

On the Convergence of Tâtonnement for Linear Fisher Markets

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Abstract

Tâtonnement is a simple, intuitive market process where prices are iteratively adjusted based on the difference between demand and supply. Many variants under different market assumptions have been studied and shown to converge to a market equilibrium, in some cases at a fast rate. However, the classical case of linear Fisher markets have long eluded the analyses, and it remains unclear whether tâtonnement converges in this case. We show that, for a sufficiently small step size, the prices given by the tâtonnement process are guaranteed to converge to equilibrium prices, up to a small approximation radius that depends on the stepsize. To achieve this, we consider the dual Eisenberg-Gale convex program in the price space, view tâtonnement as subgradient descent on this convex program, and utilize last-iterate convergence results for subgradient descent under error bound conditions. In doing so, we show that the convex program satisfies a particular error bound condition, the quadratic growth condition, and that the price sequence generated by tâtonnement is bounded above and away from zero. We also show that a similar convergence result holds for tâtonnement in quasi-linear Fisher markets. Numerical experiments are conducted to demonstrate that the theoretical linear convergence aligns with empirical observations.

1 Introduction

Market equilibrium (ME) is a central solution concept in economics. It describes a steady state of the market in which supply is equal to demand. In recent year, it has found various applications in resource allocation in large-scale online and physical market places due to its fairness, efficiency in allocating items, and tractability under various real-world-inspired settings. Since ME was first studied by Léon Walras in 1874, extensive work has been undertaken to establish the existence and uniqueness of ME in various market models (Walras 1874; Arrow and Debreu 1954; Cole et al. 2017). In this paper, we focus on the *Fisher market* setting, a well-known market model involving m divisible items which are to be allocated among n buyers. Each item has a fixed supply (usually assumed to be 1) and each buyer is endowed with a fixed budget of money. In a Fisher market, an equilibrium is a set of prices for the items, and an allocation of

items to buyers, such that each buyer is allocated the optimal bundle subject to the prices and her budget, and each item is allocated at its supply.

Alongside the development of market settings and equilibrium solution concepts, market equilibrium computation, by itself or as part of an end-to-end approach to real-world resource allocation problems, has seen a growing interest in various research communities in economics, computer science, and operations research (Scarf et al. 1967; Kantorovich 1975; Daskalakis, Goldberg, and Papadimitriou 2009; Bateni et al. 2022; Babaioff, Nisan, and Talgam-Cohen 2019; Othman, Sandholm, and Budish 2010; Cole et al. 2017; Birnbaum, Devanur, and Xiao 2011; Kroer et al. 2019, 2021; Gao and Kroer 2020; Gao, Peysakhovich, and Kroer 2021). This is especially so in the past two decades, as the emergence of internet marketplaces, including advertising systems, require approximating or maintaining equilibrium variables in large-scale market settings.

Tâtonnement is a simple, natural, and decentralized price-adjustment process where prices of items are adjusted to reflect buyers' demands: the price of a good increases if the demand exceeds the supply under current prices, and decreases if the supply exceeds the demand. It can also be viewed as an algorithm for computing market equilibria. Intuitively, if the tâtonnement process is guaranteed to converge to a market equilibrium, then it lends credence to market equilibrium as a solution concept for the economic allocation of goods. Indeed, for various market settings, including Fisher markets with non-linear *constant-elasticity-of-substitution* (CES) utilities, the convergence of tâtonnement has been established and is well-understood (Cole and Fleischer 2008; Cheung, Cole, and Devanur 2019; Avigdor-Elgrabli, Rabani, and Yadgar 2014). A prominent exception is the simplest, and arguably most prevalent, setting of linear Fisher markets (LFM), where buyers have linear (additive) utilities over items. It has long been known that, for LFMs, tâtonnement does not converge to an equilibrium in the strictest sense, that is, in terms of a vanishing metric or norm; see, e.g., Cole and Tao (2019) and Section 4 for further discussion and examples. However, whether tâtonnement demonstrates any form of *approximate* convergence remains largely unexplored.

One important fact about an LFM is that its market equilibria are captured by the well-known Eisenberg-Gale (EG)

convex program:

$$\max_{x \in \mathbb{R}^{n \times m}} \sum_{i=1}^n B_i \log \langle v_i, x_i \rangle \quad \text{s.t.} \quad \sum_{i=1}^n x_i \leq \mathbf{1}, \quad (\text{EG-Primal})$$

where $B_i \in \mathbb{R}$, $v_i \in \mathbb{R}^m$, $x_i \in \mathbb{R}^m$ denotes budget, valuation vector, and allocation vector for buyer i , respectively, in an LFM composed of n buyers and m items. It turns out that tâtonnement in the LFM is equivalent to the subgradient descent method applied to the dual program of Eq. (EG-Primal) (See, e.g., Cheung, Cole, and Devanur (2013)). This can be shown as follows: by taking the Lagrangian of Eq. (EG-Primal), we obtain that the dual problem to EG is to minimize the following function

$$\begin{aligned} \varphi(p) &= \max_{x \geq 0} \mathcal{L}(x, p) \\ &= \max_{x \geq 0} \sum_{i=1}^n \left(B_i \log \langle v_i, x_i \rangle - \sum_{j=1}^m p_j x_{ij} \right) + \sum_{j=1}^m p_j, \end{aligned} \quad (1)$$

which is a max-type nonsmooth convex function w.r.t. p by construction. Every subgradient of $\varphi(p)$ is of the form $\mathbf{1} - \sum_i x_i^*(p)$, where $x_i^*(p)$ is a maximizer of $\max_{x_i \geq 0} (B_i \log \langle v_i, x_i \rangle - \sum_{j=1}^m p_j x_{ij})$. By first-order optimality conditions, for any p , $x_i^*(p)$ is the demand of buyer i under p . Thus, tâtonnement (in the form of $p^{t+1} \leftarrow p^t + \eta(\sum_i x_i^*(p^t) - \mathbf{1})$, where $\eta \in \mathbb{R}_{>0}$) is equivalent to applying subgradient descent on $\min_{p \in \mathbb{R}_{\geq 0}^m} \varphi(p)$. We will use this relation to establish a linear convergence rate for tâtonnement in LFMs.

Contributions. We consider a classic form of the tâtonnement process in LFMs, which updates prices as follows:

$$p^{t+1} \leftarrow p^t + \eta z^t, \quad \forall t, \quad (2)$$

where z denotes excess demand at time t . We give the first positive answers to the question of whether tâtonnement converges in LFMs: we show that the prices generated by the process converge at a fast linear rate (i.e. exponentially fast) to a small neighborhood around the true equilibrium prices. Moreover, we show that the size of this neighborhood shrinks to 0 as the stepsize η goes to 0. Thus, for any approximation level $\epsilon > 0$, we can find an $\eta > 0$ such that tâtonnement converges linearly to an ϵ -approximate equilibrium. Our results are achieved by leveraging the connection between tâtonnement and subgradient methods, and from there showing that the dual of the EG convex program satisfies the quadratic growth condition around its optimum. Moreover, we must then show bounded subgradients, which we establish by showing that the tâtonnement process is guaranteed to be bounded above and away from zero for its entire trajectory, assuming a sufficiently small stepsize. As is explained in Section 2, many existing works on tâtonnement use some modified version of the process where prices are generally kept away from zero either through projection to a region bounded away from zero, or by artificially upper-bounding excess demands. Our results show that for LFMs, such modifications are not necessary as long as stepsizes

are sufficiently small. Finally, we extend our analysis framework to an important variant of LFMs: quasi-linear Fisher markets. For such markets, we again show that tâtonnement achieves a linear convergence rate to approximate equilibrium.

Notation. Throughout this paper, we use $\|\cdot\|$ to denote Euclidean distance and $\langle \cdot, \cdot \rangle$ to denote the standard inner product. We use $\Pi_{\mathcal{X}}(x)$ to denote the Euclidean projection of x onto set \mathcal{X} . We use $\|\cdot\|_1$ to denote the ℓ_1 norm. We use Argmax to denote the set of values that reach the maximum. The subdifferential of a function f at x is denoted as $\partial f(x)$. We use ∇f to denote the differential of f if f is smooth. We denote $[m] = \{1, \dots, m\}$.

2 Related Work

Convergence of tâtonnement. Initially, economists studied the convergence of tâtonnement in continuous time. There, Arrow, Block, and Hurwicz (1959) proved that a continuous-time version of tâtonnement converges to ME for markets satisfying *weak gross substitutability* (WGS). In the remainder of the literature review, we focus on the discrete-time version of tâtonnement, which is the focus of our paper. Codenotti, McCune, and Varadarajan (2005) considered a discrete version of tâtonnement and proved that it converges to an approximate equilibrium in polynomial time for any exchange economy satisfying WGS. Though their tâtonnement updates are additive and similar to ours, they need to apply tâtonnement on a transformed market and use price projection in order to restrict prices to a region that is bounded and bounded away from zero. Secondly, they assume that demand is unique, which is not true for LFMs. Cole and Fleischer (2008) considered a multiplicative tâtonnement with artificially upper-bounded excess demands, and showed polynomial convergence for it on non-linear CES markets satisfying WGS ($0 \leq \rho < 1$). Cheung, Cole, and Rastogi (2012) extended this analysis to some non-WGS markets.

Cheung, Cole, and Devanur (2019) proved that the entropic tâtonnement is equivalent to generalized gradient descent with KL divergence, and thus is guaranteed to reach ϵ fraction of initial distance to ME measured in a potential function in $\mathcal{O}(\frac{1}{\epsilon})$ iterations for the Leontief Fisher market, and $\mathcal{O}(\log \frac{1}{\epsilon})$ iterations for the complementary CES Fisher market ($-\infty < \rho < 0$). Avigdor-Elgrabli, Rabani, and Yadgar (2014); Cheung, Cole, and Rastogi (2012) considered multiplicative tâtonnement and showed convergence to an approximate equilibrium for Fisher markets with nested non-linear CES-type utilities.

Cole and Tao (2019) studied convergence to approximate equilibrium for the LFM, using a modified entropic tâtonnement process. In particular, they considered the update rule $p^{t+1} \leftarrow p^t e^{\eta \min\{z^t, 1\}}$, where p^t and z^t denote the price and excess demand at time t . They showed a linear convergence rate under what they call a “large market assumption,” which requires that for buyers with linear or close-to-linear utilities, the spending (of budget) on a single item does not vary too much as prices change. This is unnatural for LFMs as the elasticity of demand can be infinity

Tâtonnement	Papers	Market Type	Result
Additive tâtonnement* $p_j^{t+1} = \Pi_{\Delta}(p_j^t + \eta z_j^t)$	Codenotti, McCune, and Varadarajan (2005)	Exchange market WGS, unique demand	Poly-time to approx ME
Multiplicative tâtonnement $p_j^{t+1} = p_j^t(1 + \eta z_j^t)$	Avigdor-Elgrabli, Rabani, and Yadgar (2014)	Nested CES-Leontief $\rho \in (-\infty, 0) \cup (0, 1)$	$\mathcal{O}(\frac{1}{\epsilon^3})$
Multiplicative tâtonnement* $p_j^{t+1} = p_j^t(1 + \eta \min\{z_j^t, 1\})$	Cole and Fleischer (2008)	Exchange market WGS	$\mathcal{O}(E \log \frac{1}{\epsilon})$
	Cheung, Cole, and Rastogi (2012)	CES ($-1 < \rho \leq 0$) Nested non-lin. CES	$\mathcal{O}(\log \frac{1}{\epsilon})$
Entropic tâtonnement* $p_j^{t+1} = p_j^t e^{\eta \min\{z_j^t, 1\}}$	Cheung, Cole, and Devanur (2019)	CES ($-\infty \leq \rho < 0$)	$\mathcal{O}(\log \frac{1}{\epsilon})$ $\mathcal{O}(\frac{1}{\epsilon})$ if $\rho = -\infty$
	Goktas, Zhao, and Greenwald (2023)	CCNH	$\mathcal{O}(\frac{1+E^2}{\epsilon})$
Entropic tâtonnement* $p_j^{t+1} = \Pi_{\Delta}(p_j^t e^{\eta \min\{z_j^t, 1\}})$	Cole and Tao (2019)	Linear Large market assum.	Linear conv. to approx ME
Additive tâtonnement $p_j^{t+1} = p_j^t + \eta z_j^t$	This work	Linear	Linear conv. to approx ME

Table 1: Comparison of results of different tâtonnement methods. The superscript * denotes a modified version of the update rule. Note that only the last two papers consider linear Fisher markets. Cole and Fleischer (2008) and Goktas, Zhao, and Greenwald (2023) consider a broader class of utilities, but their parameter E is infinity for linear utilities. $\Pi_{\Delta}(\cdot)$ denotes projection onto a bounded region. E denotes the upper bound of the elasticity of demand.

in the LFM. Moreover, their quality of the approximation is proportional to a particular squared parameter in their assumption, while our results do not even require any such assumption.

Goktas, Viqueira, and Greenwald (2021) studied the dual of an Eisenberg-Gale-like convex program from the perspective of the expenditure minimization problem (EMP). Goktas, Zhao, and Greenwald (2023) showed an $\mathcal{O}((1 + E^2)/T)$ convergence for an entropic tâtonnement process on Fisher markets with *concave, continuous, nonnegative, and 1-homogeneous*¹ (CCNH) utilities, where E denotes the upper bound of the elasticity of demand. Note that in the case of LFM, the elasticity of demand is unbounded, and thus this result gives an infinite upper bound.

Fleischer et al. (2008) study the convergence of the *averaged* iterates generated by a tâtonnement-like algorithm in a class of Fisher markets that include linear, Leontief, and CES utilities. Note that we focus on the *last* iterate, which is the only type of convergence that allows us to say whether the day-to-day prices are converging. Secondly, last-iterate convergence implies average-iterate convergence.

We summarize the above related work on the convergence of tâtonnement in Table 1. The table provides an overview of the different tâtonnement methods considered in the literature, along with a high-level description of convergence rates.

Convergence of subgradient descent methods. Subgradient descent methods (SubGDs) have been extensively studied since their introduction in the 1960s. Historically,

¹A utility function $u : \mathbb{R}_{>0}^m \rightarrow \mathbb{R}_{\geq 0}$ is α -homogeneous if $u(ax) = a^\alpha u(x)$ for any $a \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}_{>0}^m$.

SubGD was only known to have an $\mathcal{O}(1/\sqrt{T})$ convergence rate for the averaged iterate, where T is the number of iterations. Moreover, this convergence was in terms of the optimality gap, and not in terms of the distance to the optimal solution. For tâtonnement, we are interested in the latter, as we want to know how quickly the prices converge to the market equilibrium prices. Recently, Zamani and Glineur (2023) showed a nearly $\mathcal{O}(1/\sqrt{T})$ optimal rate for the *last* iterate, but again only in terms of optimality gap. It is known, however, that SubGD can achieve a faster convergence rate when the objective satisfies various notions of *error-bound* conditions (along with a bounded gradient condition). For prior work on SubGDs under error-bound conditions, see Johnstone and Moulin (2020) and references therein. Particularly relevant to us, SubGD is known to have linear convergence under the *quadratic growth condition* (QG) (Nedić and Bertsekas 2001; Karimi, Nutini, and Schmidt 2016), in terms of the distance to the optimal solution. This condition also allows one to connect convergence in optimality gap to convergence in iterates. We will show that this condition holds for LFMs, and thus SubGD has linear convergence in our setting.

3 Linear Fisher Markets and Tâtonnement

Linear Fisher markets. We consider a linear Fisher market with n buyers and m divisible items. Each buyer i has a budget $B_i > 0$ and valuation $v_{ij} \geq 0$ for each item j . Denote $B = \sum_{i \in [n]} B_i$ and $B_{\min} = \min_{i \in [n]} B_i$. We denote $v_i = (v_{i1}, \dots, v_{im})$ for all i and v as the valuation matrix whose i -th row is v_i . An *allocation* (or *bundle*) of items $x_i \in \mathbb{R}_{\geq 0}^m$ gives buyer i a utility of $u_i(x_i) = \langle v_i, x_i \rangle$. Without loss of generality, we assume the following:

- There is a unit supply of each item j .
- $\|v_i\|_1 = 1$ for all i .
- $v \in \mathbb{R}_{\geq 0}^{n \times m}$ does not have all-zero rows or columns. In other words, each item j is wanted by at least one buyer, and each buyer i wants at least one item.

Given a set of prices $p \in \mathbb{R}_{\geq 0}^m$ for the items, the *demand set* of buyer i is defined as

$$\mathcal{D}_i(p) := \underset{x_i}{\text{Argmax}} \{ \langle v_i, x_i \rangle : \langle p, x_i \rangle \leq B_i, x_i \geq 0 \}. \quad (3)$$

We use $d_i(p) \in \mathcal{D}_i(p)$ to denote an arbitrary demand vector of buyer i under p . We use $z_j(p) = \sum_{i \in [n]} d_{ij}(p) - 1$ to denote the *excess demand* for item j under p . The *excess supply* is $-z_j(p)$.

In the LFM (or its variants), buyers only buy items with *maximum bang-per-buck* (MBB). We call such items the MBB items for a buyer, i.e., j is an MBB item for buyer i if $\frac{v_{ij}}{p_j} \geq \frac{v_{ij'}}{p_{j'}}$ for all $j' \in [m]$.

A Fisher market is said to reach a *market equilibrium* if allocations $x_i^* \in \mathbb{R}_{\geq 0}^m$, $\forall i \in [n]$ and prices $p^* \in \mathbb{R}_{\geq 0}^m$ satisfy

- Buyer optimality: $x_i^* \in \mathcal{D}_i(p^*)$ for all $i \in [n]$,
- Market clearance: $\sum_{i=1}^n x_{ij}^* = 1$ for all $j \in [m]$.

Tâtonnement. We consider the classic, discrete-time tâtonnement dynamics that adjust prices at time t as follows:

- Pick $x_i^t \in \mathcal{D}_i(p^t) \quad \forall i \in [n]$
- $z_j^t = \sum_{i=1}^n x_{ij}^t - 1 \quad \forall j \in [m]$ (4)
- $p_j^{t+1} = p_j^t + \eta^t z_j^t \quad \forall j \in [m]$.

Typically, many authors have considered some form of projection to ensure $p^{t+1} \geq 0$, or even such that it is strictly bounded away from zero. We will show that for LFMs, it is possible to select η^t such that the prices are ensured to be strictly bounded away from zero without projection. This is critical for our later convergence results. We will focus on the case where stepsizes are constant, i.e. $\eta^t = \eta$ for some $\eta > 0$.

As shown in Section 1, Eq. (4) is equivalent to applying subgradient descent method on the following non-smooth convex minimization problem:

$$\min_{p \in \mathbb{R}_{\geq 0}^m} \varphi(p). \quad (5)$$

In particular, starting with some feasible point $p^0 \in \mathbb{R}_{\geq 0}^m$, at time t , we update p as follows:

- Pick $g(p^t) \in \partial \varphi(p^t)$
- $p^{t+1} = \Pi_{\mathbb{R}_{\geq 0}^m}(p^t - \eta g(p^t)).$ (6)

4 Convergence of Tâtonnement

In this section, we establish our main result: showing that the tâtonnement process converges to an arbitrarily small neighborhood of the equilibrium price vector at a linear rate, where the size of the neighborhood is determined by the stepsize. To do this, we show that the corresponding SubGD guarantees such a linear convergence. Specifically,

we first show that the prices generated by the tâtonnement dynamics are lower bounded by positive constants if stepsizes are small enough (without any other modifications to tâtonnement). Because prices are bounded away from zero, the demands are all upper bounded, and in turn this implies that tâtonnement is equivalent to SubGD with bounded subgradients. This is crucial, because bounded subgradients are required in all last-iterate convergence results for SubGD that we are aware of. Next, we establish a quadratic growth (QG) property of the non-smooth EG dual objective function. Finally, we obtain the desired linear convergence by a standard analysis for SubGD under QG and bounded subgradients. Then, we show an example where the prices cycle in a small neighborhood of the equilibrium prices, exactly as predicted by our theory once we hit the neighborhood implied by the stepsize.

Price lower bounds. The expression for $\varphi(p)$ in Eq. (1) can be simplified into $\sum_{j=1}^m p_j - \sum_{i=1}^n B_i \log(\min_j \frac{p_j}{v_{ij}})$ up to a constant, which yields the following dual formulation of Eq. (EG-Primal):

$$\min_{p \in \mathbb{R}_{\geq 0}^m} \varphi(p) = \sum_{j=1}^m p_j - \sum_{i=1}^n B_i \log\left(\min_{k \in [m]} \frac{p_k}{v_{ik}}\right). \quad (7)$$

Here, the two expressions for φ differ only by a constant. As a result, tâtonnement in an LFM is equivalent to SubGD on $\min_{p \in \mathbb{R}_{\geq 0}^m} \varphi(p)$. In Appendix A, we show how to directly derive this well-known equivalence in terms of $\varphi(p)$.

As we mentioned, to ensure that the tâtonnement process is well-defined and converges to a market equilibrium, some past work (e.g. Cole and Tao (2019)) assumed an artificial lower bound on prices. This is to ensure that we do not encounter unbounded demands (which occur as prices tend to zero). Next, we show that such artificial lower bounds are not necessary: the tâtonnement process itself guarantees that prices are bounded away from zero, as long as our stepsize is not too large. This implies a global upper bound on demands at each step. For the proof of this result and other omitted proofs in this section, see Appendix B.

Lemma 1. *Let \tilde{p} be an m -dimensional vector where*

$$\tilde{p}_j = \frac{B_{\min}}{4m} \max_i \frac{v_{ij}}{\|v_i\|_{\infty}}, \quad \text{for all } j \in [m]. \quad (8)$$

Assume that we adjust prices in the LFM with Eq. (4), starting from an initial price vector $p^0 \geq \tilde{p}$, with any stepsize $\eta < \frac{1}{2m} \min_j \tilde{p}_j$. Then, we have for all $t \geq 0$,

- (i) $p_j^t \geq \tilde{p}_j - 2m\eta$ for all $j \in [m]$;
- (ii) $\|z(p^t)\| \leq \frac{B}{\min_j \tilde{p}_j - 2m\eta} + m$;
- (iii) $p_j^t \leq \left(1 + \frac{\eta}{\tilde{p}_j - 2m\eta}\right) B$ for all $j \in [m]$.

Note that demands in LFMs are non-unique for a given set of prices, and thus there are multiple trajectories that the prices can take. Our analysis does not impose any assumptions on the choices of demands and works for all possible trajectories. We also show that the lower bound on price in Lemma 1 is tight, in terms of its order dependence on m .

Lemma 2. *There is an instance for which $p^t = \tilde{p} - (m-1)\eta$ for some item at some time step t .*

Quadratic growth of the dual EG objective. To guarantee a fast convergence rate for (sub)gradient descent methods, we need the objective function to satisfy some form of metric regularity condition. See Karimi, Nutini, and Schmidt (2016, Appendix A) for a comprehensive comparison of different types of regularity conditions. Here, we focus on the *strong convexity* (SC) and *quadratic growth* (QG) conditions. A differentiable convex function f is said to be μ -strongly convex if

$$f(y) \geq f(x) + \langle \nabla f(x), y-x \rangle + \frac{\mu}{2} \|y-x\|^2 \quad \forall x, y \in \text{dom } f.$$

A closed convex function φ is said to have the α -quadratic growth property if

$$\varphi(x) - \varphi(\Pi_{\mathcal{X}^*}(x)) \geq \alpha \|x - \Pi_{\mathcal{X}^*}(x)\|^2 \quad \forall x \in \text{dom } \varphi,$$

where \mathcal{X}^* is the set of the minima of φ . It is easy to see that μ -strong convexity implies $\frac{\mu}{2}$ -quadratic growth. Next, we show that $\varphi(p)$ in Eq. (7) satisfies QG.

Lemma 3. *The convex function*

$$\varphi(p) = \sum_{j=1}^m p_j - \sum_{i=1}^n B_i \log \left(\min_{k \in [m]} \frac{p_k}{v_{ik}} \right)$$

satisfies the quadratic growth condition with modulus $\alpha = \min_j \frac{p_j^*}{2(1 + \frac{\eta}{\bar{p}_j - 2m\eta})^2 B^2}$, where p^* denotes the unique equilibrium price vector, and \tilde{p} is defined in Lemma 1.

Proof. Let $\bar{p}_j = (1 + \frac{\eta}{\bar{p}_j - 2m\eta})B$ denote the price upper bound for item j . First, we construct an auxiliary function

$$h(p) = \sum_{j=1}^m p_j - \sum_{j=1}^m \sum_{i=1}^n p_j^* x_{ij}^* \log \left(\frac{p_j}{v_{ij}} \right),$$

where (p^*, x^*) is a pair of equilibrium prices and allocations. Since $h(p)$ is strictly convex, it has a unique minimum. Note that, since $\nabla_{p_j} h(p) = 1 - \frac{p_j^*}{p_j} \sum_i x_{ij}^* = 1 - \frac{p_j^*}{p_j}$ for all $j \in [m]$, by first-order optimality conditions we have that p^* is the minimum of $h(p)$. Then, we have

$$\begin{aligned} \varphi(p) &= \sum_{j=1}^m p_j - \sum_{i=1}^n B_i \log \left(\min_{k \in [m]} \frac{p_k}{v_{ik}} \right) \\ &= \sum_{j=1}^m p_j - \sum_{i=1}^n \sum_{j=1}^m p_j^* x_{ij}^* \log \left(\min_{k \in [m]} \frac{p_k}{v_{ik}} \right) \\ &\quad (\sum_{j=1}^m p_j^* x_{ij}^* = B_i) \\ &\geq \sum_{j=1}^m p_j - \sum_{j=1}^m \sum_{i=1}^n p_j^* x_{ij}^* \log \left(\frac{p_j}{v_{ij}} \right) = h(p). \end{aligned}$$

Also, note that

$$\begin{aligned} \varphi(p^*) &= \sum_{j=1}^m p_j^* - \sum_{j=1}^m \sum_{i: x_{ij}^* > 0} p_j^* x_{ij}^* \log \left(\min_{k \in [m]} \frac{p_k^*}{v_{ik}} \right) \\ &= \sum_{j=1}^m p_j^* - \sum_{j=1}^m \sum_{i: x_{ij}^* > 0} p_j^* x_{ij}^* \log \left(\frac{p_j^*}{v_{ij}} \right) = h(p^*), \end{aligned}$$

where the second equality follows because $x_{ij}^* > 0$ implies $p_j^*/v_{ij} = \min_{k \in [m]} p_k^*/v_{ik}$.

Since p is upper bounded by \bar{p} , h is strongly convex with modulus $\mu = \min_j p_j^*/\bar{p}_j^2$ (see Appendix B for derivation). Since strong convexity implies the quadratic growth condition, we obtain $h(p) - h(p^*) \geq (\min_j p_j^*/2\bar{p}_j^2) \|p - p^*\|^2$ for all $p \leq \bar{p}$. Since $\varphi(p) \geq h(p)$ for all p , and $\varphi(p^*) = h(p^*)$ at the unique minimum p^* ,

$$\varphi(p) - \varphi(p^*) \geq h(p) - h(p^*) \geq \min_j \frac{p_j^*}{2\bar{p}_j^2} \|p - p^*\|^2, \quad \forall p \leq \bar{p}.$$

Therefore, φ is a convex function satisfying the QG condition with modulus $\alpha = \min_j \frac{p_j^*}{2\bar{p}_j^2}$. \square

Approximate linear convergence. Next we show that tâtonnement converges at a linear rate, due to our results on bounded prices and quadratic growth. The proof of this theorem is typical for subgradient descent methods under these conditions, and easily follows from our structural results on tâtonnement and the EG dual. We defer it to Appendix B.

Theorem 1. *Assume that we adjust prices in the linear Fisher market with Eq. (4), starting from an initial price vector $p^0 \geq \tilde{p}$, with any stepsize $\eta < \frac{1}{2m} \min_j \tilde{p}_j$, where \tilde{p}_j is defined in Lemma 1. Then, we have*

$$\|p^t - p^*\|^2 \leq (1 - 2\eta\alpha)^t \|p^0 - p^*\|^2 + e, \quad \text{for all } t \geq 0,$$

where α is defined in Lemma 3 and $e = \frac{\eta}{2\alpha} \left(\frac{B}{\min_j \tilde{p}_j - 2m\eta} + m \right)^2$.

Theorem 1 shows that the price vector p^t converges linearly to an ϵ neighborhood around the equilibrium price vector p^* , where the size of the neighborhood is determined by the stepsize η . Suppose instead that we want to get arbitrarily close to p^* . Next, we show a corollary of Theorem 1 for this case.

Corollary 1. *For a given $\epsilon > 0$, let $\eta \leq \min \left\{ \frac{\min_j \tilde{p}_j}{4m}, \frac{2\epsilon \min_j p_j^*}{9B^2 \left(\frac{2B}{\min_j \tilde{p}_j} + m \right)^2} \right\}$. Then, starting from any $p^0 \geq \tilde{p}$ such that $\sum_{j=1}^m p_j^0 = B$, tâtonnement with stepsize η generates a price vector p such that $\|p - p^*\|^2 \leq \epsilon$ in $\mathcal{O}(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$ iterations. If $\epsilon \geq \frac{9B^2 \min_j \tilde{p}_j}{8m \min_j p_j^*} \left(\frac{2B}{\min_j \tilde{p}_j} + m \right)^2$, the time complexity is on the order of $\mathcal{O}(\log \frac{1}{\epsilon})$.*

Note that convergence measured via $\|p - p^*\|^2$ implies convergence to an approximate equilibrium. This is explained in Appendix C. We also discuss the possibility of showing convergence of tâtonnement without considering regularity conditions (e.g., QG) in Appendix C.

A non-convergence example and discussion. The non-convergence of tâtonnement under linear utilities has been discussed in previous work, e.g., Cole and Fleischer (2008, Section 5), Cheung, Cole, and Devanur (2019, Section 6), Avigdor-Elgrabli, Rabani, and Yadgar (2014, Section 6). Cole and Tao (2019) gives a simple market instance (Example 1 in that paper) showing that a *multiplicative*

form of tâtonnement results in prices cycling between off-equilibrium prices. Here, we provide a similar, simple instance to demonstrate the non-convergence of the additive form of tâtonnement Eq. (4) under *any* stepsize $\eta > 0$. Consider a single buyer with a *unit* budget and two items with valuations $v_{11} = v_{12} = 0.5$. The initial prices are chosen as $p^0 = (0.5, 0.5 + \eta)$. It can be easily verified that the equilibrium prices $p^* = (0.5, 0.5)$. For a small $\eta > 0$, tâtonnement gives

$$p^t = \begin{cases} (0.5 + \eta, 0.5), & t = 1, 3, 5, \dots \\ (0.5, 0.5 + \eta), & t = 2, 4, 6, \dots \end{cases}$$

and, as can be verified via straightforward calculations, $p_1^{2t+1} \rightarrow 0.5$, $p_1^{2t} \rightarrow 0.5 + \eta$. Note, though, that the region of cycling around the equilibrium prices is proportional to η , which is consistent with Theorem 1.

Although prices cycle with a fixed stepsize (as expected since e in Theorem 1 is proportional to the stepsize), they do converge to the exact equilibrium prices with adaptive stepsizes. See Appendix C for more details on this point.

5 Quasi-Linear Fisher Markets

The quasi-linear Fisher market (QLFM) is an extension of the linear Fisher market which considers buyers who have outside value for money. Formally, given a set of prices $p = \{p_j\}_{j \in [m]} \geq 0$ on all items, the *demand* of buyer i is

$$D_i^q(p) = \arg \max_{x_i} \{ \langle v_i - p, x_i \rangle : \langle p, x_i \rangle \leq B_i, x_i \geq 0 \}.$$

In the QLFM, if the price for every item is greater than the value to the buyer, the buyer prefers not to spend her money. Thus, unlike for the LFM, buyers may not spend all their money in equilibrium. However, as in the LFM, buyers only spend money on their MBB items, and the market clears in equilibrium.

Recently, there has been increasing interest in the QL Fisher market, due to its relation to budget management in internet advertising markets. Conitzer et al. (2022) showed that the *first price pacing equilibrium* (FPPE) concept, which captures steady-state outcomes of typical budget pacing algorithms for a first-price auction setting, is equivalent to market equilibrium in the QLFM. Consequently, FPPE can be computed efficiently by solving EG-like convex programs.

As shown in Chen, Ye, and Zhang (2007); Devanur et al. (2016), market equilibria in the QLFM can be captured by the following pair of EG-like primal and dual problems:

$$\begin{aligned} \max_{x \in \mathbb{R}_{\geq 0}^{n \times m}, u, y \in \mathbb{R}_{\geq 0}^n} & \sum_i (B_i \log(u_i) - y_i) \\ \text{s.t.} & u_i \leq \langle v_i, x_i \rangle + y_i \quad \forall i \in [n] \quad (\text{ql-P}) \\ & \sum_i x_{ij} \leq 1 \quad \forall j \in [m], \end{aligned}$$

$$\begin{aligned} \min_{\beta \in \mathbb{R}_{\geq 0}^n, p \in \mathbb{R}_{\geq 0}^m} & \sum_j p_j - \sum_i B_i \log \beta_i \\ \text{s.t.} & v_{ij} \beta_i \leq p_j \quad \forall i \in [n], j \in [m] \quad (\text{ql-D}) \\ & \beta_i \leq 1 \quad \forall i \in [n]. \end{aligned}$$

Equivalence between tâtonnement and SubGD. Similar to Eq. (1), the dual problem can be expressed as minimizing a max-type nonsmooth convex function $\varphi^q(p)$ as follows:

$$\max_{x, y \geq 0} \sum_{i=1}^n \left(B_i \log(\langle v_i, x_i \rangle + y_i) - y_i - \sum_{j=1}^m p_j x_{ij} \right) + \sum_{j=1}^m p_j.$$

By the first-order optimality conditions, we have that for any p , any maximizer \bar{x} of the inner maximization problem corresponds to a set of demands under price p , and the corresponding maximizer \bar{y} denotes leftover budgets under p . Since any subgradient of $\varphi^q(p)$ is given by $1 - \sum_{i=1}^n \bar{x}_i$, the subgradient equals the excess supply under price p .

The EG dual problem for QLFM can be simplified as

$$\min_{p \in \mathbb{R}_{\geq 0}^m} \varphi^q(p) = \sum_{j=1}^m p_j - \sum_{i=1}^n B_i \log \left(\min \left\{ \min_{k \in [m]} \frac{p_k}{v_{ik}}, 1 \right\} \right). \quad (9)$$

Again, the two expressions for $\varphi^q(p)$ only differ by a constant. Thus, we have $\partial \varphi^q(p) = -z^q(p)$ for any p . Consequently, tâtonnement in the QLFM is equivalent to SubGD on Eq. (9).

Convergence analysis. By leveraging the same framework of analysis as for LFMs, we show a linear convergence rate and corresponding time complexity for tâtonnement in QLFMs. We show these results in Theorem 2 and Corollary 2, and defer all the analyses and proofs to Appendix D.

Theorem 2. *Let $\mathbf{b}(j)$ be the buyer with the minimum index such that*

$$\mathbf{b}(j) \in \underset{i}{\text{Argmax}} \frac{v_{ij}}{\|v_i\|_\infty}$$

and $v_{\min} = \min_j v_{\mathbf{b}(j)j}$. Assume that we adjust prices in the QLFM with Eq. (4), starting from any initial price $p^0 \geq \hat{p}$, with stepsize $\eta < \frac{1}{2m} \min_j \hat{p}_j$, where \hat{p} is an m -dimensional price vector where $\hat{p}_j = \min \left\{ \frac{B_{\min}}{4m}, \frac{v_{\min}}{2} \right\} \max_i \frac{v_{ij}}{\|v_i\|_\infty}$, $\forall j \in [m]$. Then, we have

$$\|p^t - p^*\|^2 \leq (1 - 2\eta\alpha)^t \|p^0 - p^*\|^2 + e \quad \text{for all } t \geq 0, \quad (10)$$

where $\alpha = \min_j \frac{p_j^}{2(\min\{\max_i v_{ij}, B\} + \frac{\eta B}{p_j - 2m\eta})^2}$ and $e = \frac{\eta}{2\alpha} \left(\frac{B}{\min_j \hat{p}_j - 2m\eta} + m \right)^2$.*

Corollary 2. *For a given $\epsilon > 0$, let $\eta \leq \min \left\{ \frac{\min_j \hat{p}_j}{4m}, \frac{2\epsilon \min_j p_j^*}{9B^2(\frac{2B}{\min_j \hat{p}_j} + m)^2} \right\}$. Then, starting from any $p^0 \geq \hat{p}$ such that $\sum_j p_j^0 = B$, tâtonnement with stepsize η generates a price vector p such that $\|p - p^*\|^2 \leq \epsilon$ in $\mathcal{O}(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$ iterations. If $\epsilon \geq \frac{9B^2 \min_j \hat{p}_j}{8m \min_j p_j^*} (\frac{2B}{\min_j \hat{p}_j} + m)^2$, the time complexity is on the order of $\mathcal{O}(\log \frac{1}{\epsilon})$.*

6 Numerical Experiments

In this section, we demonstrate the convergence of tâtonnement for the LFM and QLFM through numerical experiments. To compute the “true” equilibrium prices, we run proportional response (PR) dynamics (Zhang 2011) for

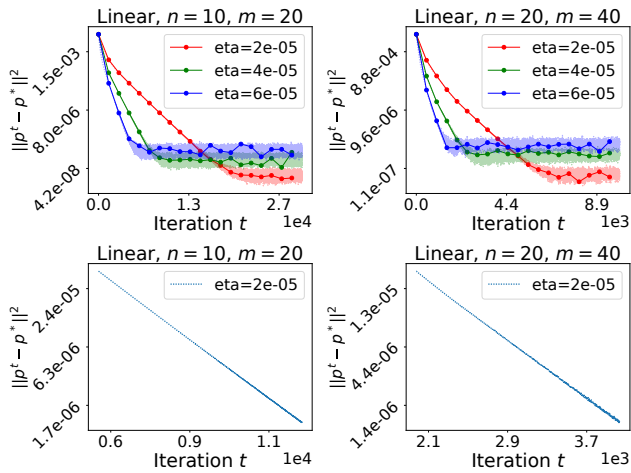


Figure 1: Convergence of squared error norms on random generated instances (v is generated from the exponential distribution with scale 1) of different sizes under linear utilities. The bottom row zooms in on partial iterations to better show the initial linear convergence.

LFM, and the quasi-linear variant of PR (Gao and Kroer 2020), which gives highly accurate solutions for our instances. We use the *squared error norm*, $\|p^t - p^*\|^2$, as the measure of convergence. For each market class, instance size, and step size, we run tâtonnement for a large number of iterations until the error residuals do not decrease further.

Synthetic data. We first test tâtonnement on randomly generated instances. We draw budgets $\{B_i\}_{i \in [n]}$ from the $[0, 1]$ uniform distribution and valuations $\{v_{i,j}\}_{i \in [n], j \in [m]}$ from various distributions, and then normalize them to $\sum_{j \in [m]} v_{ij} = 1$ for each i . In the LFM, we also normalize the budgets to $\sum_{i \in [n]} B_i = 1$. In Figs. 1 and 2, we plot $\|p^t - p^*\|^2$ across iterations for tâtonnement with different stepsize in the LFM and QLFM, respectively. The transparent dotted lines represent the values of $\|p^t - p^*\|^2$ for all iterations, and the solid lines represent those values for uniformly spaced iterations. See the rest of results for more instance sizes and distributions in Appendix E.

Real data. We next test tâtonnement on a large-scale instance constructed from a movie rating dataset (Dooms, De Pessemier, and Martens 2013; Nan, Gao, and Kroer 2023). In this instance, we regard each movie as an item and each rater as a buyer. The valuation of a buyer for an item is the rating she gives to the movie. See a full description of this instance in Appendix E. We run tâtonnement in both linear and quasi-linear settings and plot the results in Fig. 3.

Discussion. As can be seen from Figs. 1 to 3, under both linear utilities and QL utilities, tâtonnement with all stepsizes converges to a small neighborhood around the equilibrium prices for all instances. Moreover, larger stepsizes lead to faster initial convergence, but higher final error levels. This is consistent with our theoretical results: when

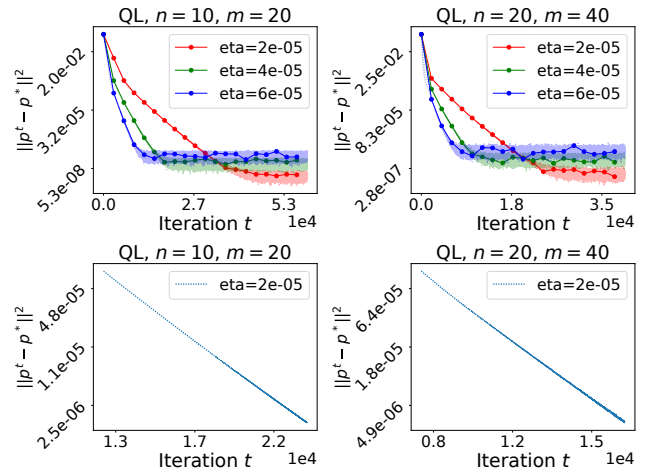


Figure 2: Convergence of squared error norms on random generated instances of different sizes under QL utilities.

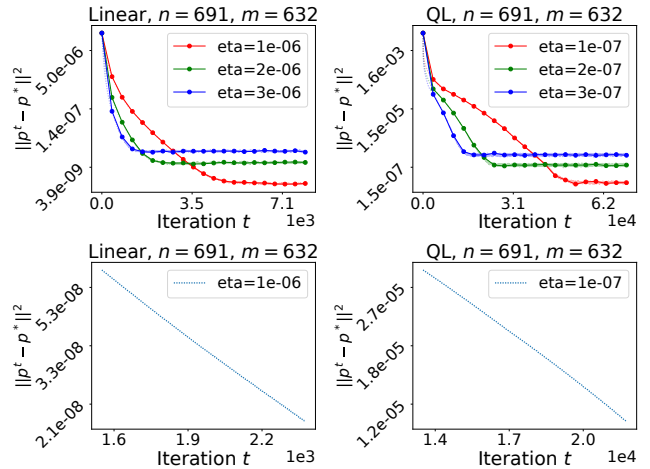


Figure 3: Convergence of squared error norms on our instances generated from a movie rating dataset.

$\eta \ll \min_j \tilde{p}_j - 2m\eta$, then $\alpha \approx \min_j p_j^*/2B^2$ is approximately a constant. Thus, $1 - 2\eta\alpha$ linearly decreases as η increases; and the final error level e increases as η increases.

In Figs. 1 to 3, We also zoom into the initial time steps of tâtonnement. As can be seen, the lines exhibit an approximately linear convergence pattern. This demonstrates that our linear convergence results align with the practical performance of tâtonnement. That is to say, we discover the “true” underlying convergence rate of tâtonnement prior to hitting the oscillating neighborhood.

We also compare the performance of different variants of tâtonnement (e.g., those in Table 1) in Appendix F.

Ethical Statement

There is no ethical concern in this work.

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