A Variable Occurrence-Centric Framework for Inconsistency Handling

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Abstract

In this paper, we introduce a syntactic framework for analyzing and handling inconsistencies in propositional bases. Our approach focuses on examining the relationships between variable occurrences within conflicts. We propose two dual concepts: Minimal Inconsistency Relation (MIR) and Maximal Consistency Relation (MCR). Each MIR is a minimal equivalence relation on variable occurrences that results in inconsistency, while each MCR is a maximal equivalence relation designed to prevent inconsistency. Notably, MIRs capture conflicts overlooked by minimal inconsistent subsets. Using MCRs, we develop a series of non-explosive inference relations. The main strategy involves restoring consistency by modifying the propositional base according to each MCR, followed by employing the classical inference relation to derive conclusions. Additionally, we propose an unusual semantics that assigns truth values to variable occurrences instead of the variables themselves. The associated inference relations are established through Boolean interpretations compatible with the occurrence-based models.

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Introduction

Logical formalisms commonly used to represent knowledge and beliefs, particularly classical logic, adhere to the explosion principle. According to this principle, any conclusion can be derived from a contradiction, which makes inconsistent belief bases non informative. This issue is critical in real-world applications, where inconsistencies often arise from various factors including noisy data, vagueness, context dependency, uncertainty, and information from multiple sources. This situation shows the need for analytical tools that can identify the causes of conflicts, ensure recovery of consistency, and enable reasoning under inconsistency. Key approaches for addressing this issue include paraconsistency (Tanaka et al. 2013; Priest, Tanaka, and Weber 2022), argumentation (Besnard and Hunter 2008), belief revision (Gärdenfors 1992), and inconsistency measurement (Hunter and Konieczny 2010; Thimm 2018).

When exploring syntactic[1] approaches to reasoning under inconsistency, two core concepts are frequently used: minimal inconsistent subsets (MISes) and maximal consistent subsets (MCSes). An example of this is the inference relation proposed by Rescher and Manor, which determines conclusions from formulas entailed by all MCSes (Rescher and Manor 1970). A number of additional MCS-based inference relations have been proposed in the literature (Brewka 1989; Benferhat, Dubois, and Prade 1997, 1999; Konieczny, Marquis, and Vesic 2019). The fundamental principle behind using MCSes is to minimize alterations to the available information.

A significant limitation of using MISes and MCSes, however, lies in their inability to account for the specific syntactic representation of each formula: they do not look inside the formulas. For instance, a MIS maintains its status when any formula within it is replaced by an equivalent one. This drawback limits the ability to identify critical syntactic details, particularly when conflicts stem from factors such as imprecision or encoding errors. This issue is notably highlighted in the context of inconsistency measurement where free formulas can influence the inconsistency degree (Thimm 2018; Besnard 2014). To illustrate, consider the propositional base $K = \{p \wedge q, \neg p \wedge r, \neg q \vee \neg r\}$ provided in (Besnard 2014). Here, there is a single MIS $\{p \wedge q, \neg p \wedge r\}$; neglecting the presence of q and r in these formulas fails to capture the conflict involving q, r and $\neg q \vee \neg r$.

In this article, we introduce new syntactic concepts for analyzing and handling propositional inconsistency. Our approach particularly focuses on the role of variable occurrences within conflicts. Initially, it distinguishes between occurrences of the same variable, then identifies equivalences between these occurrences that contribute to conflicts. This approach allows us to capture the interactions of variable occurrences in inconsistent propositional bases and provides a new strategy for restoring consistency.

We first introduce a concept called Minimal Inconsistency Relation (MIR), which corresponds to a minimal equivalence relation on variable occurrences that leads to incon-

[1]We use the term "syntactic" to refer to approaches in the literature that focus on the form of belief bases, including methods based on consistent subsets and proof theory. In contrast, other approaches use new semantics to ensure that even inconsistent belief bases have models (e.g., see (Priest 1991))

sistency. For instance, the conflict between q, r and $\neg q \vee \neg r$ in the previous propositional base K is an MIR: the equivalence relation pairs the occurrences of q as well as the occurrences of r.

Beyond the interactions between variable occurrences, MIRs also highlight additional aspects of the captured conflicts. Specifically, they reveal the variables implicated and the specific formulas they are part of, given that each occurrence appears in a unique formula. In particular, when we focus on the formulas associated with MIRs, we capture at least as many conflicts as MISes, and generally more.

We also introduce a dual concept called Maximal Consistency Relation (MCR), which corresponds to a maximal equivalence relation on variable occurrences constructed to avoid inconsistency. We provide a duality property between MIRs and MCRs, analogous to the minimal hitting set duality observed between MISes and the complements of MCSes (Reiter 1987; Bailey and Stuckey 2005).

Using MCRs, we introduce several non-explosive inference relations. Similar to MCS-based approaches, the essential principle behind using MCRs is to maintain the propositional base as close to the original as possible after alterations. However, unlike MCS-based methods, our approach retains every formula and variable present in the original propositional base. An MCS does not alter any original formula but excludes some to restore consistency, whereas an MCR is used to modify certain formulas without excluding any to achieve the same goal.

Additionally, we introduce a new semantics that assigns truth values to variable occurrences rather than to the variables themselves. This approach mirrors the technique discussed earlier and can be seen as another strategy for restoring consistency. We specifically demonstrate that each occurrence-based model corresponds uniquely to an MCR. The entailment in our framework of occurrence-based semantics is realized through Boolean interpretations that are compatible with these occurrence-based interpretations.

Preliminaries

The language of classical propositional logic is constructed by inductive definition, starting with a countably infinite set of propositional variables denoted PV. We employ the usual connectives $\wedge$ and $\neg$ to express logical relationships. The set of well-formed formulas is referred to as PF. The connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined in terms of $\wedge$ and $\neg$ as usual.

We use the letters p, q and r, along with x, y and z, each possibly modified with subscripts, to denote propositional variables. For formulas, we employ Greek letters such as ϕ, ψ, and χ. When discussing a syntactic entity X (e.g., a formula or a set of formulas), the set of variables contained in X is denoted by $\mathsf{var}(X)$.

Considering formulas $\phi, \psi_1, \ldots, \psi_k$, along with variables $p_1, \ldots, p_k$ in $\mathsf{var}(\phi)$, we use $\phi[p_1/\psi_1, \ldots, p_k/\psi_k]$ to denote the result of simultaneously substituting $p_1, \ldots, p_k$ with $\psi_1, \ldots, \psi_k$, respectively.

A *Boolean interpretation*, or simply interpretation, is a function ω that assigns a truth value in $\{0, 1\}$ to each formula in PF and meets the following conditions: $\omega(\neg\phi) = 1 - \omega(\phi)$ and $\omega(\phi \wedge \psi) = \omega(\phi) \times \omega(\psi)$.

Given an interpretation ω, a variable p, and $v \in \{0, 1\}$, define $\omega_{|p \mapsto v}$ as the same as ω except that it assigns v to p. For variables $p_1, p_2, \ldots, p_k$ and truth values $v_1, v_2, \ldots, v_k$, we use $\omega_{|p_1 \mapsto v_1, p_2 \mapsto v_2, \ldots, p_k \mapsto v_k}$ as a shorthand notation for $(\cdots((\omega_{|p_1 \mapsto v_1})_{|p_2 \mapsto v_2})\cdots)_{|p_k \mapsto v_k}$.

An interpretation ω is said to be a *model* of ϕ, denoted by $\omega \models \phi$, if and only if $\omega(\phi) = 1$. The set of models for ϕ is denoted by $\mathsf{mod}(\phi)$. Conversely, ω is a *countermodel* of ϕ, denoted by $\omega \not\models \phi$, if and only if $\omega(\phi) = 0$.

A formula is considered *consistent* if it admits at least one model; otherwise, it is referred to as *inconsistent*.

A *propositional base* (PB) is a finite subset of PF.

We say that a PB K *entails* ϕ, written $K \vdash \phi$, when for any interpretation ω, if $\omega \models \bigwedge K$ (the conjunction of all formulas in K), it holds that $\omega \models \phi$; note that $\omega \models \bigwedge \emptyset$ holds for every interpretation ω. For cases where K contains only a single formula ψ, we sometimes use $\psi \vdash \phi$.

A *minimal inconsistent subset* (MIS) of a PB K is a subset M of K where M is inconsistent, and for every $\phi \in M$, $M \setminus \{\phi\}$ is consistent. A *maximal consistent subset* (MCS) of K is a subset M of K where M is consistent, and for every $\phi \in K \setminus M$, $M \cup \{\phi\}$ is inconsistent.

Given a formula ϕ, we establish the polarity of each subformula occurrence in ϕ using the following rules:

- The polarity of the occurrence ϕ is defined as positive.
- If a subformula occurrence $\psi \wedge \chi$ is positively (resp. negatively) polarized, then both ψ and χ inherit the same positive (resp. negative) polarity.
- If a subformula occurrence $\neg\psi$ is positively (resp. negatively) polarized, then ψ is assigned a negative (resp. positive) polarity.

A propositional variable that occurs with only one polarity in ϕ is called *pure* in ϕ.

In the proposition below, both the first two properties and the last two properties can be proven simultaneously through mutual induction on the structure of ϕ.

Proposition 1. *Let ϕ be a propositional formula and p a variable in $\mathsf{var}(\phi)$. Then, the following holds:*

1. *If p is a positive pure variable in ϕ and ω is a model of ϕ, then $\omega_{|p \mapsto 1}$ is a model of ϕ.*
2. *If p is a negative pure variable in ϕ and ω is a countermodel of ϕ, then $\omega_{|p \mapsto 1}$ is a counter-model of ϕ.*
3. *If p is a negative pure variable in ϕ and ω is a model of ϕ, then $\omega_{|p \mapsto 0}$ is a model of ϕ.*
4. *If p is a positive pure variable in ϕ and ω is a countermodel of ϕ, then $\omega_{|p \mapsto 0}$ is a counter-model of ϕ.*

We define $nbOcc(p, \phi)$ as the number of occurrences of variable p in ϕ. To unambiguously identify and reference each occurrence of p in ϕ, we employ integers ranging from 1 to $nbOcc(p, \phi)$. Indeed, for a formula ϕ in a PB, a variable p occurring in ϕ, and an index i ranging from 1 to $nbOcc(p, \phi)$, $\langle p, \phi, i \rangle$ is used to denote the ith occurrence of p in ϕ (ordered from left to right). Given a PB $K = \{\phi_1, \ldots, \phi_n\}$ and a variable p occurring in K, we sometimes use p_i^s to denote the ith occurrence of p in $\phi_1 \wedge \cdots \wedge \phi_n$, where s is the polarity of this occurrence.

Additionally, we use $Occ(K)$ to denote the set of all variable occurrences in K, and $Occ(p, K)$ to denote the set of occurrences of p in K. We also use $PosOcc(K)$ and $NegOcc(K)$ to represent the positive and negative variable occurrences in K, respectively. Similarly, $PosOcc(p, K)$ and $NegOcc(p, K)$ correspond to the positive and negative occurrences of p in K.

For instance, if $K = \{p \wedge q, \neg p \wedge r, \neg q \vee \neg r\}$, then $Occ(K) = \{p_1^+, p_2^-, q_1^+, q_2^-, r_1^+, r_2^-\}$, $PosOcc(K) = \{p_1^+, q_1^+, r_1^+\}$, $NegOcc(K) = \{p_2^-, q_2^-, r_2^-\}$, and $Occ(p, K) = \{p_1^+, p_2^-\}$.

Let us recall that an equivalence relation on a set S is a binary relation $\sim$ on S that is reflexive, symmetric, and transitive. The equivalence class of $e \in S$ under $\sim$ is represented by $[e]$. The set of all equivalence classes is denoted $S/\sim$.

Variable Occurrence-based Conflicts

To achieve a fine-grained representation of inconsistency in a PB, we focus on the occurrences of variables and the conflicts resulting from the relationships between them. Our approach becomes particularly valuable when variable occurrences that are supposed to convey identical information end up representing conflicting data due to various factors such as data errors or interpretive differences. Consider, for instance, the case of conflicting medical reports for the same patient but from different laboratories, where each symptom is represented by a propositional variable (each variable occurrence is linked to a specific laboratory result). A symptom like "high temperature" might be differently reported depending on the laboratory's threshold: a temperature of $39°C$ might be classified as high by a laboratory with a threshold of $39°C$, but not by one with a threshold of $40°C$.

A method for resolving all conflicts in a PB involves substituting every occurrence of a variable with a distinct, new variable. Given a PB K, a *C-renaming* of K is a function R that assigns a distinct, new variable to each occurrence o in $Occ(K)$. We use $\mathsf{R}(K)$ to denote the modified version of K achieved by replacing each occurrence o with $\mathsf{R}(o)$. This concept of C-renaming is applied identically to formulas.

Since the choice of C-renaming does not affect our definitions, we consider this function to be fixed and denoted R for any PB and for any formula. To enhance readability, we use the letters p, q, and r to denote the variables occurring in the original PB, and x, y, and z for those in $\mathsf{R}(K)$.

Proposition 2. *A PB K is inconsistent iff $\bigwedge \mathsf{R}(K) \wedge \bigwedge_{p \in \mathsf{var}(K)} \bigwedge_{o,o' \in Occ(p,K)} (\mathsf{R}(o) \leftrightarrow \mathsf{R}(o'))$ is inconsistent.*

Considering a PB K, we define an equivalence relation $\sim_c^K$ on $Occ(K)$ as follows: $o \sim_c^K o'$ iff $\mathsf{var}(o) = \mathsf{var}(o')$.

Let us now introduce the main concept we use to represent conflicts.

Definition 1 (Minimal Inconsistency Relation)**.** *A* Minimal Inconsistency Relation *(MIR) of a PB K is an equivalence relation $\sim$ on $Occ(K)$ satisfying the following conditions:*

1. *(Compliance) for all occurrences $o, o' \in Occ(K)$, if $o \sim o'$, then $\mathsf{var}(o) = \mathsf{var}(o')$;*
2. *(Inconsistency) $\mathsf{R}(K) \wedge \bigwedge_{(o,o') \in \sim} (\mathsf{R}(o) \leftrightarrow \mathsf{R}(o'))$ is inconsistent; and*
3. *(Minimality) there exists no equivalence relation $\sim'$ on $Occ(K)$ that satisfies Properties (1) and (2), and $\sim'$ is a proper subset of $\sim$ (i.e., $\sim' \subsetneq \sim$).*

The set of all MIRs of K is represented by $\mathsf{MIRs}(K)$.

The Compliance condition states that equivalence can only occur between occurrences of the same variable. The Inconsistency condition says that every MIR must result in inconsistency. The Minimality condition guarantees that every MIR is minimal with respect to set inclusion.

Using the Compliance condition, it is evident that $\sim \subseteq \sim_c^K$ holds for any PB K and any MIR $\sim$ of K.

Example 1. *Consider the inconsistent PB $K_1 = \{p \wedge q, \neg p \wedge r, \neg q \vee \neg r\}$ and $\mathsf{R}(K_1) = \{x_1 \wedge y_1, \neg x_2 \wedge z_1, \neg y_2 \vee \neg z_2\}$. The PB K_1 admits two MIRs $\sim_1^i$ and $\sim_2^i$: $Occ(K_1)/\sim_1^i = \{\{p_1^+, p_2^-\}, \{q_1^+\}, \{q_2^-\}, \{r_1^+\}, \{r_2^-\}\}$ and $Occ(K_1)/\sim_2^i = \{\{p_1^+\}, \{p_2^-\}\{q_1^+, q_2^-\}, \{r_1^+, r_2^-\}\}$.*

The next theorem is mainly derived from Proposition 2.

Theorem 1. *A PB is inconsistent iff it admits at least one MIR.*

The proposition below demonstrates that each equivalence class with more than one element includes both positive and negative occurrences.

Proposition 3. *Let K be a PB. For every MIR $\sim$ of K and every $C \in Occ(K)/\sim$ with $|C| \geq 2$, it holds that $C \cap PosOcc(K) \neq \emptyset$ and $C \cap NegOcc(K) \neq \emptyset$.*

Proof. Assume for contradiction that there exists an equivalence class $C = \{o_1, \ldots, o_k\}$ s. t. $k \geq 2$ and $C \cap NegOcc(K) = \emptyset$. Consider refining the equivalence relation to $\sim'$, where $Occ(K)/\sim' = ((Occ(K)/\sim) \setminus C) \cup \bigcup_{o \in C} \{\{o\}\}$. Using Minimality, $\sim'$ does not satisfy Inconsistency. Thus $\mathsf{R}(K) \wedge \bigwedge_{(o,o') \in \sim'} (\mathsf{R}(o) \leftrightarrow \mathsf{R}(o'))$ admits a model ω. Since for every o in C, $R(o)$ is positive in $\mathsf{R}(K)$, and applying Proposition 1, $\omega_{|R(o_1) \mapsto 1, \ldots, R(o_k) \mapsto 1}$ is also a model of $\mathsf{R}(K)$. Consequently, $\mathsf{R}(K) \wedge \bigwedge_{(o,o') \in \sim} (\mathsf{R}(o) \leftrightarrow \mathsf{R}(o'))$ is consistent, leading to a contradiction.

A similar contradiction arises if we assume $C \cap PosOcc(K) = \emptyset$. This is obtained by using the truth value 0 instead of 1. $\square$

Given a PB K and an equivalence relation $\sim$ on $Occ(K)$, define $PN(\sim)$ as the set $\{(o, o') \in PosOcc(K) \times NegOcc(K) : o \sim o'\}$.

The following theorem shows that the core of conflicts fundamentally stems from the interactions between positive and negative occurrences. It is primarily a consequence of Proposition 3.

Theorem 2. *Let K be a PB and $\sim$ an equivalence relation on $Occ(K)$. The relation $\sim$ is an MIR iff it satisfies the properties of Compliance, Inconsistency, and the following property: (Minimality-2) for every equivalence relation $\sim'$ on $Occ(K)$ that satisfies Compliance and Inconsistency, $PN(\sim')$ is not a proper subset of $PN(\sim)$.*

MIRs can provide a more detailed representation than MISes, even in scenarios where the objective is to identify conflicts as subsets of formulas.

Given a PB K and an MIR $\sim$ on $Occ(K)$, $\mathsf{form}_{mir}(\sim)$ represents the set $\{\phi \in K : \exists C \in Occ(K)/\sim, \exists p \in \mathsf{var}(\phi), \exists i \in \mathbb{N} \text{ s. t. } |C| \geq 2 \text{ and } \langle p, \phi, i\rangle \in C\}$. Informally, $\mathsf{form}_{mir}(\sim)$ corresponds to the formulas that contain the occurrences that are paired with $\sim$ (excluding reflexive links).

Definition 2 (O-MIS). *An O-MIS of a PB K is a subset M of K for which there exists an MIR $\sim$ of K such that $M = \mathsf{form}_{mir}(\sim)$.*

Given that $\mathsf{form}_{mir}(\sim)$ is inconsistent for every MIR $\sim$, it follows that every O-MIS includes a MIS.

Proposition 4. *Given a PB K, if M is a MIS of K, then M is also an O-MIS of K.*

Proof. According to Theorem 1, M is associated with at least one MIR $\sim$. Let $\sim'$ be the equivalence relation on $Occ(K)$ obtained by extending $\sim$ to pair each occurrence in $K \setminus M$ with only itself. Since $\sim$ is an MIR of M, it follows that $\sim'$ is an MIR of K. We know that $\mathsf{form}_{mir}(\sim')$ is inconsistent. If $\mathsf{form}_{mir}(\sim') \subsetneq M$, it would contradict the minimality of M. Therefore, we must have $\mathsf{form}_{mir}(\sim') = M$, implying that M is an O-MIS. □

Example 2. *Consider again the PB K_1 from Example 1. With respect to the two possible MIRs, $\sim_1^i$ and $\sim_2^i$, K_1 admits two O-MISes: $M_1 = \{p \wedge q, \neg p \wedge r\}$ and $M_2 = \{p \wedge q, \neg p \wedge r, \neg q \vee \neg r\}$. However, K_1 has only one MIS, namely M_1.*

The concepts of MIR and O-MIS can have an interesting application in the quantification of inconsistency, following an approach similar to that of MISes (e.g. see (Thimm 2018)). For instance, this can be done by defining measures based on the number of MIRs, their sizes, or the number of equivalence classes.

A Dual Notion: Maximal Consistency Relation

This section focuses on a dual notion to the MIR. This notion is defined as a maximal equivalence relation on variable occurrences constructed to avoid inconsistency.

Definition 3 (Maximal Consistency Relation). *A* Maximal Consistency Relation *(MCR) of a PB K is an equivalence relation $\sim$ on $Occ(K)$ satisfying the following conditions:*

1. *(Compliance) for all occurrences $o, o' \in Occ(K)$, if $o \sim o'$, then $\mathsf{var}(o) = \mathsf{var}(o')$;*
2. *(Consistency) $\bigwedge \mathsf{R}(K) \wedge \bigwedge_{(o,o')\in\sim}(\mathsf{R}(o) \leftrightarrow \mathsf{R}(o'))$ is consistent; and*
3. *(Maximality) there exists no equivalence relation $\sim'$ on $Occ(K)$ that satisfies Properties (1) and (2), and $\sim$ is a proper subset of $\sim'$.*

Let $\mathsf{MCRs}(K)$ represent the set of all MCRs of K.

Like MCSes, MCRs are uniquely determined in consistent PBs.

Proposition 5. *For every consistent PB K, the relation $\sim_c^K$ is the unique MCR of K.*

Example 3. *Revisiting the inconsistent PB K_1 from Example 1, we find that it admits two MCRs $\sim_1^c$ and $\sim_2^c$, where $Occ(K_1)/\sim_1^c = \{\{p_1^+\}, \{p_2^-\}, \{q_1^+, q_2^-\}, \{r_1^+\}, \{r_2^-\}\}$ and $Occ(K_1)/\sim_2^c = \{\{p_1^+\}, \{p_2^-\}, \{q_1^+\}, \{q_2^-\}, \{r_1^+, r_2^-\}\}$.*

Forgetting is a well-established method for restoring consistency (Lin and Reiter 1994; Lang and Marquis 2002; Besnard 2016). In particular, Besnard (Besnard 2016), in the context of defining an inconsistency measure, proposes substituting variable occurrences with constants to achieve consistency. In this context, the MCR concept can further enhance this perspective by providing a more holistic view.

For example, consider the PB $K = \{p_1 \wedge \cdots \wedge p_n, \neg p_1 \wedge \cdots \wedge \neg p_n\}$. A forgetting-based approach would produce 2^n minimal repairs, since each variable p_i could be individually forgotten to resolve the contradiction between p_i and $\neg p_i$ (this involves replacing either p_i or $\neg p_i$ with $\top$). In contrast, our approach yields a single MCR $\sim$ with $Occ(K)/\sim = \{\{p_i\}, \{\neg p_i\} : i = 1, \ldots, n\}$.

Given a PB K, an equivalence relation $\sim$ on $Occ(K)$, and p in $\mathsf{var}(K)$, $\mathsf{EqC}(p, \sim)$ denotes the set of equivalence classes in $Occ(K)/\sim$ that contain an occurrence of p.

Note that a variable can be associated with at most two equivalence classes in an MCR. This mainly arises from the observation that in any model ω of $\mathsf{R}(K)$, each occurrence is associated with one of two possible truth values, 0 and 1.

Proposition 6. *Let K be a PB. For every MCR $\sim$ of K and every variable $p \in \mathsf{var}(K)$, the cardinality of $\mathsf{EqC}(p, \sim)$ is at most two, i.e., $|\mathsf{EqC}(p, \sim)| \leq 2$.*

Consider the following complementary definition of Minimality-2:

- (Maximality-2) for every MCR $\sim'$ of K, $PN(\sim)$ is not a proper subset of $PN(\sim')$.

The goal in the foregoing condition is to maximize the equivalence between positive and negative occurrences. This is grounded in the understanding that the essence of conflicts mainly arises from the interactions between positive and negative occurrences.

Definition 4 (BMCR). *A BMCR of a PB K is an MCR $\sim$ of K that satisfies Maximality-2.*

In the definition of MIR, we demonstrated that incorporating Minimality-2 does not affect the concept, as replacing Minimality with Minimality-2 yields the same notion. However, the following example illustrates that MCRs are not always BMCRs.

Example 4. *Consider the PB $K_2 = \{p, \neg p, \neg p \vee q\}$. It admits two MCRs $\sim_1^c$ and $\sim_2^c$, where $Occ(K_2)/\sim_1^c = \{\{p_1^+, p_3^-\}, \{p_2^-\}, \{q_1^+\}\}$ and $Occ(K_2)/\sim_2^c = \{\{p_1^+\}, \{p_2^-, p_3^-\}, \{q_1^+\}\}$. The unique BMCR is $\sim_1^c$ because $PN(\sim_1^c) = \{(p_1^+, p_3^-)\}$ and $PN(\sim_2^c) = \emptyset$.*

The minimal hitting set duality between MISes and the complements of MCSes asserts that every MIS is a minimal hitting set of the set of all complements of MCSes, and vice versa (e.g., see (Reiter 1987; Bailey and Stuckey 2005; Liffiton et al. 2005)). This property is particularly useful for computing all MISes and MCSes. Here, we show that a similar duality property exists between MIRs and MCRs.

Let U be a set of elements and $S = \{S_1, \ldots, S_k\}$ a collection of subsets of U. A *hitting set* of S is a set $H \subseteq U$ that intersects with every element of S, i.e., for every $S_i \in S$,

$S_i \cap H \neq \emptyset$. A hitting set is said to be *minimal* if no proper subset of it can also serve as a hitting set.

To establish the duality properties, we need some preliminary notions.

Definition 5 (H-Maximality)**.** *Let K be a PB and $H \subseteq \sim_c^K$. An equivalence relation $\sim$ on $Occ(K)$, where $\sim \subseteq \sim_c^K$, is said to be H-maximal if it meets the following conditions: (i) $\sim \cap H = \emptyset$, and (ii) for any equivalence relation $\sim'$ on $Occ(K)$ with $\sim' \subseteq \sim_c^K$ and $\sim \subsetneq \sim'$, $\sim' \cap H \neq \emptyset$.*

Definition 6 (H-Minimality)**.** *Let K be a PB and $H \subseteq \sim_c^K$. An equivalence relation $\sim$ on $Occ(K)$, where $\sim \subseteq \sim_c^K$, is said to be H-minimal if it meets the following conditions: (i) $H \subseteq \sim$, and (ii) for any equivalence relation $\sim'$ on $Occ(K)$ with $\sim' \subsetneq \sim$, $H \not\subseteq \sim'$.*

Alternatively stated, an equivalence relation $\sim \subseteq \sim_c^K$ is H-maximal if and only if it excludes all elements of H and is maximal with respect to set inclusion. The relation $\sim$ is H-minimal if and only if it includes all elements of H and is minimal with respect to set inclusion.

Definition 7 (C-MCR)**.** *A C-MCR of a PB K is a relation θ on $Occ(K)$ such that $\sim_c^K \setminus \theta$ is an MCR.*

Let $\mathsf{CMCRs}(K)$ represent the set of all C-MCRs of K.

For simplicity, the duality theorem references MCRs in one property and C-MCRs in the other.

Theorem 3. *Let K be a PB and $\sim$ an equivalence relation on $Occ(K)$ s.t. $\sim \subseteq \sim_c^K$. Then, the following properties hold:*

1. *$\sim$ is an MCR of K iff there exists a minimal hitting set H of $\mathsf{MIRs}(K)$ such that $\sim$ is H-maximal.*
2. *$\sim$ is an MIR iff there exists a minimal hitting set H of $\mathsf{CMCRs}(K)$ such that $\sim$ is H-minimal.*

Proof. We provide a proof for Property 1 only, as the other is supported by a symmetrical proof. In this proof, $(R)^*$ represents the transitive and symmetric closure of R.

We begin by proving the *if* part. Let H be a minimal hitting set of $\mathsf{MIRs}(K)$ s. t. $\sim$ is an H-maximal equivalence relation. Note that $(o, o) \notin H$ for any occurrence o; otherwise, Property (i) from Definition 5 would be violated, as $\sim \cap H \neq \emptyset$. Assume, for contradiction, that $\sim$ does not satisfy Consistency. This would imply that $\mathsf{R}(K) \wedge \bigwedge_{(o,o') \in \sim}(\mathsf{R}(o) \leftrightarrow \mathsf{R}(o'))$ is inconsistent, meaning there exists an MIR $\sim'$ of K such that $\sim' \subseteq \sim$. This leads to a contradiction since $\sim \cap H = \emptyset$ and there exists $(o, o') \in H \cap \sim'$. Thus, $\sim$ satisfies Consistency. Now, suppose $\sim$ does not satisfy Maximality. This implies that there exists $(o, o') \in \sim_c^K \setminus \sim$ s. t. $(\sim \cup (o, o'))^* \cap H = \emptyset$, resulting in a contradiction since $\sim$ is H-maximal.

Next, we prove the *only if* part. Assume $\sim$ is an MCR of K. Let $H = \sim_c^K \setminus \sim$ (the C-MCR associated with $\sim$). Suppose H is not a hitting set of $\mathsf{MIRs}(K)$. Then there exists an MIR $\sim'$ of K such that $\sim' \subseteq \sim$, which leads to a contradiction since $\sim$ does not include any MIR. Let H' be an arbitrary minimal hitting set of $\mathsf{MIRs}(K)$ s. t. $H' \subseteq H$. Suppose $\sim$ is not H'-maximal. Then there exists $(o, o') \in H$ s.t. $(\sim \cup \{(o, o')\})^*$ does not intersect with H'. Therefore, $(\sim \cup (o, o'))^*$ satisfies Consistency since it does not include any MIR, leading to a contradiction with Maximality. □

MCR-based Inference Relations

Similar to how MCSes are used to define inference relations, new non-explosive inference relations can be established using (B)MCRs. The key idea is to restore consistency by modifying the propositional base according to each MCR. This involves assigning a distinct variable to each equivalence class. Following this assignment, we apply the classical inference relation to derive conclusions.

The main advantage of our approach compared to those based on MCSes is that it retains every formula, including inconsistent ones, and every variable when employing the classical inference relation. In particular, our inference relations, unlike those based on MCSes, do not distinguish between a base and its corresponding formula.

We define an *inference relation* (IR) as a binary relation between PBs and formulas. An IR is considered *more cautious* than another if it is not identical to the other, and every conclusion entailed by the first relation can also be entailed by the second (Pinkas and Loui 1992).

Considering an MCR $\sim$ of K, we use the following preliminary notions and notations:

- A $\sim$-renaming function is a function $\rho_\sim$ that assigns a distinct propositional variable to each equivalence class in $Occ(K)/\sim$, where for every $p \in \mathsf{var}(K)$, if C is an equivalence class of $\sim$ containing all occurrences of p, then $\rho_\sim(C) = p$. For an occurrence o, we often write $\rho_\sim(o)$ to denote $\rho_\sim([o])$.
- $\rho_\sim(K)$ represents a PB constructed by replacing each variable occurrence o with $\rho_\sim(o)$.
- $\lceil \rho_\sim \rceil$ is a function that maps each variable p to the set $\{\rho_\sim(o) : o \in Occ(p, K)\}$.
- For a tuple of distinct variables $S = (p_1, \ldots, p_m)$, we define $\mathsf{P}(\rho_\sim, S)$ as the set of tuples $\lceil \rho_\sim \rceil(p_1) \times \cdots \times \lceil \rho_\sim \rceil(p_m)$.

The choice of the $\sim$-renaming function does not impact any of our IRs. Thus, for every MCR $\sim$, we assume that this function is fixed and denoted $\rho_\sim$.

Example 5. *Consider again the PB $K_2 = \{p, \neg p, \neg p \vee q\}$ from Example 4. Renaming functions associated with $\sim_1^c$ and $\sim_2^c$ can be defined as follows:*

- $\rho_{\sim_1^c} = \{\{p_1^+, p_3^-\} \mapsto x_1, \{p_2^-\} \mapsto x_2, \{q_1^+\} \mapsto q\}$
- $\rho_{\sim_2^c} = \{\{p_1^+\} \mapsto x_1, \{p_2^-, p_3^-\} \mapsto x_2, \{q_1^+\} \mapsto q\}$.

We obtain $\rho_{\sim_1^c}(K_2) = \{x_1, \neg x_2, \neg x_1 \vee q\}$ and $\rho_{\sim_2^c}(K) = \{x_1, \neg x_2, \neg x_2 \vee q\}$. Moreover, we have $\lceil \rho_{\sim_1^c} \rceil = \lceil \rho_{\sim_2^c} \rceil = \{p \mapsto \{x_1, x_2\}, q \mapsto \{q\}\}$. Finally, $\mathsf{P}(\rho_{\sim_1^c}, (p, q)) = \mathsf{P}(\rho_{\sim_2^c}, (p, q)) = \{(x_1, q), (x_2, q)\}$.

We now introduce four MCR-based IRs:

- $K \mathrel{\vdash\!\!\circleddash}_1 \phi$ iff for every MCR $\sim$ of K, there exists a tuple $(q_1, \ldots, q_m) \in \mathsf{P}(\rho_\sim, S)$ such that $\rho_\sim(K) \vdash \phi[p_1/q_1, \ldots, p_m/q_m]$.
- $K \mathrel{\vdash\!\!\circleddash}_2 \phi$ iff for every MCR $\sim$ of K and for every tuple $(q_1, \ldots, q_m) \in \mathsf{P}(\rho_\sim, S)$, $\rho_\sim(K) \vdash \phi[p_1/q_1, \ldots, p_m/q_m]$.

- $K \mid\!\ominus_1^B \phi$ iff for every BMCR $\sim$ of K, there exists a tuple $(q_1, \ldots, q_m) \in \mathsf{P}(\rho_\sim, S)$ such that $\rho_\sim(K) \vdash \phi[p_1/q_1, \ldots, p_m/q_m]$.
- $K \mid\!\ominus_2^B \phi$ iff for every BMCR $\sim$ of K and for every tuple $(q_1, \ldots, q_m) \in \mathsf{P}(\rho_\sim, S)$, $\rho_\sim(K) \vdash \phi[p_1/q_1, \ldots, p_m/q_m]$.

Here, $S = (p_1, \ldots, p_m)$ represents a tuple of distinct variables such that $\mathsf{var}(K) \cap \mathsf{var}(\phi) = \{p_1, \ldots, p_m\}$.

The relation $\mid\!\ominus_1$ asserts that a formula ϕ is a consequence of K if, for every MCR $\sim$, the PB derived from K by applying a renaming (which assigns a distinct variable to each equivalence class of $\sim$) classically entails at least one version of ϕ that is renamed in a similar way. In the case of $\mid\!\ominus_2$, we require that all versions of ϕ be entailed by the PB after renaming. The relations $\mid\!\ominus_1^B$ and $\mid\!\ominus_2^B$ are analogous to $\mid\!\ominus_1$ and $\mid\!\ominus_2$, respectively, but use BMCRs instead of MCRs.

Example 6. *Returning to Example 5, $K_2 \mid\!\ominus_1 p$ and $K_2 \mid\!\ominus_1^B p$ hold because $\rho_{\sim_1^c}(K_2) \vdash x_1$ and $\rho_{\sim_2^c}(K_2) \vdash x_1$. However, $K_2 \not\mid\!\ominus_2 p$ and $K_2 \not\mid\!\ominus_2^B p$ hold since $\rho_{\sim_1^c}(K_2) \nvdash x_2$ and $\rho_{\sim_2^c}(K_2) \nvdash x_2$. Additionally, both $K_2 \mid\!\ominus_1^B q$ and $K_2 \mid\!\ominus_2^B q$ hold because $\rho_{\sim_1^c}(K_2) \vdash q$, and $\sim_1^c$ is the unique BMCR of K_2. Using $\rho_{\sim_1^c}(K_2) \nvdash q$, we obtain $K_2 \not\mid\!\ominus_1 q$ and $K_2 \not\mid\!\ominus_2 q$.*

Note that while our analysis focuses on a limited set of principles to define IRs, our approach can be adapted to establish numerous other IRs by employing principles similar to those used in the case of MCSes (e.g., see (Pinkas and Loui 1992)). For instance, one can derive alternative IRs by considering conclusions derived from at least one MCR or from preferred MCRs, taking into account different preference criteria, such as prioritizing the largest MCRs.

Since the unique (B)MCR of a consistent PB K is $\sim_c^K$, we deduce that our four IRs coincide with the classical IR in the case of consistent PBs.

Proposition 7. *For every consistent PB K and every formula ϕ, the following properties are equivalent: $K \vdash \phi$, $K \mid\!\ominus_1 \phi$, $K \mid\!\ominus_2 \phi$, $K \mid\!\ominus_1^B \phi$, and $K \mid\!\ominus_2^B \phi$.*

Next, we examine the relationships between the introduced IRs. We can clearly see the following: $\mid\!\ominus_2 \subseteq \mid\!\ominus_1$, $\mid\!\ominus_2^B \subseteq \mid\!\ominus_1^B$, $\mid\!\ominus_1 \subseteq \mid\!\ominus_1^B$, and $\mid\!\ominus_2 \subseteq \mid\!\ominus_2^B$. Further, using the properties related to the conclusion p in Example 6, we establish that $\mid\!\ominus_2 \subsetneq \mid\!\ominus_1$, $\mid\!\ominus_2^B \subsetneq \mid\!\ominus_1^B$, and $\mid\!\ominus_1 \not\subseteq \mid\!\ominus_2^B$. By considering the properties related to the conclusion q, we find that $\mid\!\ominus_1 \subsetneq \mid\!\ominus_1^B$, $\mid\!\ominus_2 \subsetneq \mid\!\ominus_2^B$, and $\mid\!\ominus_2^B \not\subseteq \mid\!\ominus_1$. Consequently, $\mid\!\ominus_2$ is more cautious than the other three IRs, whereas $\mid\!\ominus_1^B$ is less cautious than the remaining three IRs.

Now, we exhibit some relationships between our IRs and Priest's Minimally Inconsistent Logic of Paradox (LP_m) (Priest 1991).

An LP_m interpretation is a function λ that assigns a value in $\{\{0\}, \{1\}, \{0, 1\}\}$ to each formula in PF and meets the following conditions: $\lambda(\neg\phi) = \{1 - v : v \in \lambda(\phi)\}$ and $\lambda(\phi \wedge \psi) = \{v \times v' : v \in \lambda(\phi), v' \in \lambda(\psi)\}$. We use $\lambda!$ to denote the set of variables p such that $\lambda(p) = \{0, 1\}$.

We say that λ is an LP_m model of ϕ, written $\lambda \models_{LP_m} \phi$, if and only if $1 \in \lambda(\phi)$.

An LP_m model of a formula ϕ is said to be *minimal* iff, for any LP_m model λ' of ϕ, it does not hold that $\lambda'! \subsetneq \lambda!$.

A PB K entails ϕ in LP_m, written $K \vdash_{LP_m} \phi$, iff for every minimal LP_m model λ of $\bigwedge K$, $\lambda \models_{LP_m} \phi$ holds.

First, we have $\{p, \neg p\} \not\mid\!\ominus_1^B p \wedge \neg p$ and $\{p, \neg p\} \vdash_{LP_m} p \wedge \neg p$. Then, we have $\vdash_{LP_m} \not\subseteq \mid\!\ominus_1^B$, which implies $\vdash_{LP_m} \not\subseteq \mid\!\ominus_1$, $\vdash_{LP_m} \not\subseteq \mid\!\ominus_2$, and $\vdash_{LP_m} \not\subseteq \mid\!\ominus_2^B$.

Furthermore, both $\mid\!\ominus_1^B \not\subseteq \vdash_{LP_m}$ and $\mid\!\ominus_2^B \not\subseteq \vdash_{LP_m}$ hold. This is demonstrated by the fact that $\{p, \neg p, \neg p \vee q\} \mid\!\ominus_1^B q$ and $\{p, \neg p, \neg p \vee q\} \mid\!\ominus_2^B q$, whereas $\{p, \neg p, \neg p \vee q\} \nvdash_{LP_m} q$.

Given an LP_m model λ of K, $\sim_\lambda$ represents an equivalence relation on $Occ(K)$ defined by $Occ(K)/\sim_\lambda = \{Occ(p, K) : |\lambda(p)| = 1\} \cup \{PosOcc(p, K), NegOcc(p, K) : \lambda(p) = \{0, 1\}\}$.

The next proposition highlights that an MCR can be derived from every model in LP_m.

Proposition 8. *If λ is a minimal LP_m model of K, then $\sim_\lambda$ is an MCR of K.*

Considering that $\vdash_{LP_m} \not\subseteq \mid\!\ominus_1$, the following theorem shows that $\mid\!\ominus_1$ is more cautious than $\vdash_{LP_m}$.

Theorem 4. *For every PB K and every formula ϕ, if $K \mid\!\ominus_1 \phi$, then $K \vdash_{LP_m} \phi$.*

Occurrence-based Semantics

Building on the approach of differentiating between occurrences of the same variable to restore consistency, we introduce an unusual semantics that assigns truth values to the occurrences of variables rather than to the variables themselves. The entailment is established through Boolean interpretations that align with the occurrence-based models.

An *occurrence-based interpretation* (o-interpretation for short) of a formula ϕ is a function μ mapping each occurrence in $Occ(\phi)$ to either 0 or 1. We say that μ is an *o-model* of ϕ if and only if $\omega_\mu \models \mathsf{R}(\phi)$, where ω_μ is any boolean interpretation such that, for each $o \in Occ(\phi)$, $\omega_\mu(\mathsf{R}(o)) = \mu(o)$. An o-model of a PB K is an o-model of its corresponding formula $\bigwedge K$.

To define our IRs in the framework of occurrence-based semantics, we examine two minimality properties in o-models: a-minimality and b-minimality.

We denote by $\mathsf{diff}_a(\mu)$ the set of ordered pairs $\{(o, o') \in Occ(\phi) \times Occ(\phi) : \mathsf{var}(o) = \mathsf{var}(o'), \mu(o) \neq \mu(o')\}$. We then define the preorder relation $\preceq_a$ on the o-models of ϕ such that $\mu \preceq_a \mu'$ if and only if $\mathsf{diff}_a(\mu) \subseteq \mathsf{diff}_a(\mu')$. The corresponding strict preorder is denoted by $\prec_a$.

An o-model μ of ϕ is considered *a-minimal* if it is minimal with respect to $\preceq_a$, i.e., for any o-interpretation μ' of ϕ where $\mu' \prec_a \mu$, μ' is not an o-model of ϕ.

Observe that for every a-minimal o-model μ of a consistent formula, $\mathsf{diff}_a(\mu) = \emptyset$. This indicates that every a-minimal o-model in such cases can be regarded as a Boolean interpretation: all occurrences of each variable have the same truth value.

A relationship between MCRs and a-minimal o-models is established in the following proposition.

Proposition 9. *For every formula ϕ, if μ is an a-minimal o-model of ϕ, then $\sim_t^\phi \setminus \mathsf{diff}_a(\mu)$ is an MCR of ϕ.*

Given an MCR $\sim$ of ϕ and a model ω of $\Psi = \mathsf{R}(\phi) \wedge \bigwedge_{(o,o') \in \sim} (\mathsf{R}(o) \leftrightarrow \mathsf{R}(o'))$, we define μ_ω as an o-interpretation such that $\mu(o) = 1$ if and only if $\omega(\mathsf{R}(o)) = 1$. Then, the set $\mathsf{OM}(\sim)$ consists of the o-interpretations $\{\mu_\omega : \omega \in \mathsf{mod}(\Psi)\}$.

Proposition 10. *For every formula ϕ, if $\sim$ is an MCR of ϕ, then $\mathsf{OM}(\sim)$ is a set of a-minimal o-models of ϕ.*

We say that a Boolean interpretation ω is *compatible with an o-interpretation* μ of ϕ if and only if for every propositional variable p occurring in ϕ, there exists an occurrence o of this variable such that $\omega(p) = \mu(o)$.

We define the IRs $\mid\!\sim_{a1}$ and $\mid\!\sim_{a2}$ as follows:

- $K \mid\!\sim_{a1} \phi$ iff for each a-minimal o-model μ of K, there exists a Boolean interpretation ω that is compatible with μ such that $\omega \models \phi$.
- $K \mid\!\sim_{a2} \phi$ iff for each a-minimal o-model μ of ϕ, and for each Boolean interpretation ω which is compatible with μ, it follows that $\omega \models \phi$.

Using mainly Proposition 9, we derive the following property.

Proposition 11. *For every PB K and every formula ϕ, if $K \mid\!\sim_1 \phi$, then $K \vdash_{a1} \phi$.*

In fact, $\mid\!\sim_1$ is more cautious than $\mid\!\sim_{a1}$. We can illustrate this using $K = \{p, \neg p, q \vee r\}$ and $\phi = (\neg p \wedge (\neg q \vee \neg r)) \vee (p \wedge q \wedge r)$. Indeed, we have $K \mid\!\sim_{a1} \phi$ but $K \not\mid\!\sim_1 \phi$.

The distinction between $\mid\!\sim_{a1}$ and $\mid\!\sim_1$ mainly arises from the fact that each MCR $\mid\!\sim_1$ necessitates finding an adaptation of the conclusion that can be entailed, while $\mid\!\sim_{a1}$ requires an adaptation for each o-interpretation corresponding to an MCR. For instance, the previous PB K admits a single MCR $\sim$, defined by $Occ(K)/\sim = \{\{p_1^+\}, \{p_2^-\}, \{q_1^+\}, \{r_1^-\}\}$. It does not hold that $K \mid\!\sim_1 \phi$ because $\{x_1, \neg x_2, q \vee r\}$ does not entail any of the possible adaptations $(\neg x_1 \wedge (\neg q \vee \neg r)) \vee (x_1 \wedge q \wedge r)$ and $(\neg x_2 \wedge (\neg q \vee \neg r)) \vee (x_2 \wedge q \wedge r)$. We have three a-minimal o-models $\mu_1 = \{p_1^+ \mapsto 1, p_2^- \mapsto 0, q_1^+ \mapsto 1, r_1^- \mapsto 0\}$, $\mu_2 = \{p_1^+ \mapsto 1, p_2^- \mapsto 0, q_1^+ \mapsto 0, r_1^- \mapsto 1\}$ and $\mu_3 = \{p_1^+ \mapsto 1, p_2^- \mapsto 0, q_1^+ \mapsto 1, r_1^- \mapsto 1\}$. The conclusion ϕ can be derived using three respective interpretations ω_1, ω_2, and ω_3, which satisfy the following conditions: $\{p \mapsto 0, q \mapsto 1, r \mapsto 0\} \subseteq \omega_1$, $\{p \mapsto 0, q \mapsto 0, r \mapsto 1\} \subseteq \omega_2$, and $\{p \mapsto 1, q \mapsto 1, r \mapsto 1\} \subseteq \omega_3$.

Interestingly, the IR $\mid\!\sim_{a2}$ coincides with $\mid\!\sim_2$.

Proposition 12. *For every PB K and any formula ϕ, $K \mid\!\sim_{a2} \phi$ iff $K \vdash_2 \phi$.*

Given an LP_m interpretation λ and a PB K, we define μ_λ^K as the o-interpretation over $Occ(K)$ using the following criteria: for each occurrence $o = \langle p, \phi, i \rangle$, if $\lambda(p) = \{0\}$, then $\mu_\lambda^K(o) = 0$; if $\lambda(p) = \{1\}$, then $\mu_\lambda^K(o) = 1$; Otherwise, if $\lambda(p) = \{0, 1\}$, then $\mu_\lambda^K(o) = 0$ if o is a negative occurrence, and $\mu_\lambda^K(o) = 1$ if o is positive.

Proposition 13. *Let K be a PB and λ an LP_m interpretation s.t. $|\lambda(p)| = 1$ for every variable $p \notin \mathsf{var}(K)$. Then, λ is a minimal LP_m model of K iff μ_λ^K is an a-minimal o-model of K.*

The forgoing proposition leads to the following theorem.

Theorem 5. *For every PB K and any formula ϕ, if $K \mid\!\sim_{a1} \phi$ then $K \vdash_{LP_m} \phi$.*

Given the previous theorem and the fact that $\{p, \neg p\} \vdash_{LP_m} p \wedge \neg p$ and $\{p, \neg p\} \not\mid\!\sim_{a1} p \wedge \neg p$, we conclude that $\mid\!\sim_{a1}$ is more cautious than $\vdash_{LP_m}$.

IRs similar to $\mid\!\sim_1^B$ and $\mid\!\sim_2^B$ can also be defined in our framework of occurrence-based semantics. We use $\mathsf{diff}_b(\mu)$ to denote the set of ordered pairs $\{(o, o') \in PosOcc(\phi) \times NegOcc(\phi) : \mathsf{var}(o) = \mathsf{var}(o'), \mu(o) \neq \mu(o')\}$. We define a preorder relation $\preceq_b$ on the o-models of ϕ as follows: $\mu \preceq_b \mu'$ if and only if $\mathsf{diff}_b(\mu) \subseteq \mathsf{diff}_b(\mu')$. Its associated strict preorder is denoted by $\prec_b$.

An o-model μ of ϕ is said to be *b-minimal* if and only if it is a-minimal and, for any o-model μ' of ϕ, $\mu' \not\prec_b \mu$ holds.

The IRs $\mid\!\sim_{b1}$ and $\mid\!\sim_{b2}$ are defined in the same way as $\mid\!\sim_{a1}$ and $\mid\!\sim_{a2}$, respectively, but using b-minimal o-model instead a-minimal o-models.

The relation $\mid\!\sim_{b2}$ also coincides with $\mid\!\sim_2^B$. Moreover, since every b-minimal o-model is also a-minimal, it follows that $\mid\!\sim_{a1} \subseteq \mid\!\sim_{b1}$. Using $\{p, \neg p, \neg p \vee q\} \mid\!\sim_{b1} q$, $\{p, \neg p, \neg p \vee q\} \not\mid\!\sim_{a1} q$ and $\{p, \neg p, \neg p \vee q\} \nvdash_{LP_m} q$, we deduce $\mid\!\sim_{a1} \subsetneq \mid\!\sim_{b1}$ and $\mid\!\sim_{b1} \not\subseteq \vdash_{LP_m}$.

Let us summarize the cautiousness relationships:

- $\mid\!\sim_2 \subsetneq \mid\!\sim_1$, $\mid\!\sim_2 \subsetneq \mid\!\sim_2^B$, $\mid\!\sim_1 \not\subseteq \mid\!\sim_2^B$, $\mid\!\sim_2^B \not\subseteq \mid\!\sim_1$, $\mid\!\sim_1 \subsetneq \mid\!\sim_1^B$, $\mid\!\sim_2^B \subsetneq \mid\!\sim_1^B$, $\mid\!\sim_1 \subsetneq \vdash_{LP_m}$, $\mid\!\sim_2^B \not\subseteq \vdash_{LP_m}$, and $\vdash_{LP_m} \not\subseteq \mid\!\sim_1^B$.
- $\mid\!\sim_1 \subsetneq \mid\!\sim_{a1}$, $\mid\!\sim_{a2} = \mid\!\sim_2$, $\mid\!\sim_1^B \subsetneq \mid\!\sim_{b1}$, $\mid\!\sim_{b2} = \mid\!\sim_2^B$, $\mid\!\sim_{a1} \subsetneq \mid\!\sim_{b1}$, $\mid\!\sim_{a1} \not\subseteq \mid\!\sim_1^B$, $\mid\!\sim_2^B \not\subseteq \mid\!\sim_{a1}$, $\mid\!\sim_{a1} \subsetneq \vdash_{LP_m}$, and $\vdash_{LP_m} \not\subseteq \mid\!\sim_{b1}$.

Since all our IRs are grounded in the classic inference relation and Boolean interpretations, they exhibit the following properties: (1) all consequences derived from our IRs are consistent; (2) any formula equivalent to a derived consequence is itself also a consequence.

These two properties highlight the key distinctions between our approach and that of LP_m. In fact, apart from $\vdash_{LP_m}$ leading to contradictions, a result motivated by dialetheism, $\vdash_{LP_m}$ can also yield a consistent formula without ensuring that its equivalent formulas are consequences as well. For instance, we have $\{p, \neg p, \neg q\} \vdash_{LP_m} (p \vee q) \wedge \neg p$ but $\{p, \neg p, \neg q\} \nvdash_{LP_m} q \wedge \neg p$.

Conclusion and Perspectives

We presented a variable occurrence-based framework for resolving propositional inconsistencies. We introduced two complimentary concepts: Minimal Inconsistency Relations (MIRs) and Maximal Consistency Relations (MCRs). Using MCRs, we defined several non-explosive inference relations. We proposed further non-explosive inference relations using an occurrence-based semantics.

For future work, we aim to investigate computational problems associated with MIRs, MCRs, and our inference relations. Moreover, we intend to develop additional inference relations by focusing on specific MCRs. Our plans also include broadening our approach to include some non-classical logics.

References

Bailey, J.; and Stuckey, P. J. 2005. Discovery of Minimal Unsatisfiable Subsets of Constraints Using Hitting Set Dualization. In Hermenegildo, M. V.; and Cabeza, D., eds., *Practical Aspects of Declarative Languages, 7th International Symposium, PADL 2005, Long Beach, CA, USA, Proceedings*, volume 3350 of *Lecture Notes in Computer Science*, 174–186. Springer.

Benferhat, S.; Dubois, D.; and Prade, H. 1997. Some Syntactic Approaches to the Handling of Inconsistent Knowledge Bases: A Comparative Study Part 1: The Flat Case. *Studia Logica*, 58(1): 17–45.

Benferhat, S.; Dubois, D.; and Prade, H. 1999. Some syntactic approaches to the handling of inconsistent knowledge bases : A comparative study. Part 2 : the prioritized case. In *Logic at work: Essays Dedicated to the Memory of Helena Rasiowa*, volume 24 of *Studies in Fuzziness and Soft Computing*, 473–511. Physica-Verlag, Heidelberg.

Besnard, P. 2014. Revisiting Postulates for Inconsistency Measures. In Fermé, E.; and Leite, J., eds., *Logics in Artificial Intelligence - 14th European Conference, JELIA 2014, Funchal, Madeira, Portugal. Proceedings*, volume 8761 of *Lecture Notes in Computer Science*, 383–396. Springer.

Besnard, P. 2016. Forgetting-Based Inconsistency Measure. In Schockaert, S.; and Senellart, P., eds., *Scalable Uncertainty Management - 10th International Conference, SUM 2016, Nice, France. Proceedings*, volume 9858 of *Lecture Notes in Computer Science*, 331–337. Springer.

Besnard, P.; and Hunter, A. 2008. *Elements of Argumentation*. MIT Press.

Brewka, G. 1989. Preferred Subtheories: An Extended Logical Framework for Default Reasoning. In Sridharan, N. S., ed., *Proceedings of the 11th International Joint Conference on Artificial Intelligence. Detroit, MI, USA*, 1043–1048. Morgan Kaufmann.

Gärdenfors, P., ed. 1992. *Belief Revision*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.

Hunter, A.; and Konieczny, S. 2010. On the measure of conflicts: Shapley Inconsistency Values. *Artificial Intelligence*, 174: 1007–1026.

Konieczny, S.; Marquis, P.; and Vesic, S. 2019. Rational Inference Relations from Maximal Consistent Subsets Selection. In Kraus, S., ed., *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2019. Macao, China*, 1749–1755. ijcai.org.

Lang, J.; and Marquis, P. 2002. Resolving Inconsistencies by Variable Forgetting. In Fensel, D.; Giunchiglia, F.; McGuinness, D. L.; and Williams, M., eds., *Proceedings of the Eights International Conference on Principles and Knowledge Representation and Reasoning, KR'02. Toulouse, France*, 239–250. Morgan Kaufmann.

Liffiton, M. H.; Moffitt, M. D.; Pollack, M. E.; and Sakallah, K. A. 2005. Identifying Conflicts in Overconstrained Temporal Problems. In Kaelbling, L. P.; and Saffiotti, A., eds., *Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence, IJCAI 2005. Edinburgh, Scotland, UK*, 205–211. Professional Book Center.

Lin, F.; and Reiter, R. 1994. Forget It! In *Proceedings of the AAAI Fall Symposium on Relevance*, 154–159.

Pinkas, G.; and Loui, R. P. 1992. Reasoning from Inconsistency: A Taxonomy of Principles for Resolving Conflict. In Nebel, B.; Rich, C.; and Swartout, W. R., eds., *Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning, KR'92. Cambridge, MA, USA*, 709–719. Morgan Kaufmann.

Priest, G. 1991. Minimally inconsistent LP. *Studia Logica*, 50: 321–331.

Priest, G.; Tanaka, K.; and Weber, Z. 2022. Paraconsistent Logic. In Zalta, E. N., ed., *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2022 edition.

Reiter, R. 1987. A Theory of Diagnosis from First Principles. *Artificial Intelligence*, 32(1): 57–95.

Rescher, N.; and Manor, R. 1970. On inference from inconsistent premisses. *Theory and Decision*, 1: 179–217.

Tanaka, K.; Berto, F.; Mares, E. D.; and Paoli, F., eds. 2013. *Paraconsistency: Logic and Applications*, volume 26 of *Logic, Epistemology, and the Unity of Science*. Springer.

Thimm, M. 2018. On the Evaluation of Inconsistency Measures. In Grant, J.; and Martinez, M. V., eds., *Measuring Inconsistency in Information*, volume 73 of *Studies in Logic*. College Publications.