $i \notin \mathrm{CR}(v, \pi)$, then we can infer $i \notin \mathrm{CR}(v, \pi')$, so $\phi_i^{\mathrm{WVS}}(v, \pi') = \phi_i^{\mathrm{WVS}}(v, \pi) = 0$.
- If $i \in \mathrm{CR}(v, \pi)$ but not the marginal player, we first show that $\mathrm{CR}(v_{|\hat{S}(v,\pi)}, \pi_{|\hat{S}(v,\pi)}) \subseteq \mathrm{CR}(v_{|\hat{S}(v,\pi')}, \pi_{|\hat{S}(v,\pi')})$. We denote the marginal player of π as i^*, then i^* is also the marginal player of π' following Observation 1. Notice that, if i^* is moved one position forward simultaneously in π and π', we still have $p^\pi(i) = p^{\pi'}(i)$, and the difference only happens when i^* is the next player of i in π. As $i \in \mathrm{CR}(v, \pi)$, we have $i \in \hat{S}(v, \pi)$, which means i^* would not be moved before i when finding $\hat{S}(v, \pi)$. In π', i is also included in $\hat{S}(v, \pi)$, but j may be excluded. Therefore, $\hat{S}(v, \pi') \subseteq \hat{S}(v, \pi)$, so $\mathrm{CR}(v_{|\hat{S}(v,\pi)}, \pi_{|\hat{S}(v,\pi)}) \subseteq \mathrm{CR}(v_{|\hat{S}(v,\pi')}, \pi_{|\hat{S}(v,\pi')})$ following Observation 2. This suffices to show $\phi_{i^*}^{\mathrm{WVS}}(v, \pi) \ge \phi_{i^*}^{\mathrm{WVS}}(v, \pi')$. As we have $\mathrm{CR}(v, \pi) = \mathrm{CR}(v, \pi')$ following Observation 1 and i arrives earlier in π', $\phi_i^{\mathrm{WVS}}(v, \pi') \ge \phi_i^{\mathrm{WVS}}(v, \pi)$.
- If i is the marginal player of π, but not the marginal player of π', then j is the marginal player of π', and $\mathrm{CR}(v, \pi) = \mathrm{CR}(v, \pi')$. As we mentioned, i's share in π is determined by $\mathrm{CR}(v_{|\hat{S}(v,\pi)}, \pi_{|\hat{S}(v,\pi)})$ and $|\mathrm{CR}(v_{|\hat{S}(v,\pi)}, \pi_{|\hat{S}(v,\pi)})| \ge |\mathrm{CR}(v, \pi)|$. Note that i is the last player in $\mathrm{CR}(v_{|\hat{S}(v,\pi)}, \pi_{|\hat{S}(v,\pi)})$, but the penultimate player in $\mathrm{CR}(v, \pi')$, so $\phi_i^{\mathrm{WVS}}(v, \pi) \le \phi_i^{\mathrm{WVS}}(v, \pi')$.
- If i is the marginal player of π, π', then following the recursively determined share of i, we have $\phi_i^{\mathrm{EVS}}(v, \pi) = \phi_i^{\mathrm{EVS}}(v, \pi')$ except for the case $\mathrm{CR}(v, \pi) = 1 < \mathrm{CR}(v, \pi')$. However, as we mentioned, this would only happen when v is unsolvable.

MOS. Given solvable v and π, let i be the marginal player and j be the penultimate critical player, the proof is equivalent to show i's share is weakly less than j's. Recall Observation 2, we have $\mathrm{CR}(v, \pi) \subseteq \mathrm{CR}(v_{|\hat{S}(v,\pi)}, \pi_{|\hat{S}(v,\pi)})$so the weight of i is less than j and the total weight determining i's share is larger than j's, which is sufficient for the proof. □

EVS Maximizes EW

We prove that EVS maximizes the EW comparing to all mechanisms satisfying SF, OIR, I4EA and MOS.

Proof of Theorem 3. Let $\pi = [..., j, i, ...]$ and $\pi' = [..., i, j, ...]$ where i is the marginal player of π. Following the constraint of I4EA, we require $\phi_i^{\mathcal{M}}(v, \pi) \le \phi_i^{\mathcal{M}}(v, \pi')$. Now we discuss by categories as follows:

- If i is not the marginal player of π', we point out that EVS maximized $\phi_i(v, \pi)$ as all critical players have the same share in (v, π). If one mechanism assigns a strictly higher share of i, then there exists $i' \in \mathrm{CR}(v, \pi) \setminus \{i\}$ that $\phi_{i'}(v, \pi) < \phi_i(v, \pi)$, which leads to a contradiction.
- If i is the marginal player of π', then $\phi_i(v, \pi)$ is maximized by EVS as it assigns $\phi_i^{\mathcal{M}}(v, \pi) = \phi_i^{\mathcal{M}}(v, \pi')$.

Therefore, EVS maximizes the share of the marginal player recursively, so it maximizes the EW. □

EVS Minimizes Expected SD

We prove that EVS minimizes the expected SD comparing to all mechanisms satisfying SF, OIR, I4EA and MOS.

Proof of Theorem 4. Recall the partition $\mathcal{P}(i, \Pi) = \{\Pi_{i0}, \Pi_{i1}, ..., \Pi_{im}\}$ constructed in the Proof of Theorem 2 and $\sum_{\pi \in \Pi_{ik}} \phi_i^{\mathcal{M}}(v, \pi) = 1$ where $\mathcal{M}$ is SF, OIR, I4EA and MOS. Now, consider the expected SD of a mechanism $\mathcal{M}$:

$$\begin{aligned}
\mathbb{E}_\pi[\mathrm{SD}^{\mathcal{M}}(v, \pi)] &= \frac{1}{n!} \sum_{\pi \in \Pi(N)} \sum_{i \in \pi} (\phi_i^{\mathcal{M}}(v, \pi) - \mathrm{SV}_i(v))^2 \\
&= \frac{1}{n!} \sum_{i \in N} \sum_{\Pi_{ik} \in \mathcal{P}(i,\Pi)} \sum_{\pi \in \Pi_{ik}} (\phi_i^{\mathcal{M}}(v, \pi) - \mathrm{SV}_i(v))^2.
\end{aligned}$$

The innermost term $\sum_{\pi \in \Pi} (\phi_i^{\mathcal{M}}(v, \pi) - \mathrm{SV}_i(v))^2$ is minimized by EVS. Since $\sum_{\pi \in \Pi_{ik}} \phi_i^{\mathcal{M}}(v, \pi) = 1$ and SV_i is fixed, this term only varies with $\sum_{\pi \in \Pi_{ik}} (\phi_i^{\mathcal{M}}(v, \pi))^2$, i.e., it can be written as

$$-2\,\mathrm{SV}_i(v) + \sum_{\pi \in \Pi_{ik}} \left((\phi_i^{\mathcal{M}}(v, \pi))^2 + (\mathrm{SV}_i(v))^2\right)$$

which is minimized when $\{\phi_i^{\mathcal{M}}(v, \pi)\}_{\pi \in \Pi_{ik}}$ are as equal as possible. Following the requirements of MOS and anonymous, the minimal term in $\{\phi_i^{\mathcal{M}}(v, \pi)\}_{\pi \in \Pi_{ik}}$ is the share when i is the marginal player in π. Since EVS maximizes the egalitarian share, and other shares in $\{\phi_i^{\mathcal{M}}(v, \pi)\}_{\pi \in \Pi_{ik}}$ are equal to each other, the term $\sum_{\pi \in \Pi} (\phi_i^{\mathcal{M}}(v, \pi) - \mathrm{SV}_i(v))^2$ is minimized. Therefore, EVS minimized expected SD. □

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