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Nearly Tight Bounds on Approximate Equilibria in Spatial Competition on the Line

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Abstract

In Hotelling's model of spatial competition, a unit mass of voters is distributed in the interval $[0, 1]$ (with their location corresponding to their political persuasion), and each of m candidates selects as a strategy their distinct position in this interval. Each voter votes for the nearest candidate, and candidates choose their strategy to maximize their votes. It is known that if there are more than two candidates, equilibria may not exist in this model. It was unknown, however, how close to an equilibrium one could get. Our work studies approximate equilibria in this model, where a strategy profile is an (additive) ϵ-equilibria if no candidate can increase their votes by ϵ, and provides tight or nearly-tight bounds on the approximation ϵ achievable.

We show that for 3 candidates, for any distribution of the voters, $\epsilon \geq 1/12$. Thus, somewhat surprisingly, for any distribution of the voters and any strategy profile of the candidates, at least $1/12$th of the total votes is always left "on the table." Extending this, we show that in the worst case, there exist voter distributions for which $\epsilon \geq 1/6$, and this is tight: one can always compute a $1/6$-approximate equilibria in polynomial time. We then study the general case of m candidates, and show that as m grows large, we get closer to an exact equilibrium: one can always obtain a $1/(m + 1)$-approximate equilibria in polynomial time. We show this bound is asymptotically tight, by giving voter distributions for which $\epsilon \geq 1/(m + 3)$.

Introduction

In elections, strategic positioning by candidates is a common phenomenon. Candidates try to estimate where voters lie on a particular issue, such as through polls, past experience, or media reports, and then adopt positions accordingly. Depending on the opinion of the voters, candidates may position themselves to appear more conservative, liberal, or moderate, often even contradicting their earlier or stated positions.

Hotelling's seminal model, introduced in 1929, has been very influential in studying such strategic positioning by candidates (Hotelling 1929). The initial model by Hotelling studied two merchants selling an identical good that sought to attract customers distributed in the interval $[0, 1]$. Each merchant chose both a strategic location and a price for the identical good, and each customer then chose to purchase from the merchant which minimized the sum of distance and price paid. The primary message of Hotelling's work was that, in such competition in duopolies, competition leads to *minimum differentiation*, where at equilibrium the two competitors end up being very close to each other.

Downs adopted Hotelling's basic model to examine electoral competition, even in scenarios involving more than two candidates (Downs 1957). Since then, the model, and variations of it, have been very influential in political economics. This is then a spatial competition among the candidates, where each candidate seeks to maximize the number of votes they receive, in the presence of the other candidates. With more than three candidates, however, there may not be an equilibrium (and in fact, for three candidates there is never an equilibrium!) (Eaton and Lipsey 1975; Osborne 1995).

Despite its shortcomings, equilibria is a fundamental and very appealing solution. However, exact equilibrium may be too demanding a notion and the non-existence of equilibria may not be interesting if one could get very close to an equilibrium. On the other hand, if any strategy profile is far from an equilibrium, then the game is highly unstable, and equilibrium may not be a suitable model for the behaviour of agents. In this work, we hence focus on the question of *how close can one get to an equilibrium* or *how stable the instance is* within the basic model of spatial competition motivated by Hotelling's work. For a unit mass of voters we call a strategy profile an (additive) ϵ-equilibrium if no candidate can increase their votes by ϵ, and provide tight or nearly-tight bounds on the approximation ϵ achievable. Despite the vast literature on this model and extensions to it, work on quantifying the instability in the absence of equilibria is lacking.

We note that in many practical applications where exact solutions do not exist or are difficult to obtain, additive approximate solutions are commonly used. An example of this from social choice theory is the use of EF1 (envy-free up to one good) allocations for indivisible goods, which is an additive approximation (Caragiannis et al. 2019). For mixed Nash equilibria in bimatrix (i.e., two-person normal-form) games, additive approximations are more commonly studied. In bimatrix games, for constant ϵ, quasi-polynomial time algorithms giving an additive ϵ-approximation to Nash equilibria were known prior to the seminal PPAD-completeness result (Chen and Deng 2006; Lipton, Markakis, and Mehta 2003). The PPAD-completeness result was immediately followed by the hardness of computing an additive ϵ-approximation, for polynomially small ϵ (Chen, Deng, and Teng 2006).

Related Work

The paper closest to ours in studying spatial competition with multiple candidates is by Eaton and Lipsey (Eaton and Lipsey 1975). Their objective is to study conditions in which the minimum differentiation shown by Hotelling holds. Among other results, they describe necessary and sufficient conditions for the existence of equilibria and show that if there are three candidates, then an equilibrium does not exist. For $m \geq 4$ candidates, they show that if m exceeds twice the number of local maxima in the distribution of voters, there is no equilibrium. Hence in particular for an unimodal distribution of voters (such as a truncated Gaussian distribution), there does not exist an equilibrium for any $m \geq 3$ candidates. They also study equilibria when the market is a circle, rather than the interval $[0, 1]$, as well as the multidimensional setting when the customers are distributed in a disc. The existence of equilibrium is also studied by Fournier and Scarsini (Fournier and Scarsini 2019), albeit in a more general model where the voters are present in a graph. They show that for a *uniform* distribution of voters, if the number of candidates is large enough, then an equilibrium exists, and they also study the inefficiency (i.e., the Price of Anarchy and Price of Stability) of equilibria.

Several variants of Hotelling's model are also studied. This literature is quite vast, and hence we give some pointers rather than a comprehensive survey. One variant studies the case of competition between parties, each of which can choose one of multiple candidates located on the line (Harrenstein et al. 2021; Deligkas, Eiben, and Goldsmith 2022). The goal of each party is to maximize their votes, given the candidates chosen by other parties. Other studied variations include models where candidates can enter and exit the election (Feddersen, Sened, and Wright 1990; Sengupta and Sengupta 2008), when candidates strategize to win the election, rather than maximizing their votes (Chisik and Lemke 2006), when some candidates have fixed positions and do not strategize (Jones, Sirianni, and Fu 2022), and when voters do not vote at all when there is no candidate sufficiently close to their position (Feldman, Fiat, and Obraztsova 2016; Shen and Wang 2017; Jones, Sirianni, and Fu 2022). For a more detailed survey of these results, we refer to Eiselt, Marianov, and Drezner (2019), and Enelow and Hinich (1990).

Our Contribution

Our results are summarised in Tables 1 and 2. We first focus on the case of approximate equilibria with 3 candidates. This is the simplest case beyond 2 candidates and is interesting because, as shown earlier, there is no equilibrium irrespective of the distribution of voters (Eaton and Lipsey 1975). We first show tight bounds on the distance of any strategy profile from equilibria, for any voter distribution. We show that for *any* distribution, one cannot obtain better than a $1/12$-approximate equilibrium. Thus for any distribution of the voters, and for any location of the three candidates, some candidate can increase her votes by at least $1/12$ (or 8.5% of the total vote) by choosing a different position. Given that modern elections often hinge on a small percentage of the total vote, our findings suggest that any scenario involving three candidates is highly *unstable*. We also show that this is the worst possible: there exists a distribution where one can in fact obtain a $1/12$-approximate equilibrium.

The bound of $1/12$ is true for all distributions; an immediate question then is about the worst-case voter distribution. We next show that there exists a voter distribution where one cannot obtain better than a $1/6$-approximate equilibrium. We also show that the approximation $1/6$ is tight: for any distribution, one can obtain a $1/6$-approximate equilibrium.

Moving beyond 3 candidates, we then consider approximate equilibria for m candidates. It is known that there are distributions for which equilibria exist for $m \geq 4$ candidates (Eaton and Lipsey 1975), hence we are concerned with worst-case distributions. Here, we again show nearly tight bounds. Specifically, we show that for any distribution, one can obtain a $\frac{1}{m+1}$-approximate equilibrium, and there exist distributions for which one cannot obtain better than a $\frac{1}{m+3}$-approximate equilibrium. Thus our results present a quantitative perspective on previous results on the nonexistence of equilibria. The upper bounds of our model indicate that as the number of candidates increases, the model becomes increasingly *stable*, regardless of the distribution of voters. Overall our results present tight or nearly tight bounds on approximate equilibria. Our upper bounds are via polynomial time algorithms that have access to an oracle that supports two functions: (i) given $x \in [0, 1]$, returns $F(x)$, the total voters until the location x, and (ii) given $x, v \in [0, 1]$, returns a location $y = \text{Cut}(x, v) \geq x$ so that there are exactly v voters in the interval $[x, y]$.[1] These are the same as Eval and Cut queries used in the Robertson-Webb query model for fair cake division (Robertson and Webb 1998). In fact, the simplicity of our algorithm in determining the upper bound of $\frac{1}{m+1}$ indicates that a simpler access method suffices. Specifically, an oracle that provides the positions of the mth quantiles of the voter distribution for any m suffices. This is similar to the oracle utilized by the GLIME mechanism (Ben-Porat et al. 2019).

Lastly, we note that our results are for the case where candidates may occupy locations near each other, but cannot occupy the same location in the $[0, 1]$ interval, and for continuous voter distributions. A natural question then is if the results change if these assumptions do not not hold. We study two variants. In the first variant, multiple candidates are allowed to occupy the same location. In the prior case, even for candidates close to each other, we assume that voters can perfectly distinguish between them. This variant relaxes this assumption, and models the case where when candidates are very close to each other, voters may not be able to distinguish between them and their vote get equally divided between the candidates. We show this variant may give differ-

[1]Note that the second query — Cut(x,v) — can be simulated to arbitrary precision by running a binary search using just the first query.

	$m = 3$ candidates		$m \geq 4$ candidates	
	$\exists$ distrib.	$\forall$ distribs.	$\exists$ distrib.	$\forall$ distribs.
Lower bound	$\epsilon \geq 1/6$	$\epsilon \geq 1/12$	$\epsilon \geq 1/(m+3)$	0
Upper bound	$\epsilon \leq 1/12$	$\epsilon \leq 1/6$	0	$\epsilon \leq 1/(m+1)$

Table 1: Our results on the additive approximation achievable. The "$\exists$ distrib." columns show results for worst-case distributions for lower bounds and best-case distributions for upper bounds. The "$\forall$ distribs." columns show results over all distributions. Our bounds are tight for 3 candidates, and asymptotically tight for 4 or more candidates.

	$m = 3$ candidates	
	$\exists$ distrib.	$\forall$ distribs.
Lower bound	$\epsilon \geq 1/7$	?
Upper bound	?	$\epsilon \leq 1/7$

Table 2: Our results on the additive approximation achievable for the variant where multiple candidates can occupy the same location. Question marks indicate open questions. Note the better approximation achievable, as compared to Table 1.

ent results as illustrated in Table 2. Specifically, for scenarios involving three candidates, we show that regardless of the distribution of voters, a $1/7$-approximate equilibrium can be achieved, thus offering greater stability compared to the previous model, for which there exist distributions for which ϵ must be at least $1/6$. Furthermore, this bound is tight, as there exists a distribution where obtaining a better approximation than $1/7$ is impossible. We leave further study in this model for future work.

Secondly, we study the case where there is a finite population of voters, rather than a continuous distribution. Intuitively, if the population of voters is large enough, they should approximate the continuous distribution. For $m \geq 4$ candidates, we show that this is indeed the case. With a sufficiently large population of voters, we obtain the results in Table 1, even for discrete voters.

All proofs missing in the paper are given in the full version (Bhaskar and Pyne 2024).

Preliminaries and Notation

A unit mass of voters is distributed in the interval $[0, 1]$ according to a density function $f : [0, 1] \to \mathbb{R}^+ \cup \{0\}$ such that f is integrable, bounded so that $f(z) \leq M$ for some finite M, and $\int_0^1 f(z)\,dz = 1$. Let $F(y) = \int_0^y f(z)\,dz$ be the integral. Since $f(z) \leq M$, for any interval of length δ, the total voters in the interval is at most $M\delta$.

There are m candidates. Each candidate i chooses a real number x_i from $[0, 1]$ as their strategy, hence $X = (x_1, x_2, ..., x_m)$ is a strategy profile. We restrict the candidates to occupy distinct positions in the interval, so that $\min_{i,j} |x_i - x_j| \geq \delta$ for some small δ. It is helpful to think of δ as approaching zero. In particular, we will assume $M\delta < 10^{-3}$. For a strategy profile X, we will use X_{-i} to denote the position of candidates other than i.

Given a strategy profile X, assume without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_m$.[2] Then for a candidate $1 < i < m$, the voters located between $(x_{i-1} + x_i)/2$ on the left and $(x_i + x_{i-1})/2$ on the right are nearer candidate i than any other candidate j, and hence vote for i. We define the *left* and *right votes* (or *utility*) of candidate i as

$$U_i^L(X) = \int_{\frac{x_{i-1}+x_i}{2}}^{x_i} f(z)\,dz, \text{ and } U_i^R(X) = \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} f(z)\,dz.$$

The utilities for candidate 1 and m are defined separately.

$$U_1^L(X) = \int_0^{x_1} f(z)\,dz, \text{ and } U_1^R(X) = \int_{x_1}^{\frac{x_1+x_2}{2}} f(z)\,dz\,.$$

$$U_m^L(X) = \int_{\frac{x_{m-1}+x_m}{2}}^{x_m} f(z)\,dz, \text{ and } U_m^R(X) = \int_{x_m}^{1} f(z)\,dz.$$

Thus $U_1^L(X) = F(x_1)$, and $U_m^R(X) = 1 - F(x_m)$. The total votes or utility of candidate $i \in [m]$, is $U_i(X) = U_i^L(X) + U_i^R(X)$. Note that the total votes summed over all candidates is always 1.

A strategy profile X is an equilibrium if for all candidates i, and all locations x' that satisfy $|x' - x_j| \geq \delta$ for all $j \neq i$, $U_i(X) \geq U_i(x', X_{-i})$. It is known that for two candidates, the positions $x_1 = \mu - \delta/2$, $x_2 = \mu + \delta/2$ are an equilibrium, where μ is the median (i.e., $F(\mu) = 1/2$). However, for three candidates, no equilibrium exists. Our goal in this paper is to study approximate equilibria.

Definition 1 (ϵ-equilibrium)**.** *Given an $\epsilon \geq 0$, $X = (x_1, x_2, ..., x_m)$ is an ϵ-equilibrium if for any candidate $i \in [m]$ and any location $x'_i \in [0, 1]$ such that $\forall j \neq i$, $|x'_i - x_j| \geq \delta$,*

$$\lim_{\delta \to 0} (U_i(X') - U_i(X)) \leq \epsilon$$

where $X' = (x'_i, X_{-i})$.

[2]We will make this assumption whenever possible to avoid cumbersome notation.

Note that X is a *pure Nash equilibrium* if the condition holds for $\epsilon = 0$.

For our algorithms to access the voter density function f, we assume access to an oracle that supports the queries:

- $F(z)$: Returns $F(z)$, the total voters in the interval $[0, z]$.
- Cut(z, v): Given a location $z \in [0,1]$ and a value $v \in [0,1]$, returns a location $y \geq z$ so that $F(y) - F(z) = v$, or returns 1 if there is no such y. If there are multiple such locations, return one arbitrarily but consistently.

Approximate Equilibria for Three Candidates

In this section, we are interested in three candidates, i.e., $m = 3$. Each candidate $i \in \{1, 2, 3\}$ chooses a real number x_i from $[0, 1]$ as their strategy such that $x_1 < x_2 < x_3$. We begin by showing a lower bound of $1/12$ on ϵ.

Theorem 1. *If $X = (x_1, x_2, x_3)$ is an ϵ-equilibrium, then $\epsilon \geq \frac{1}{12} - M\delta$.*

The proof of the theorem is based on three claims. The first two bound the left and right votes for each candidate, and the last bounds the votes between x_1 and x_3.

Claim 1. *$U_2^L(X)$ and $U_2^R(X)$ are both at most $\epsilon + M\delta$.*

Proof sketch. For a contradiction, assume $U_2^L(X) > \epsilon + M\delta$. Now let $x_1' = x_2 - \delta$. Then candidate 1 gets her earlier votes plus the left votes of candidate 2 minus at most $M\delta$, an increase of more than ϵ, which is a contradiction. A similar argument shows $U_2^R(X) \leq \epsilon + M\delta$ as well. □

Claim 2. *$U_1^L(X)$ and $U_3^R(X)$ are both at most $3(\epsilon + M\delta)$.*

Proof. From Claim 1, candidate 2 has utility at most $2(\epsilon + M\delta)$. Now if $U_1^L(X) > 3(\epsilon + M\delta)$, let $x_2' = x_1 - \delta$, and consider the strategy profile $X' = (x_1, x_2', x_3)$. Then, candidate 2's vote is at least $F(x_1 - \delta) \geq U_1^L(X) - M\delta$, and since $U_1^L(X) > 3(\epsilon + M\delta)$, this is greater than $3\epsilon + 2M\delta$. Hence, X cannot be an ϵ-equilibrium, giving a contradiction. Similarly, we obtain $U_3^R(X) \leq 3(\epsilon + M\delta)$. □

Claim 3. *$F((x_1 + x_3)/2) - F(x_1)$ and $F(x_3) - F((x_1 + x_3)/2)$ are both at most $3(\epsilon + M\delta)$.*

The proof is similar to Claim 2. We show that if the claim does not hold, then for candidate 2, shifting to either $x_1 + \delta$ or $x_3 - \delta$ increases her votes by more than ϵ.

We now complete the proof of the theorem. Figure 1 shows the bounds on the votes from Claim 1 and Claim 2.

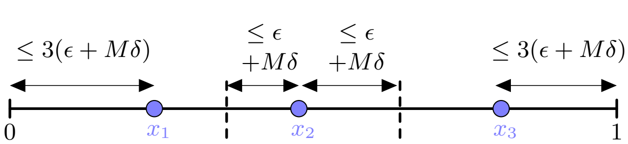

Figure 1: Figure showing bounds on the votes, as shown by Claim 1 and Claim 2. Candidates are shown by blue circles. The dashed lines are the mid-points of x_1, x_2 and x_2, x_3.

Proof of Theorem 1. We can write the total votes as

$$1 = F(x_1) + (F((x_1 + x_3)/2) - F(x_1)) + (F(x_3) - F((x_1 + x_3)/2)) + (1 - F(x_3)) \ . \quad (1)$$

Now, from Claim 2 we get that $F(x_1) = U_1^L(X) \leq 3(\epsilon + M\delta)$, and $1 - F(x_3) = U_3^R(X) \leq 3(\epsilon + M\delta)$. From Claim 3 we get that the remaining two terms are also at most $3(\epsilon + M\delta)$. Substituting in (1), we get that $1 \leq 12(\epsilon + M\delta)$, or $\epsilon \geq \frac{1}{12} - M\delta$, as required. □

We now show that the bound in Theorem 1 is tight: we can't obtain a worse bound than $1/12$ that holds for all distributions. In particular, we exhibit a distribution for which a $1/12$-equilibrium is obtainable. Define $\epsilon := 1/12$. The distribution for the upper bound is shown in Figure 2. Theorem 2 proves that the strategy profile $X = (x_1, x_2, x_3) = (\frac{3}{22}, \frac{1}{2}, \frac{19}{22})$ is in fact an ϵ-equilibrium.

Theorem 2. *There exists a distribution f of voters for which there exists a $\frac{1}{12}$-equilibrium.*

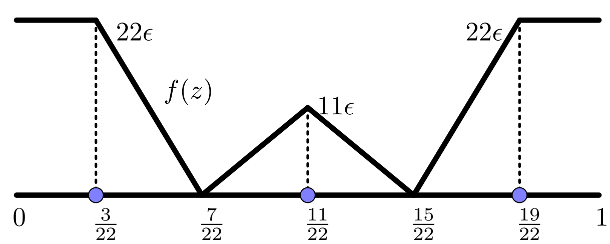

Figure 2: Distribution of voters and candidates for the lower bound of $1/12$ in Theorem 2. Candidates are indicated by blue circles.

The upper bound in Theorem 2 holds irrespective of the voter distribution. We next consider worst-case voter distributions. How much further from an exact equilibrium are we pushed? Our next two theorems give tight bounds of $1/6$ on the worst-case approximation.

Theorem 3. *For any constant $\epsilon < \frac{1}{6}$, there exists a distribution f of voters for which there does not exist an ϵ-equilibrium with three candidates as $\delta \to 0$.*

This theorem is a special case of the more general result proved in Theorem 7, where we show a lower bound of $1/(m+3)$ in the worst case if there are m candidates. We thus defer the proof until later. Our next theorem however gives the required upper bound.

Theorem 4. *Given any distribution f of voters, a $(1/6 + M\delta)$-equilibrium can be found for three candidates in polynomial time.*

Proof. Let $x_1 = \text{Cut}(0, 1/3)$ and $x_3 = \text{Cut}(0, 2/3)$. Then $F(x_1) = F(x_3) - F(x_1) = 1 - F(x_3) = 1/3$. Keeping x_1 and x_3 fixed as the equilibrium locations for candidates 1 and 3, we will try and find a location for candidate 2 to obtain an equilibrium. Notice that either $F((x_1 + x_3)/2) - F(x_1) \geq 1/6$, or $F(x_3) - F((x_1 + x_3)/2) \geq 1/6$. Without loss of generality, assume $F((x_1+x_3)/2) - F(x_1) \geq 1/6$. Since F is continuous, there must exist $x \in (x_1, x_3)$ so that $F((x+x_3)/2) - F(x) = 1/6$. Let x_2 be such a location x. If $|x_2 - x_1| < \delta$, let $x_2 = x_1 + \delta$.[3] We will now show that $X = (x_1, x_2, x_3)$ is in fact a $1/6$-equilibrium.

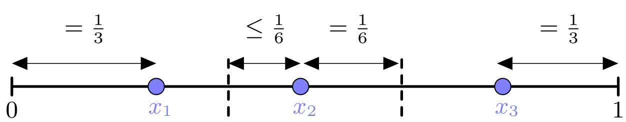

Figure 3: Figure showing the votes, as shown by Theorem 4. Blue circles show candidates. The dashed lines are the midpoints of x_1, x_2 and x_2, x_3.

For candidate 2, $U_2^R(X) = F((x_2+x_3)/2) - F(x_2) \geq 1/6 - M\delta$. Locations x_1 and x_3 equally partition the set of voters, hence the maximum votes that candidate 2 can get is $1/3$. Thus she cannot increase her votes by more than $1/6 + M\delta$.

For candidate 1, $U_1^L(X) = 1/3$. Since $U_2^R(X) \geq 1/6 - M\delta$ and $U_3^R(X) = 1/3$, it follows that $F(x_2) \leq 1/2 + M\delta$. Hence x_2 and x_3 partition the set of voters so that at most $1/2 + M\delta$ are to the left of x_2, at most $1/3$ between x_2 and x_3, and $1/3$ to the right of x_3. It follows that candidate 1 cannot increase her votes by more than $1/6 + M\delta$.

For candidate 3, $U_1^L(X) = 1/3$. Again we note that x_1 and x_2 partition the set of voters so that $1/3$ voters are to the left of x_1, and at most $1/3$ between x_1 and x_2. Hence deviating to the left of x_2 cannot increase her votes. If she deviates to the right of x_2, she can at most gain the right votes of candidate 2 which is $U_2^R(X) \leq 1/6$. Hence candidate 3 also cannot increase her votes by more than $1/6$, and hence X is a $(1/6 + M\delta)$-equilibrium. □

Approximate Equilibria for m Candidates

How do the previous results change as we consider more candidates? As previously shown (Hotelling 1929), for $m \geq 4$, there exist voter distributions that admit equilibria, hence in the best case $\epsilon = 0$. Hence we focus on worst case voter distributions and show that the approximation improves as m increases.

Theorem 5. *Given a distribution f of voters, a $\frac{1}{m+1}$-equilibrium can be found in polynomial time.*

The proof shows that the strategy profile where the candidates form an equipartition of the voters — i.e., $x_1 = \text{Cut}(0, \frac{1}{m+1})$, and $x_k = \text{Cut}(x_{k-1}, \frac{1}{m+1})$ for $k = 2, \ldots, m$ — is in fact a $\frac{1}{m+1}$-equilibrium.

We now define a particular rapidly decreasing function that forms the basis for the voter density functions in the lower bounds for approximate equilibria.

[3] Since $F(x_3) - F(x) \geq 1/6$, and we assume $M\delta < 10^{-3}$, clearly $x < x_3 - \delta$.

Theorem 6. *Given $A > \gamma > 0$ and $l \in [0,1]$, there exists an integrable and bounded function $g : [0,l] \to \mathbb{R}^+ \cup \{0\}$ that satisfies (i) $\int_0^l g(z)\,dz = A$, and (ii) for all $y \in [0,l]$, $\int_{y/2}^y g(z)\,dz \leq \gamma$.*

Proof. Define g as:

$$g(z) = \begin{cases} 0 & \text{for } z < le^{-\frac{A}{\gamma}}, \\ \frac{\gamma}{z} & \text{for } le^{-\frac{A}{\gamma}} \leq z \leq l. \end{cases}$$

Let $G(y) = \int_0^y g(z)\,dz$. For $y \leq le^{-\frac{A}{\gamma}}$, $G(z) = 0$, and for $z \geq le^{-\frac{A}{\gamma}}$, $G(z) = A + \gamma \log\left(\frac{z}{l}\right)$. Notice that $G(l) = A$, and for any $y \in [0,l]$, $G(y) - G(y/2) = \gamma \ln 2 < \gamma$. □

The function derived from Theorem 6 is discontinuous, but can be smoothed in the intuitive way so that it is continuous and differentiable (though the derivatives may be large), and all other required properties are maintained. We skip a formal description of the resulting function to avoid the additional complexity.

Theorem 7. *Given m candidates, there exists a voter distribution f for which $\epsilon \geq \frac{1}{m+3}$ for any ϵ-equilibrium as $\delta \to 0$.*

Proof. Fix $\gamma > 0$. We will eventually take the limit $\gamma \to 0$, to obtain the bound of $\frac{1}{m+3}$. We choose the distribution f of the voters as per Theorem 6, so that f is bounded, integrable, $\int_0^1 f(z)dz = 1$, and for any $y \in [0,1]$, $F(y) - F(y/2) \leq \gamma$.

We note one property that follows from the distribution f. For all candidates $k \geq 2$,

$$\begin{aligned} U_k^L(X) &= F(x_k) - F((x_k + x_{k-1})/2) \\ &\leq F(x_k) - F(x_k/2) \leq \gamma. \end{aligned}$$

We next show an upper bound on the utility from voters on the right for candidates $k \leq m-1$. These utilities are depicted in Figure 4.

Claim 4. *For candidates $k \leq m-1$, $U_k^R(X) \leq \epsilon + \gamma + M\delta$*

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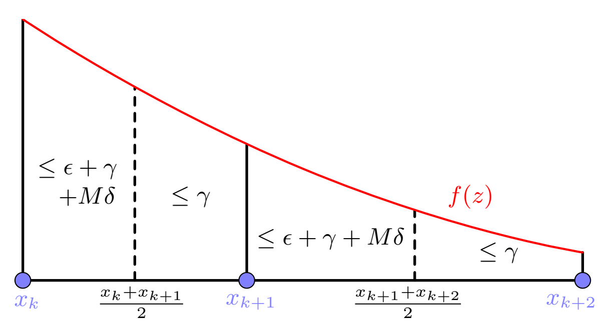

Figure 4: Bounds on the utility of candidate $k+1$ in Theorem 7.

Thus for all candidates $k \in \{2, \ldots, m-1\}$, the total utility is $U_k(X) = U_k^L(X) + U_k^R(X) \leq \epsilon + 2\gamma + M\delta$. It follows then that for candidates 1 and m, $U_1^L(X)$ and $U_m^R(X)$ are at most $\leq 2\epsilon + 2\gamma + 2M\delta$.

Finally, note that for any $k \leq m-1$, $F(x_{k+1}) - F(x_k) = U_k^R(X) + U_{k+1}^L(X) \leq \epsilon + 2\gamma + M\delta$. Thus,

$$\begin{aligned} 1 &= F(x_1) + \sum_{k=1}^{m-1} (F(x_{k+1}) - F(x_k)) + (1 - F(x_m)) \\ &\leq (2(\epsilon + \gamma + M\delta) + (m-1)(\epsilon + 2\gamma + 2M\delta) + \\ &\qquad\qquad 2(\epsilon + \gamma + M\delta)) \\ &= (m+3)\epsilon + 2(m+1)(\gamma + M\delta) \end{aligned}$$

and hence, as $\gamma, \delta \to 0$, $\epsilon \geq 1/(m+3)$. □

Allowing Multiple Candidates at a Location

Our previous results require that two candidates can be arbitrarily close, but cannot occupy the same location. In effect, even for candidates that are nearly identical, voters are able to perfectly differentiate between the two. It is natural to wonder if our results hold in the limit. That is, if we relax this assumption, and allow multiple candidates to occupy the same position (and thus voters fail to differentiate between these candidates), do our results hold? While it may appear that upper bounds on approximate equilibria in the original model should hold for this model as well, due to the larger space of strategies allowed, we do not know if this is indeed the case. In this section, we however give some evidence that this is indeed true. In particular, for the case of 3 candidates, we show that for any distribution, we can obtain a $1/7$-equilibrium, and this is tight: there exist distributions for which we cannot do better than a $1/7$-equilibrium.

A slight redefinition of utilities is required in this model. If there are r candidates at a given location x, and x is closer to γ voters than any other candidate, then each candidate at x gets γ/r votes. Formally, given a strategy profile $X = (x_1, \ldots, x_k)$, where $x_1 \leq \ldots \leq x_k$, say $x_{i-1} < x_i = \ldots = x_j < x_{j+1}$. Then candidates $i, \ldots, j$ get utility

$$\frac{F\left((x_{j+1} + x_j)/2\right) - F\left((x_i + x_{i-1})/2\right)}{j - i + 1}\,.$$

The definition of ϵ-equilibrium gets modified appropriately.

Definition 2 (ϵ-equilibrium)**.** *Given an $\epsilon \geq 0$, $X = (x_1, x_2, ..., x_m)$ is an ϵ-equilibrium if for any candidate $i \in [m]$, and any location $x_i' \in [0,1]$, with $X' = (x_i', X_{-i})$,*

$$U_i(X') - U_i(X) \leq \epsilon\,.$$

Theorem 8. *For 3 candidates, if we allow multiple candidates to choose the same locations, then given a distribution f of voters, there always exists a $\frac{1}{7}$-equilibrium.*

Proof sketch. For the proof, we define $\delta > 0$ as a small constant, so that $\delta M < 10^{-3}$. We will actually show a $\frac{1}{7} + M\delta$-equilibrium, and then take $\delta \to 0$. We define three key points on the interval: $z_{2/7} = \text{Cut}(0, 2/7)$, $z_{5/7} = \text{Cut}(0, 5/7)$, and $\hat{z_2}$ is the point between $(z_{2/7} + \delta)$ and $(z_{5/7} - \delta)$ with maximum votes. Thus $\hat{z_2} = \arg\max_{z \in [z_{2/7}+\delta, z_{5/7}-\delta]} U_2((z_{2/7}, z, z_{5/7}))$. The strategy profile $X^0 = (z_{2/7}, \hat{z_2}, z_{5/7})$ is our initial strategy profile, which we will modify to get a $1/7$-equilibrium.

We first show that in this initial strategy profile, candidate 2 gets at least $1/7$ votes.

Claim 5. $U_2(X^0) \geq 1/7$.

Proof. Note that $F(z_{5/7}) - F(z_{2/7}) = 3/7$, hence, placing candidate 2 either just after $z_{2/7}$ or just before $z_{5/7}$ should give him at least half of these votes. That is, either $F((z_{2/7} + z_{5/7})/2) - F(z_{2/7}) \geq 1.5/7$ or $F(z_{5/7}) - F((z_{2/7} + z_{5/7})/2) \geq 1.5/7$. In either case, since $\hat{z_2}$ is the position in $[z_{2/7} + \delta, z_{5/7} - \delta]$ where candidate 2 gets maximum votes, she must get at least $1/7$ votes at $\hat{z_2}$. □

In the complete proof, we will consider two cases: the first, where $U_2(X^0) \leq 2/7$, and the second, where $U_2(X^0) > 2/7$. Here we only prove the (easy) first case, leaving the second case for the full version (Bhaskar and Pyne 2024).

Case I: $U_2(X^0) \leq 2/7$**.** Here, we consider two subcases. For the easy case, suppose both $U_2^L(X^0) \leq 1/7$ and $U_2^R(X^0) \leq 1/7$. Then we claim that in fact $X^0 = (z_{2/7}, \hat{z_2}, z_{5/7})$ is a $1/7$-equilibrium.

1. For candidate 2, note that she cannot improve her votes in the interval $(z_{2/7}, z_{5/7})$ by more than δM. At any position before $z_{2/7}$ or after $z_{5/7}$, she would get at most $2/7$ votes, and could thus improve by at most $1/7$. At the positions $z_{2/7}$ and $z_{5/7}$, she would get at most $(U_2(X^0) + \delta M + 2/7)/2 \leq 1/7 + U_2(X^0) + \delta M/2$.
2. For candidates 1 and 3, since $U_2^L(X^0) \leq 1/7$ and $U_2^R(X^0) \leq 1/7$, they can at best increase their votes by $1/7$.

For the second subcase, either $U_2^L(X^0) > 1/7$ or $U_2^R(X^0) > 1/7$. Clearly, both cannot hold, since we assume $U_2(X^0) \leq 2/7$. Without loss of generality, assume $U_2^R(X^0) > 1/7$ and $U_2^L(X^0) \leq 1/7$. Then by continuity of the function F, there must exist some $\eta > 0$ so that at the strategy profile $X' = (z_{2/7}, \hat{z_2} + \eta, z_{5/7})$, candidate 2 gets exactly $1/7$ votes from the right. That is, $U_2^R(X') = 1/7$. Further, $U_2^L(X') \leq 1/7$ (because she receives a maximum of $2/7$ votes at $\hat{z_2}$), and $U_1^L(X') = U_3^R(X') = 2/7$. Note that $U_2(X') \geq 1/7$. We claim that in this case, X' is a $1/7$-equilibrium.

1. For candidate 2, in the interval $(z_{2/7}, z_{5/7})$ she gets at most $2/7 + M\delta$ votes (since at $\hat{z_2}$ she gets at most $2/7$ votes), and hence cannot improve by more than $1/7 + \delta M$. Similarly, before $z_{2/7}$ and after $z_{5/7}$ she gets at most $2/7$ votes. At the positions $z_{2/7}$ and $z_{5/7}$, she would get at most $(U_2(X^0) + 2/7 + \delta M)/2 \leq 1/7 + U_2(X') + \delta M/2$.
2. For candidates 1 and 3, as previously, since $U_2^L(X') \leq 1/7$ and $U_2^R(X') \leq 1/7$, they can at best increase their votes by $1/7$.

Thus, if $U_2(X^0) \leq 2/7$, either X^0 or X' is a $(1/7+M\delta)$-equilibrium. As mentioned, we leave the other case for the full version. □

Note that the proof of Theorem 8 only shows the existence of a $1/7$-approximate equilibrium. However, to obtain a polynomial time algorithm that achieves a $1/7$-approximate equilibrium, we require a more robust oracle. This oracle should perform the following tasks: given an interval $[x, y]$, it returns a sub-interval $[x', y'] \subset [x, y]$ such that (i) $y' - x' = (y - x)/2$, and (ii) $F(y') - F(x')$ is maximized among all such sub-intervals. The location $\hat{z_2}$ as defined in the proof of Theorem 8 corresponds to the midpoint $(x' + y')/2$ of such an interval. Unfortunately, it remains unclear how to identify such a sub-interval in polynomial time using only Cut and Eval queries.

The subsequent theorem asserts the tightness of this $1/7$-approximate equilibrium.

Theorem 9. *For 3 candidates, if we allow multiple candidates to choose the same location, then there exists a distribution f of voters such that for any ϵ-equilibrium, $\epsilon \geq \frac{1}{7}$.*

Proof sketch. For some $\gamma > 0$, we choose the distribution f of the voters as per Theorem 6, so that for any $y \in [0, 1]$, $F(y) - F(y/2) \leq \gamma$. Let $X = (x_1, x_2, x_3)$ be an ϵ-approximate equilibrium. We consider four cases based on the maximum number of candidates occupying the same location. In case I, all three candidates are in different positions, and the proof follows from Theorem 7. In case II, all three candidates are in the same position and get $1/3$ votes. Deviating slightly can get a candidate at least $1/2$ votes, hence $\epsilon \geq 1/6$. In case III, candidates 1 and 2 choose the same location x_1, while candidate 3 chooses a different location x_3. In this case, we show that $F(x_1) \leq 3\epsilon$, $F(x_3) - F(x_1) \leq (\epsilon + \gamma)$, and $1 - F(x_3) \leq 3\epsilon$. Therefore, we obtain:

$$1 = F(x_1) + (F(x_3) - F(x_1)) + (1 - F(x_3)) \ \leq \ 7\epsilon + \gamma\,.$$

Therefore, for $\gamma \to 0$, $\epsilon \geq \frac{1}{7}$. We leave the remaining proof for the full version (Bhaskar and Pyne 2024). □

Approximate Equilibria for the Discrete Variant

In this variant, n voters are distributed according to the positions in $P = \{p_1, p_2, ..., p_n\} \subseteq [0, 1]$ where p_i is the position of the i-th voter. There are m candidates who select distinct positions from $[0, 1]$. For a given strategy profile $X = (x_1, x_2, ..., x_m)$, $N_i(X)$ represents the set of candidates closest to the i-th voter.

$$N_i(X) \ = \ \arg\min_j |p_i - x_j|$$

Let $I_A(x)$ be the indicator function which outputs 1 if $x \in A$ and 0 otherwise. $U_j(X)$ denotes the utility of the j-th candidate.

$$U_j(X) = \sum_{i \in [n]} \frac{I_{N_i(X)}(j)}{|N_i(X)|}$$

Definition 3 (ϵ-equilibrium)**.** *Given an $\epsilon \geq 0$, $X = (x_1, x_2, ..., x_m)$ is an ϵ-equilibrium if for any candidate $j \in [m]$ and any location $x'_j \in [0, 1]$ such that $\forall i \neq j$, $x_i \neq x'_j$,*

$$U_j(X') - U_j(X) \leq \epsilon n\,.$$

where $X' = (x'_j, X_{-j})$.

As stated earlier, for a sufficiently large population of voters, we recover the bounds for the continuous voter model when $m \geq 4$. Theorem 10 shows the upper bound, and is based on the same principle as Theorem 5: the candidates form an approximate equipartition of the voters.

Theorem 10. *Given n voters at positions $p_1 \leq p_2 \leq, ... \leq p_n$ and m candidates, a $\frac{1}{m+1}$-equilibrium can be found in polynomial time.*

The lower bound then uses a finite distribution of voters to approximate the continuous distribution from Theorem 6. In particular we choose the positions $p_1, p_2, ..., p_n$ so that $\forall i > 1$, $((p_1 + p_i)/2) > p_{i-1}$ and $p_1 \geq 0$. This is satisfied, e.g., by choosing $p_1 = \xi$ for some small $\xi > 0$, and $p_i = 2^{i-1}\xi$ for $i = 2, \ldots, n$.

Theorem 11. *For m candidates and $\nu > 0$, there exists a distribution of a finite set of voters such that for any ϵ-equilibrium, $\epsilon > \frac{1}{m+3+\nu}$.*

Conclusion

In the basic model of spatial competition in the unit interval, our work gives nearly tight best-case and worst-case bounds on how close we can get to equilibria, quantifying the instability due to competition. As in prior work, it would be interesting to see if these bounds are robust to slight changes in the model, such as if we allow multiple candidates to occupy the same location. Another assumption that we would like to relax is that of inelastic demand, which requires that each voter must vote, even if the nearest candidate is very far from her location. Given the emphasis placed in elections on "turning out the vote," it seems to us that voter abstention or apathy is a crucial aspect that should be captured. More generally, we believe that quantifying instability in this manner may be a useful and interesting line of research.

In our model, we assume that the utilities of the candidates are normalised, and we study additive approximations to equilibria. We believe this is well-motivated. For normalised utilities, additive approximations to equilibria are more robust than multiplicative approximations. For example, in our model, consider two candidate strategy profiles X, X', so that for any candidate i, $|x_i - x'_i| \leq \delta$. Then if X is an additive ϵ-approximate equilibrium, X' is an additive $(\epsilon + 4M\delta)$-equilibrium, where $M \geq f(z)$ for all $z \in [0, 1]$. However, if X is a multiplicative ϵ-approximate equilibrium, we cannot give any similar guarantees for X'. Further, lower bounds on additive approximations directly imply lower bounds on multiplicative approximations. However obtaining multiplicative bounds on approximate equilibria is clearly an important question, which remains open for future work.

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