Best of Both Worlds Guarantees for Equitable Allocations

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Abstract

Equitability is a well-studied fairness notion in fair division, where an allocation is equitable if all agents receive equal utility from their allocation. For indivisible items, an exactly equitable allocation may not exist, hence, relaxations such as EQX and EQ1 are studied. In this paper, we study equitability in the context of randomized allocations. Specifically, we aim to achieve equitability in expectation (ex ante EQ) and require that each deterministic outcome in the distribution be nearly equitable. Such an allocation is commonly known as a 'Best of Both Worlds' allocation, and has been studied, e.g., for envy-freeness and MMS.

We characterize the existence of such allocations using a geometric condition on convex combinations of allocations, and use this to give comprehensive results for existence and computation. For two agents, we show that while ex ante EQ and ex post EQX allocations do not exist, ex ante EQ and ex post EQ1 allocations do exist and can be computed in polynomial time. For three or more agents, however, such allocations may not exist, and deciding existence is NP-hard. For a relaxation of EQ1, termed equitability up to one good more-and-less (EQ^1_1), we show that ex ante EQ and ex post EQ^1_1 allocations always exist and are efficiently computable. Finally, for agents with binary valuations, we show that ex ante EQ and ex post EQ1 allocations that additionally satisfy welfare guarantees exist, and are efficiently computable.

Introduction

Allocating a set of resources among interested agents with diverse preferences is a fundamental problem, studied formally since at least the 1940s (Steinhaus 1948; Dubins and Spanier 1961). A central goal in such settings is to ensure that the allocation is *fair*, so that no individual is unduly disadvantaged by the outcome. This problem arises in many real-world settings, such as resolving border disputes (Brams and Taylor 1996), dividing rent (Su 1999; Gal et al. 2017), assigning courses to students (Budish et al. 2017), allocating subsidized houses (Kamiyama, Manurangsi, and Suksompong 2021; Madathil, Misra, and Sethia 2023), and assigning conference papers to reviewers (Lian et al. 2018). There have been multiple implementations of fair division algorithms (e.g., spliddit.org, Kajibuntan, and fairoutcomes.com (Goldman and Procaccia 2015; Igarashi and Yokoyama 2023)).

The most commonly studied notion of fairness is *envy-freeness* (EF) (Foley 1967), which requires that no agent prefers another's allocation over their own. Envy-freeness avoids interpersonal comparisons of utility. In an envy-free allocation, an agent could get very large utility, while another agent gets almost none.[1] Another important notion in fairness is *equitability* (Dubins and Spanier 1961) (EQ), which demands that all agents derive the same level of utility from their own allocation. In empirical studies with human subjects, equitability is demonstrated to have a significant impact on the perceived fairness (Herreiner and Puppe 2009, 2010). Additionally, equitability plays a crucial role in applications such as divorce settlements (Brams and Taylor 1996) and rental harmony (Gal et al. 2017). Equitable allocations are also well-studied in the literature, and previous work has studied existence (Gourvès, Monnot, and Tlilane 2014; Hosseini and Sethia 2025; Barman et al. 2024), welfare guarantees (Freeman et al. 2020, 2019; Sun, Chen, and Doan 2023; Bhaskar et al. 2023), as well as allocations that satisfy both approximate envy-freeness and equitability (Aziz 2021).

When dealing with indivisible resources, exact envy-freeness or equitability is often impossible. Thus several relaxations are studied, such as envy-freeness or equitability up to any item (EFX and EQX) (Freeman et al. 2019; Gourvès, Monnot, and Tlilane 2014; Plaut and Roughgarden 2020), the weaker notions of envy-freeness or equitability up to one item (EF1 and EQ1) (Budish 2011; Lipton et al. 2004), and further relaxations. These relaxations allow for a small degree of unfairness, permitting envy or inequity to be eliminated by hypothetically removing an item from the larger-valued bundle. While the existence of EFX allocations remains an open question, EF1 and EQX (and hence EQ1) allocations always exist, and can also be computed efficiently for a large class of valuations, including monotone valuations (Lipton et al. 2004; Budish 2011; Gourvès, Monnot, and Tlilane 2014; Barman et al. 2024).

An alternative to relaxation-based remedies for nonexis-

[1]Consider, e.g., an instance with n agents, n items, where the first agent gets utility 1 from the first item and 0 from all the others, and all other agents get utility $1/n$ from each item. Allocating item i to agent i is an envy-free allocation where the first agent gets utility 1, while all other agents get utility $1/n$.

tence is to employ randomization, aiming to achieve fairness in expectation. Both envy-freeness and equitability can be trivially satisfied in expectation by allocating all goods to a single agent chosen uniformly at random. However the realized allocation is clearly unfair, since one agent receives everything, leaving all others with nothing.

Recent work (Aziz et al. 2023) asked if randomization allowed us to get the best of both worlds. For envy-freeness, this meant a randomized allocation that is EF ex ante (in expectation, prior to realization of the random bits) and EF1 ex post (after the realization). Aziz et al. (2023) show that such allocations always exists using the Probabilistic Serial algorithm (Bogomolnaia and Moulin 2001) and Birkhoff's decomposition algorithm (Birkhoff 1946; von Neumann 1953) as subroutines. Hence, through randomization, stronger guarantees on fairness are obtainable. Subsequent works have further explored this in the context of envy-based (Feldman et al. 2024; Hoefer, Schmalhofer, and Varricchio 2023) and share-based fairness (Babaioff, Ezra, and Feige 2022). Prior work also studies randomized equitable allocations in restricted instances with chores (Sun and Chen 2025).

Given the importance of equitability, a fundamental question is whether there is a randomized ex ante EQ allocation which is ex post nearly equitable (such as EQX, or EQ1). In this work, we comprehensively address this question and present a complete landscape of existence and tractability. We show that even for two agents, an ex ante EQ and ex post EQX allocation may not exist. However for two agents an ex ante EQ and ex post EQ1 allocation always exists and can be computed efficiently. For binary valuations ($v_i(g) \in \{0, 1\}$ for any $i \in N$ and $g \in M$) and any number of agents, we exhibit existence and efficient computation coupled with strong welfare guarantees. However, beyond binary valuations, even for three agents, such allocations may not always exist, and the corresponding decision problem becomes NP-hard. Relaxing EQ1 to a notion termed equitability up to one good more-and-less (EQ_1^1) helps: there always exists an allocation that is ex ante EQ and ex post EQ_1^1, and can be computed efficiently. EQ_1^1 allows inequitability to be resolved by removing a good from the richer agent and giving a good to the poorer agent. Our results contrast with the results on envy-freeness, where ex ante EF and ex post EF1 allocations always exist. Our techniques also differ from prior work, and include a geometric characterization of best of both worlds equitable allocations and the use of duality.

We also introduce i-biased EQ1 allocations: EQ1 allocations where agent i has maximum value among all agents. While EQ1 is a notion of fairness, an i-biased EQ1 allocation asks for a small amount of bias towards agent i. We show that for two agents, existence of i-biased EQ1 allocations characterizes existence of ex ante EQ and ex post EQ1 allocations. Curiously, we show that in some instances, there may be an agent i for which an i-biased EQ1 allocation does not exist, and hence every EQ1 allocation disfavours i.

Our Contributions

We present comprehensive results on both the existence and computation of both EQ + EQ1 and EQ + EQX allocations. Our first result is a geometric characterization of instances that possess such a best of both worlds (BoBW) allocation.

Theorem 1. *Let $\mathcal{I}$ be a fair division instance with n agents and normalized valuations. The following statements are equivalent.*

1. *$\mathcal{I}$ admits an* EQ + EQ1 *allocation.*
2. *For any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_n$ with $\sum_i \lambda_i = 0$, there exists an* EQ1 *allocation A such that $\sum_{i=1}^{n} \lambda_i v_i(A_i) \geq 0$.*

Similarly, $\mathcal{I}$ admits an EQ+EQX *allocation iff for any $\lambda \in \mathbb{R}_n$ with $\sum_i \lambda_i = 0$, there exists an* EQX *allocation A such that $\sum_{i=1}^{n} \lambda_i v_i(A_i) \geq 0$.*

The characterization forms the basis for most of our results. In fact, the characterization holds for general monotone valuations, not just additive valuations.

Two Agents Using the characterization, we show first that for instances with two agents and normalized valuations, an ex ante EQ and ex post EQ1 allocation always exists. In fact, the characterization in this case is equivalent to proving the existence of an i-biased EQ1 allocation (i.e., where agent i has the largest value), for $i \in \{1, 2\}$. The proof requires a careful analysis of the allocation obtained starting from an EQX allocation.

An obvious question is if we can show the existence of an ex ante EQ and ex post EQX allocation, particularly since in the prior result we start with an EQX allocation.We show however that this is not true. Even with two agents, three items, and normalized valuations, an ex ante EQ and ex post EQX allocation may not exist.

Binary Valuations If agents have binary valuations over the set of goods, it is not difficult to see that an ex ante EQ and ex post EQ1 allocation must always exist. For this, any good that has value zero for some agent is simply assigned to the agent. This only leaves goods that have value one for every agent, which can then be assigned using a slight modification of the Birkhoff-von Neumann theorem (e.g., Aziz et al. (2023)). However, this allocation is wasteful, since goods are assigned to agents that have zero value for them.

Instead, we show a stronger result. We show that for agents with binary valuations, there exists an ex ante EQ and ex post EQ1 allocation that has optimal social welfare (or total value over the agents) over all EQ allocations. That is, we can get an ex ante EQ and ex post EQ1 allocation with social welfare equal to that of highest welfare EQ allocation (that could be fractional). Thus the restriction that the EQ allocation be supported on EQ1 allocations does not cause any loss in the social welfare. We find this result surprising, since clearly not all fractional EQ allocations can be obtained as a convex combination of EQ1 allocations.

Our result in this case is based on carefully rounding a linear program using results on LPs with bihierarchical constraint structures — a generalization of the Birkhoff-von Neumann theorem — from Budish et al. (2013).

General Instances: Existence For general instances, even for agents with normalized valuations, we show that an ex ante EQ and ex post EQ1 allocation may not exist. This is in contrast to envy-freeness, where an ex ante EF and ex post EF1 allocation always exists (Aziz et al. 2023).

In fact, our example showing the non-existence of ex ante EQ and ex post EQ1 allocations is tight in multiple regards. The instance consists of just three agents and four items — with two agents, or with three agents and three items, an ex ante EQ and ex post EQ1 allocation always exists. The example consists of just two types of agents (where agents are of the same type if they have identical valuations). For just one type of agent, i.e., when all agents have identical valuations, there always exists an ex ante EQ and ex post EQ1 allocation. Finally, there are just two types of goods as well. Note that instances with normalized valuations and a single type of good are trivial instances, where $v_i(g) = c$ for some constant c, for all agents i and goods g.

Given the negative result, we consider further relaxations of EQ1 for the ex post guarantee. A natural relaxation studied for EF1 is envy-freeness up to one good more-and-less (EF^1_1), which allows resolution of envy by removing an item from the envied agent *and* giving an item to the envious agent (Barman and Krishnamurthy 2019; Aziz et al. 2023). We study the analogous relaxation for EQ1 — equitability up to one good more-and-less (EQ^1_1) — and show that in fact an ex ante EQ and ex post EQ^1_1 allocation always exists, for any number of agents and possibly non-normalized valuations, and can be computed efficiently.

General Instances: Complexity We further study the computational complexity of determining the existence of ex ante EQ and ex post EQ1 allocations. We show that, for three agents, it is weakly NP-hard to determine if there exists an ex ante EQ and ex post EQ1 allocation, and with n agents, the problem is strongly NP-hard.

We then show that the weak NP-hardness shown is the best possible, by giving a pseudopolynomial time algorithm for determining the existence of EQ + EQ1 allocations when the number of agents is constant. This is based on a somewhat technical dynamic program. In fact, the dynamic program is quite versatile, and can be slightly modified to determine the existence of (i) EQ + EQX allocations, as well as existence of (ii) i-biased allocations, for any agent i.

Finally, we show that for general instances, determining the existence of an i-biased allocation for an agent i is NP-hard. All missing proofs are available in the full version of this paper (Bhaskar et al. 2025).

Preliminaries

An instance $\mathcal{I}$ of the fair division problem is given by a tuple $\langle N, M, \mathcal{V}\rangle$, where N is a set of n *agents*, M is a set of m indivisible *items* (or *goods*), and $\mathcal{V}$ is the *valuation profile* consisting of each agent's valuation function $\{v_i\}_{i\in N}$. For agent $i \in N$, its valuation function $v_i : M \to \mathbb{Z}^+$ specifies its value (or utility) for each good in M. Valuations are additive, i.e. $S \subseteq M$, $v_i(S) = \sum_{g\in S} v_i(g)$. If $v_i(M) = v_j(M)$ for all agents $i, j \in N$, the instance is said to be *normalized*.

Allocation. A *bundle* refers to any (possibly empty) subset of goods. An *integral* allocation $A := (A_1, \ldots, A_n)$ is a partition of the set of goods M into n bundles, one for each agent, and A_i is the bundle assigned to agent i. Note that A is also specified by an $n \times m$ binary matrix (also denoted as A) with exactly one 1 in every column. An allocation where goods can be fractionally assigned to multiple agents is called a *fractional* allocation. It is specified by an $n \times m$ column-stochastic matrix A where $A_{i,g}$ is the fraction of good g assigned to agent i. Note that the polytope of fractional allocations $\{A \in \mathbb{R}^{n\times m}_+ :$ for all $g \in M, \sum_{i\in N} A_{i,g} = 1\}$ is an integral polytope (e.g., the constraint matrix is totally unimodular, since each variable appears exactly once with coefficient $+1$). Hence, by Carathéodory's theorem, any fractional allocation can be obtained as a distribution over at most $mn + 1$ integral allocations. In the following, an allocation typically refers to an integral allocation.

A randomized allocation X is a lottery over a set of integral allocations $\{A^k\}_{k\in[\ell]}$, where allocation A^k is chosen with probability $p_k \in [0, 1]$ and $\sum_{k\in[\ell]} p_k = 1$. Note that X corresponds to the fractional allocation $\sum_{k\in[\ell]} p_k A^k$ in expectation. The integral allocations $\{A^k\}_{k\in[\ell]}$ are said to be the support of X.

Equitable Allocation. An allocation A is *equitable* (EQ) if for any agents $i, j \in N$, we have $v_i(A_i) = v_j(A_j)$ (Dubins and Spanier 1961), and *equitable up to any good* (EQX) if for any agents $i, j \in N$ such that $A_j \neq \emptyset$, we have $v_i(A_i) \geqslant v_j(A_j \setminus \{g\})$ for all goods $g \in A_j$. Allocation A is *equitable up to one good* (EQ1) — a relaxation of EQX — if for any agents $i, j \in N$ such that $A_j \neq \emptyset$, there is a good $g \in A_j$ such that $v_i(A_i) \geqslant v_j(A_j \setminus \{g\})$ (Gourvès, Monnot, and Tlilane 2014; Freeman et al. 2019). Finally, relaxing EQ1 further, an allocation A is *equitable up to one good more-and-less* (EQ^1_1) if for any agents i and j such that $v_i(A_i) < v_j(A_j)$, we have $v_i(A_i) + v_i(g_i) \geqslant v_j(A_j) - v_j(g_j)$ for some $g_i \notin A_i$ and $g_j \in A_j$.

We say that agent i is *rich* in an allocation A if $v_i(A_i) \geqslant v_j(A_j)\ \forall\ j \in N$. Analogously, an agent j is *poor* if $v_j(A_j) \leqslant v_i(A_i)\ \forall\ i \in N$. An EQ1 allocation A is said to be i-biased EQ1 if agent i is a rich agent in A.

Best of Both Worlds Fairness A randomized allocation X is *ex ante* EQ if every agent derives the same utility in expectation, i.e., $\mathbb{E}[v_i(X_i)] = \mathbb{E}[v_j(X_j)]\ \forall\, i, j \in N$. Equivalently, X is ex ante EQ if the fractional allocation corresponding to X is EQ.[2] The randomized allocation X is *ex post* EQ1 (or EQX) if it can be obtained as a distribution over EQ1 (or EQX) allocations. We use EQ + EQ1 (or EQX) to denote a randomized allocation X that is ex ante EQ and ex post EQ1 (or ex post EQX). An allocation is *Best of Both Worlds* (or BoBW) if it is either EQ + EQ1, or EQ + EQX.

Note that the assumption of normalization is crucial in the setting of indivisible items (except in Theorem 6, for EQ^1_1 allocations; and in Theorem 3 for binary valuations). Without this assumption, an EQ + EQ1 allocation may not exist, even in trivial instances. E.g., consider an instance with two agents and two items. Agent 1 has value 10 for both items,

[2]We use the fact that every fractional allocation can be obtained as a distribution over integral allocations.

and agent 2 has value 1 for both items. In any EQ1 allocation, agent 1 must get at least 1 item. But then in every EQ1 allocation, agent 1 has strictly greater value than agent 2. This must then be true of any distribution over EQ1 allocations as well, and hence any distribution over EQ1 allocations cannot give a fractional EQ allocation.

We thus assume all valuations are normalized unless otherwise stated. We further note the following trivial cases: if either (i) all agents have identical valuations, or (ii) $m = n$, there always exists an EQ + EQ1 allocation that can be computed efficiently.

Proposition 1. *Given a fair division instance with either (i) agents with identical valuations, or (ii) an equal number of goods and agents and normalized valuations, an* EQ + EQX *allocation always exists and can be computed efficiently.*

A Characterization of BoBW Instances

We now present a geometric characterization of instances that admit EQ + EQ1 (or EQ + EQX) allocations in Theorem 1. This forms the basis for many of our further results. The theorem in fact holds for general monotone valuations, though we restrict our study to additive valuations.

Theorem 1. *Let $\mathcal{I}$ be a fair division instance with n agents and normalized valuations. The following statements are equivalent.*

1. *$\mathcal{I}$ admits an* EQ + EQ1 *allocation.*
2. *For any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_n$ with $\sum_i \lambda_i = 0$, there exists an* EQ1 *allocation A such that $\sum_{i=1}^n \lambda_i v_i(A_i) \geq 0$.*

Similarly, $\mathcal{I}$ admits an EQ+EQX *allocation iff for any $\lambda \in \mathbb{R}_n$ with $\sum_i \lambda_i = 0$, there exists an* EQX *allocation A such that $\sum_{i=1}^n \lambda_i v_i(A_i) \geq 0$.*

Proof. We show the proof for EQ + EQ1 allocations. The proof for EQ + EQX allocations is very similar.

(1) $\implies$ (2): Suppose $\mathcal{I}$ admits an EQ + EQ1 allocation. Let X be a randomized EQ allocation over a support $(A^1, A^2, \ldots, A^\ell)$ of EQ1 allocations with probabilities $(p_1, p_2, \ldots, p_\ell)$. By the definition of ex ante EQ, we have $\mathbb{E}[v_i(X_i)] = \mathbb{E}[v_j(X_j)] \quad \forall i, j \in N$.

Suppose for the sake of contradiction there exists $\lambda \in \mathbb{R}_n$ with $\sum_i \lambda_i = 0$ such that for all EQ1 allocations A, $\sum_i \lambda_i v_i(A_i) < 0$. Then for the randomized allocation X,

$$\sum_i \lambda_i \mathbb{E}[v_i(X_i)] = \sum_i \lambda_i \left(\sum_{k=1}^{\ell} p_k v_i(A_i^k) \right) = \sum_{k=1}^{\ell} p_k \left(\sum_i \lambda_i v_i(A_i^k) \right) < 0\,.$$

Since X is ex ante EQ, each agent has the same expected utility, say u, and hence

$$\sum_i \lambda_i \mathbb{E}[v_i(X_i)] = u \sum_i \lambda_i = 0.$$

This gives a contradiction, and hence for every $\lambda \in \mathbb{R}_n$, with $\sum_i \lambda_i = 0$, there exists some EQ1 allocation A such that $\sum_{i=1}^n \lambda_i v_i(A_i) \geqslant 0$.

(1) $\impliedby$ (2): Let $\mathcal{E} = \{A^1, A^2, \ldots, A^\ell\}$ be the (finite) set of all EQ1 allocations. Consider the following linear program, with variables μ and $\{p_k\}_{k\in[\ell]}$:

$$\begin{aligned} \text{maximize} \quad & 0 \\ \text{subject to} \quad & -\sum_{k=1}^{\ell} p_k \cdot v_i(A_i^k) + \mu = 0 \quad \text{for all } i \in [n], \\ & \sum_{k=1}^{\ell} p_k = 1, \\ & p_k \geqslant 0 \quad \text{for all } k \in [\ell] \end{aligned}$$

This LP seeks a convex combination of EQ1 allocations such that all agents receive the same expected utility μ, i.e., the convex combination is ex ante EQ. A feasible solution to this LP gives a randomized allocation that is EQ + EQ1 (since all allocations in the support are EQ1). We show that this LP is feasible, completing the proof.

To show feasibility, we consider the dual linear program. To that end, we introduce dual variables $\lambda_i \in \mathbb{R}$ for each agent i, and $\theta \in \mathbb{R}$ for the normalization constraint.

$$\begin{aligned} \text{minimize} \quad & \theta \\ \text{subject to} \quad & -\sum_{i=1}^{n} \lambda_i \cdot v_i(A_i^k) + \theta \geqslant 0 \quad \text{for all } k \in [\ell], \\ & \sum_{i=1}^{n} \lambda_i = 0 \end{aligned}$$

Clearly, the dual has a feasible solution given by $\lambda_i = 0$ for all $i \in [n]$ and $\theta = 0$. From the dual constraints, we have

$$\theta \geqslant \max_{k\in[\ell]} \sum_{i=1}^{n} \lambda_i \cdot v_i(A_i^k)\,.$$

By assumption, if $\theta, (\lambda_i)_{i\in[n]}$ is a feasible dual solution, then there exists $k \in [\ell]$ such that for the EQ1 allocation A^k, $\sum_i \lambda_i v_i(A_i^k) \geqslant 0$, and hence $\theta \geqslant 0$. Therefore, the dual is bounded below. By strong duality, the primal is feasible. This completes the proof. □

Remark 1. *Note that the primal has an exponential number of variables owing to the potentially exponential number of* EQ1 *allocations. Since each (possibly fractional) allocation A is represented by the valuation vector $\vec{v}(A) \in \mathbb{R}^n$, if X is an ex ante* EQ *and ex post* EQ1 *(or* EQX*) allocation, then by Carathéodory's theorem, it can be obtained as a distribution over at most $n+1$* EQ1 *(or* EQX*) allocations. In particular, there is a polynomial-sized certificate for existence of* EQ + EQ1 *or* EQ + EQX *allocations in an instance, which is this succinct distribution.*

For two agents, we have the following corollary.

Corollary 1. *Let $\mathcal{I}$ be a fair division instance with $n = 2$ agents, m items, and normalized valuations. The following statements are equivalent:*

1. *$\mathcal{I}$ admits an* EQ + EQ1 *allocation.*
2. *There exists an i-biased* EQ1 *allocation for each $i \in \{1, 2\}$.*

Similarly, $\mathcal{I}$ admits an EQ + EQX *allocation iff there exists an i-biased* EQX *allocation for each $i \in \{1, 2\}$.*

To see the corollary, observe that for two agents, the second condition in Theorem 1 is equivalent to the statement that for any $\lambda \in \mathbb{R}$, there exists an EQ1 (or EQX) allocation A such that $\lambda v_1(A_1) - \lambda v_2(A_2) \geqslant 0$. Then for $\lambda > 0$, this is equivalent to the condition that there exists a 1-biased EQ1 (or EQX) allocation. For $\lambda < 0$, this is equivalent to the condition that there exists a 2-biased EQ1 (or EQX) allocation.

Two Agents

This section considers instances with two agents. We show that an EQ + EQ1 allocation always exists and can be computed in linear time. However, this result does not extend to the stronger notion of EQ + EQX; such allocations do not always exist.

Theorem 2. *Given a fair division instance with two agents with normalized valuations, an* EQ + EQ1 *allocation always exists and can be computed in time $\mathcal{O}(m)$.*

Proof sketch. We will show that for $i \in \{1, 2\}$, there always exists an i-biased EQ1 allocation. From Corollary 1, this suffices to prove the theorem. Let A be an EQX allocation; such an allocation can be obtained in time $O(m)$ by a greedy algorithm (Gourvès, Monnot, and Tlilane 2014). If $v_1(A_1) = v_2(A_2)$, then this is already an EQ + EQ1 allocation. Otherwise $v_1(A_1) < v_2(A_2)$, then A is a 2-biased allocation. We show how to modify the allocation A to obtain a 1-biased allocation.

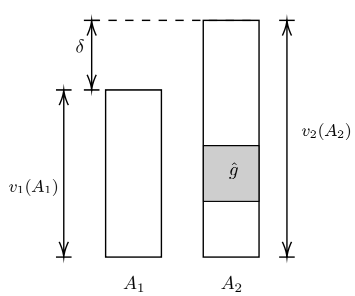

Figure 1: EQX allocation A where $v_2(A_2) - v_1(A_1) = \delta$.

Define $C = \{g \in M : v_1(g) \geq v_2(g)\}$ as the set of *compressing goods*, and $E = M \setminus C$ as the set of *expanding goods*. Our algorithm crucially uses these sets, as follows.

Let $\delta := v_2(A_2) - v_1(A_1) > 0$ (see Figure 1). Assume that there exists a good $\hat{g} \in A_2$ such that $\hat{g} \in C$, i.e., $v_1(\hat{g}) \geq v_2(\hat{g})$. Since A is an EQX allocation, removing $\hat{g}$ from A_2 must remove the inequitability, and hence $v_2(\hat{g}) \geq \delta$. Since $\hat{g} \in C$, $v_1(\hat{g}) \geq \delta$. In this case, we show that the allocation $A' = (A_2, A_1)$ obtained by swapping the allocating bundles is an 1-biased EQ1 allocation, as required.

Now consider the case where $C \cap A_2 = \emptyset$. Thus, $C \subseteq A_1$, and for every good $g \in A_2$, $v_1(g) < v_2(g)$. We now pick any good $\hat{g} \in A_2$ and transfer it to A_1. Let $A'_1 = A_1 \cup \{\hat{g}\}$, and $A'_2 = A_2 \setminus \{\hat{g}\}$. Since A was an EQX allocation, after transferring good $\hat{g}$, $v_1(A'_1) \geq v_2(A'_2)$.

Our algorithm then transfers items in $C \cap A'_1$ one-by-one from A'_1 to A'_2, ensuring that the condition $v_1(A'_1) \geq v_2(A'_2)$ holds after each item is transferred. Our proof shows that if we end up transferring all goods in $C \cap A'_1$ to agent 2 while maintaining that agent 1 is a rich agent, then the resulting allocation is a 1-biased EQ1 allocation (in particular, removal of the good $\hat{g}$ makes agent 1 poor, ensuring EQ1). However, if it holds that, on transferring some item $s^* \in C \cap A'_1$ agent 1 would become a poor agent, we transfer s^* to agent 2 and then swap the allocations. Thus, $A''_1 = A'_2 \cup \{s^*\}$, and $A''_2 = A'_1 \setminus \{s^*\}$. By normalisation, agent 1 is then a rich agent, and we show that removal of the good s^* makes agent 1 poor, ensuring that $A'' = (A''_1, A''_2)$ is the required 1-biased EQ1 allocation. □

An EQ + EQX allocation for two agents may however not exist, even with normalized valuations and just three items, as shown in Figure 2. The only EQX allocation A has $A_1 = \{g_3\}$ and $A_2 = \{g_1, g_2\}$. But this is clearly not equitable.

	g_1	g_2	g_3
1	1	3	5
2	4	3	2

Figure 2: An instance where an EQ + EQX allocations does not exist.

Binary Valuations

We now consider instances where agents have binary valuations. As mentioned, it is not difficult to obtain an EQ + EQ1 allocation in this case, where goods are assigned to agents that have value 0 for them. However such an allocation is clearly wasteful. Instead, we show a stronger result. For an instance $\mathcal{I}$, let OPT be the maximum social welfare (i.e., the total value of the agents) in any EQ allocation (possibly fractional). We show that in fact there exists an EQ + EQ1 allocation where the social welfare is OPT. Thus, the restriction that the ex ante EQ allocation is supported on EQ1 allocations does not impose any cost on the social welfare.

Theorem 3. *Given a fair division instance with binary valuations, an* EQ + EQ1 *allocation X that obtains maximum utilitarian social welfare over all fractional* EQ *allocations exists, and can be computed efficiently.*

Note that we do not require normalized valuations.

We first describe the main tool from (Budish et al. 2013) we use.[3] Given a binary matrix $G \in \{0, 1\}^{n' \times m'}$, for each row $i \in [n']$, define the set $S_i = \{j : A_{ij} = 1\}$ as the

[3]Budish et al. (2013) describe Theorem 4 differently, in terms of assignment matrices and quotas. The description here is adapted to our notation, but the technical content is the same.

columns with non-zero entries. Then a set $\mathcal{T} \subseteq [n']$ is *hierarchical* (or laminar) if for any $i, i' \in \mathcal{T}$, the sets S_i and $S_{i'}$ are either disjoint, or one is contained in the other. The matrix G is *bihierarchical* if the set $[n']$ can be partitioned into $\mathcal{T}_1$ and $\mathcal{T}_2$ so that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are hierarchical.

Theorem 4 (Budish et al. (2013))**.** *Given a binary matrix $G \in \{0,1\}^{n' \times m'}$ and integral vectors $\bar{q}, \underline{q} \in \mathbb{Z}^{n'}$ such that G is bihierarchical, if the polytope $\{x \in \mathbb{R}^{n'} : \underline{q} \leqslant Gx \leqslant \bar{q}\}$ is feasible, then it is integral. Further, any fractional solution x can be decomposed into a convex combination of integral solutions in strongly polynomial time.*

We are now ready to prove the theorem.

Proof of Theorem 3. We first write the following LP L_1 that maximizes the utilitarian social welfare among all fractional EQ allocations. Note that in an EQ allocation, each agent has the same welfare (captured by the variable w), and the social welfare is nw. Further, each $v_i(g) \in \{0,1\}$.

$$\max \quad w \tag{1}$$

$$\text{subject to:} \quad \sum_{g=1}^{m} v_i(g) x_{ig} = w, \quad \forall i \in N \tag{2}$$

$$\sum_{i=1}^{n} x_{ig} = 1, \quad \forall g \in M \tag{3}$$

$$x_{ig} \geqslant 0, \quad \forall i \in N, g \in M \tag{4}$$

We show that the above LP admits a feasible solution.

Claim 1. *The LP L_1 is feasible.*

Let X^*, w^* be an optimal solution to L_1. Clearly X^* is a fractional EQ allocation of maximum social welfare w^*. We will show that X^* can be obtained as a distribution over EQ1 allocations, with the same expected welfare. For this, consider the polytope P_2, defined as the set of feasible solutions to the following linear constraints.

$$\lfloor w^* \rfloor \leqslant \sum_{g=1}^{m} v_i(g) x_{ig} \leq \lceil w^* \rceil, \quad \forall i \in N \tag{5}$$

$$\sum_{i=1}^{n} x_{ig} = 1, \quad \forall g \in M \tag{6}$$

$$x_{ig} \geqslant 0, \quad \forall i \in N, g \in M \tag{7}$$

Comparing L_1 and P_2, for each agent $i \in N$, $\sum_g v_i(g) X^*_{ig} = w^* \in [\lfloor w^* \rfloor, \lceil w^* \rceil]$. Hence, X^* is a feasible fractional solution to P_2. In order to apply Theorem 4, we need to show that the constraint matrix is bihierarchical. For this, we define the two hierarchical constraint sets $\mathcal{T}_1$ and $\mathcal{T}_2$ as follows. Let $\mathcal{T}_1$ contain the welfare constraints (5). Note that each variable x_{ig} appears at most once in the constraints (5), hence these sets are clearly laminar (and in fact, any two sets are disjoint). Let $\mathcal{T}_2$ contain the assignment constraints (6) and nonnegativity constraints (7). In the assignment constraints, each variable appears exactly once, hence these sets are disjoint. The nonnegativity constraints each have size 1, and hence each set is contained by a set in an assignment constraint. Thus, $\mathcal{T}_2$ is laminar as well, and hence the constraint matrix is bihierarchical.

We can now apply Theorem 4, and obtain that the fractional EQ allocation X^* can be decomposed in strongly polynomial time into a convex combination of integral allocations. Let $A^1, \ldots, A^\ell$ be these integral allocations with convex coefficients $p_1, \ldots, p_\ell$. Note that each integral allocation A^k for $k \in [\ell]$ is an EQ1 allocation, since by the welfare constraints (5), each agent i has value either $\lfloor w^* \rfloor$ or $\lceil w^* \rceil$. Finally, since $X^* = \sum_{k=1}^{\ell} p_k A^k$, the expected welfare of the distribution is w^*, the optimal value for L_1. □

General Instances

We now discuss instances with more than 2 agents and valuations beyond binary. We first show that, beyond binary valuations, the existence of EQ + EQ1 allocations is not guaranteed, even in instances with just three agents and four items. The result holds true even if there are only two types of agents and two types of items. The instance is thus tight, in multiple regards: an EQ + EQ1 allocation exists with two agents (Theorem 2), with three agents and three items (Proposition 1), with a single type of agent (i.e., when all agents are identical, Proposition 1), and with a single type of good (for normalized valuations, this implies that each value $v_i(g)$ is the same).

Theorem 5. *Given a fair division instance with* 3 *agents,* 4 *items, and normalized additive valuations, an* EQ + EQ1 *allocation may not exist.*

The proof is based on the instance in Figure 3 with 3 agents and 4 items. We show that in every EQ1 allocation A, $v_1(A_1) - \frac{1}{2}(v_2(A_2) + v_3(A_3)) < 0$. It follows from Theorem 1 that there is no EQ + EQ1 allocation for this instance.

	g_1	g_2	g_3	g_4
1	1.4	2.2	2.2	2.2
2	5	1	1	1
3	5	1	1	1

Figure 3: An instance with normalized valuations where there is no EQ + EQ1 allocation.

Given the nonexistence of an EQ + EQ1 allocation, we consider a relaxed ex post guarantee termed *equitability up to one good more-and-less* (EQ^1_1), where any equitability can be removed by hypothetically removing an item from the richer agent's bundle *and* adding an item to the poorer agent's bundle. In the next result, we show the existence and efficient computation of EQ + EQ^1_1 allocations. Note that the result holds even if the valuations are not normalized.

Theorem 6. *Given a fair division instance, an* $\text{EQ}+\text{EQ}^1_1$ *allocation always exists and can be computed in polynomial time.*

The theorem follows almost directly from the following result by Aziz et al. (2023).

Proposition 2. *(Aziz et al. 2023) Given a fractional allocation X, one can compute, in strongly polynomial time, a randomized allocation implementing X whose support consists of integral allocations $A^1, \dots A^\ell$ such that for every $k \in [\ell]$ and every agent $i \in N$:*

1. *If $v_i(A_i^k) < v_i(X_i)$, then $\exists g_i^- \notin A_i^k$ with $X_{i,g_i^-} > 0$ such that $v_i(A_i^k) + v_i(g_i^-) > v_i(X_i)$.*
2. *If $v_i(A_i^k) > v_i(X_i)$, then $\exists g_i^+ \in A_i^k$ with $X_{i,g_i^+} > 0$ such that $v_i(A_i^k) - v_i(g_i^+) < v_i(X_i)$.*

To prove Theorem 6, we choose X as the fractional allocation where all agents have equal utility, and then use Proposition 2 to obtain integral allocations $A^1, \dots A^\ell$. The stated conditions imply that each of these allocations are in fact EQ_1^1.

Hardness and Computation

We next show that deciding whether an EQ + EQ1 allocation exists is NP-hard.

Theorem 7. *Given a fair division instance, deciding the existence of an* EQ + EQ1 *allocation is weakly* NP*-complete, even for three agents.*

We prove by showing a reduction from 2-PARTITION. Let $S = \{b_1, b_2, \dots b_m\}$ be an instance of 2-PARTITION, where $\sum_i b_i = 2T$. We need to decide if there is a partition of S into S_1 and S_2 such that the sum of either set is T. Our fair division instance for the reduction is shown in Figure 4, and consists of 3 agents, m *set* items $\{g_1, \dots g_m\}$ and two additional items d_1 and d_2. Note that the valuations are normalized, and every agent values the grand bundle at $(m+5)T$.

	g_1	g_2	$\dots$	g_{m-1}	g_m	d_1	d_2
a_1	T	T	$\dots$	T	T	$4T$	T
a_2	b_1	b_2	$\dots$	b_{m-1}	b_m	$5T$	$(m-2)T$
a_3	b_1	b_2	$\dots$	b_{m-1}	b_m	$5T$	$(m-2)T$

Figure 4: Reduction for the proof of Theorem 7.

When the number of agents is not constant, the problem is strongly NP-hard by reduction from 3-PARTITION.

Theorem 8. *Given a normalized fair division instance, deciding the existence of an* EQ + EQ1 *allocation is strongly* NP*-complete.*

A Pseudopolynomial Time Algorithm

Given an instance $\langle N, M, \mathcal{V} \rangle$, let $v_{\max} := \max_{i,g} v_i(g)$ be the maximum value for any good. We now show that if the number of agents is fixed, we can in pseudopolynomial time determine if a BoBW allocation exists. Hence, the weak NP-hardness shown in Theorem 7 for three agents is tight.

Theorem 9. *Given an instance $\langle N, M, \mathcal{V} \rangle$, we can determine existence of an* EQ + EQ1 *allocation in time* $\mathsf{poly}\left((m v_{\max})^n\right)$.

Proof sketch. As before, let $\mathcal{E} := \{A : A$ is an EQ1 allocation$\}$ be the set of all possible EQ1 allocations. Define $\vec{v}(A) := (v_1(A_1), \dots, v_n(A_n))$ as the vector of agent values, and $\vec{v}(\mathcal{E}) := \{\vec{v}(A) : A \in \mathcal{E}\}$. For the theorem, we consider the possible value profiles $\vec{v}(\mathcal{E})$ using a dynamic program. For n agents, the number of distinct value profiles $(v_i(A_i))_{i \in [n]}$ over *all* allocations is at most $(m v_{\max})^n$, since each agent's value lies between 0 and $m v_{\max}$. Then $|\vec{v}(\mathcal{E})| \leqslant (m v_{\max})^n$. Given the set $\vec{v}(\mathcal{E})$, for the proof, we only need to determine if there exists a convex combination of the vectors in $\vec{v}(\mathcal{E})$ where each entry in the resulting vector is equal. Clearly, such a convex combination exists iff there exists an EQ + EQ1 allocation (the convex coefficients give us the required probability distribution).

We thus need only to obtain the set $\vec{v}(\mathcal{E})$, for which we give a dynamic program algorithm. Here we briefly describe the program for the case $n = 2$. The dynamic program maintains a table $\mathcal{T} \in m \times [m v_{\max}]^2 \times [v_{\max}]^2$ where an entry $\mathcal{T}(t, w_1, w_2, h_1, h_2) = 1$ if there exists an allocation $A = (A_1, A_2)$ of the first t items so that $v_1(A_1) = w_1$, $v_2(A_2) = w_2$, $\max_{g \in A_1} v_1(g) = h_1$, and $\max_{g \in A_2} v_2(g) = h_2$. Note that A is EQ1 allocation iff $w_i - h_i \leqslant w_{3-i}$ for $i \in \{1, 2\}$. Our algorithm constructs the table, and determines if there exists a convex combination of the vectors in $\vec{v}(\mathcal{E})$ where each entry in the resulting vector is equal. ☐

Remark 2. *The pseudopolynomial time algorithm is easily modified to determine the existence* EQ + EQX *allocations (by letting h_i be the* minimum *value of a good in A_i, in table $\mathcal{T}$) and to determine the existence of i-biased* EQ1 *allocations (by checking each entry of $\vec{v}(\mathcal{E})$, from the proof of Theorem 9, to see if agent i has highest value).*

Lastly, we show that if there are more than two agents, then an i-biased EQ1 allocation may not exist, and it is NP-hard to determine if an i-biased EQ1 allocation exists. Recall that for two agents, an i-biased EQ1 allocation always exists for $i \in \{1, 2\}$, and an i-biased EQX allocation may not exist.

Theorem 10. *Given an instance with 3 agents and normalized valuations, an i-biased* EQ1 *allocation may not exist. Deciding the existence of such an allocation is NP-hard.*

Conclusion

Our work gives a geometric characterization for BoBW equitability, and presents a complete landscape of the existence and computation of such allocations. An obvious open question regards BoBW allocations with approximate equitability. Concretely, can we show existence of ex ante α-EQ and ex post EQ1 (or EQX) allocations, for $\alpha < 1$? It would also be interesting to study if our results for binary valuations extend beyond additive valuations, e.g., for matroid rank valuations. Finally, given the nonexistence of EQ+EQ1 allocations for general instances, it would be useful to study other restricted domains where such allocations do exist.

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