Achieving Maximin Share and EFX/EF1 Guarantees Simultaneously

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Abstract

We study the problem of computing *fair* divisions of a set of indivisible goods among agents with *additive* valuations. For the past many decades, the literature has explored various notions of fairness, that can be primarily seen as either having *envy-based* or *share-based* lens. For the discrete setting of resource-allocation problems, *envy-free up to any good* (EFX) and *maximin share* (MMS) are widely considered as the flag-bearers of fairness notions in the above two categories, thereby capturing different aspects of fairness herein. Due to lack of existence results of these notions and the fact that a good approximation of EFX or MMS does not imply particularly strong guarantees of the other, it becomes important to understand the compatibility of EFX and MMS allocations with one another.

In this work, we identify a novel way to simultaneously achieve MMS guarantees with EFX/EF1 notions of fairness, while beating the best known approximation factors by Chaudhury et al. and Amanatidis, et al. Our main contribution is to constructively prove the existence of (i) a partial allocation that is both $2/3$-MMS and EFX, and (ii) a complete allocation that is both $2/3$-MMS and EF1. Our algorithms run in pseudo-polynomial time if the approximation factor for MMS is relaxed to $2/3 - \varepsilon$ for any constant $\varepsilon > 0$ and in polynomial time if, in addition, the EFX (or EF1) guarantee is relaxed to $(1 - \delta)$-EFX (or $(1 - \delta)$-EF1) for any constant $\delta > 0$. In particular, we improve from the best approximation factor known prior to our work by Chaudhury et al., which computes partial allocations that are $1/2$-MMS and EFX in pseudo-polynomial time.

1 Introduction

The theory of fair division addresses the fundamental problem of allocating a set of resources among a group of individuals, with varied preferences, in a meaningfully *fair* manner. The need of fairness is a key concern in the design of many social institutions, and it arises naturally in multiple real-world settings such as division of inheritance, dissolution of business partnerships, divorce settlements, assigning computational resources in a cloud computing environment, course assignments, allocation of radio and television spectrum, air traffic management, to name a few (see (Pratt and Zeckhauser 1990; Brams and Taylor 1996b,a; Moulin 2004; Budish and Cantillon 2010)).

The numerous applications depicting the necessity of understanding the process of fairly dividing resources among multiple economic players has given rise to a formal theory of *fair division*. Such problems lie at the interface of economics, mathematics, and computer science, and they have been extensively studied for past several decades (Moulin 2019). Although the roots of fair division can be found in antiquity, for instance, in ancient Greek mythology and the Bible, its formal history stretches back to the seminal work of Steinhaus, Banach and Knaster in 1948 (Steinhaus 1948).

In this work, we study the well-studied fair division setting of allocating a set of discrete or indivisible items among agents. A fair division instance consists of a set $\mathcal{N} = [n]$ of n agents and a set $\mathcal{M}$ of m items. Every agent i specifies her preferences over the items via an *additive* valuation function $v_i : 2^{\mathcal{M}} \rightarrow \mathbb{R}_{\geq 0}$. The goal is to find a partition $X = (X_1, \ldots, X_n)$ of the given items such that every agent $i \in \mathcal{N}$, upon receiving bundle X_i, considers X to be *fair*.

Primarily, there have been two ways of defining fairness for resource-allocation settings: (i) *envy-based*, where an agent *compares* her bundle with other bundles in the allocation to decide if it is *fair* to her and (ii) *share-based*, where an agent considers an allocation to be *fair* for her through the value she obtains from her bundle (irrespective of what others receive). *Envy-freeness* (Foley 1966) is arguably the flag-bearer of envy-based notions of fairness that entails an allocation X to be *fair* if every agent values her own bundle at least as much as she value any other agent's bundle (i.e., $v_i(X_i) \geq v_i(X_j)$ for all $i, j \in [n]$). On the other hand, *proportionality* is an important share-based notion of fairness that entails an allocation to be fair when every agent $i \in [n]$ values her bundle at least as much as her proportional share value of $v_i(\mathcal{M})/n$. Both of these notions are known to exist in the setting where the resource is divisible (i.e., a cake $[0, 1]$), but unfortunately, a simple instance where a single valuable (indivisible) item is to be divided between two agents does not admit any envy-free or proportional allocation.

Within the last decade, we have seen an extensive study of various relaxations of envy-freeness and proportionality that are more suitable for the discrete setting. Among those, the most prominent relaxations, and the focus of our work, include the notions of *envy-freeness up to any good* (EFX) (Caragiannis et al. 2016), *envy-freeness up to one good* (EF1) (Lipton et al. 2004), and *maximin share* fairness (MMS) (Budish 2011). Here, EFX and EF1 relax the notion of envy-freeness, while maximin share is considered to be a natural relaxation of proportionality.

Maximin share (MMS) has been one of the most celebrated relaxations of proportionality for the discrete setting. The maximin share value ($\mathsf{MMS}_i^n(\mathcal{M})$ or MMS_i) of an agent i, is defined to be the maximum value she can obtain among all possible allocations of the set of items $\mathcal{M}$ among n agents, while receiving the minimum-valued bundle in any allocation. Since MMS allocations may not always exist for fair division instances with more than two agents (Procaccia 2015), a significant amount of research has been focused on achieving better approximation guarantees (i.e., for some $\alpha \in (0, 1)$, every agent i gets a value of α-MMS_i) for maximin share. A recent breakthrough proves the existence and develops a PTAS to compute $(\frac{3}{4} + \frac{3}{3836})$-MMS allocations for additive valuations (Akrami and Garg 2024).

On the other hand, focusing on relaxations of envy-freeness, in an EF1 allocation X, any agent $i \in [n]$ may envy another agent j, but the *envy* must vanish after removing some good from the bundle X_j (i.e., $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for some $g \in X_j$). EF1 allocations are known to exist and can be computed in polynomial time as well (Lipton et al. 2004). Later, Caragiannis et al. (2016) introduced a stronger relaxation of envy-freeness called EFX. Here, again, in an EFX allocation, any agent $i \in [n]$ may envy another agent j, but the envy now must vanish after removing *any* good from the bundle X_j (i.e., $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$). As a complete contrast to EF1, the notion of EFX is fundamentally more challenging and despite significant efforts, the community has not been able to fully understand the existential and computational guarantees of EFX allocations. For instance, the biggest open problem in fair division is to resolve the existence of EFX allocations for instances with four or more agents (Procaccia 2020).

Reasonably enough, several approximations and relaxations of EFX have been extensively studied (see Section 1.2 for more details). One of the notable results herein is pseudo-polynomial time computability of *partial* EFX allocations where at most $n - 1$ goods go to *charity* (i.e., remain unallocated) such that no agent envies the charity bundle (Chaudhury et al. 2021).

It is relevant to note that the notions of EF1/EFX and MMS capture different aspects of fairness. Either of EF1/EFX or MMS properties does not necessarily imply particularly strong approximation guarantees for the other(s) (Amanatidis, Birmpas, and Markakis 2018). In Section 2, we discuss the guarantees EFX/EF1 allocations can provide for MMS and vice versa. This is in complete contrast to the divisible setting guarantees, where any envy-free allocation is necessarily proportional as well. Hence, it becomes compelling to ask for allocations that attain good guarantees with respect to envy-based and share-based notions of fairness simultaneously. There are few works along these lines in the literature, e.g., (Barman et al. 2018; McGlaughlin and Garg 2020), some of which give purely existential guarantees (Caragiannis et al. 2016). Motivated by the above question, this work focuses on understanding the compatibility of two different classes of fairness notions, i.e., in particular, MMS with EF1/EFX.

1.1 Our Results

We study fair division instances with agents having additive valuations over a set of indivisible items. The aim of this work is to push our understanding of the compatibility between two different classes of fairness notions: EFX/EF1 with MMS guarantees. Our main contribution is developing (simple) algorithms for achieving EFX/EF1 and MMS guarantees simultaneously.

Main Theorem: For any fair division instance, we show that there exists

1. a partial allocation that is both $2/3$-MMS and EFX [see Theorem 4.2 and Algorithm 2].
2. a complete allocation that is both $2/3$-MMS and EF1 [see Theorem 5.2 and Algorithm 3].

If we relax $2/3$-MMS to $(2/3 - \varepsilon)$-MMS for any arbitrary constant $\varepsilon > 0$, then the above allocations can be computed in pseudo-polynomial time. If in addition to that, we relax EFX/EF1 to $(1 - \delta)$-EFX/$(1 - \delta)$-EF1, then the allocations can be computed in polynomial time.

We note that the above results have led to a new approach for finding desired partial EFX allocations, in particular, where we have a good bound on the amount of value each agent receives. It is known that EFX is not compatible with the economic efficiency notion of Pareto optimality (Plaut and Roughgarden 2020). Therefore, it may seem that, in order to guarantee EFX, one might have to sacrifice a lot of utility and agents may not receive bundles with high valuations. Nevertheless, using Algorithm 2, we prove that we can still guarantee their $2/3$-MMS value to every agent while finding a partial EFX allocation.

We use Algorithm 1 developed by Amanatidis et al. (2021) to compute $2/3$-MMS allocations as a starting point to have share-based guarantee. Here, as soon as an agent receives a bundle, she is taken out of consideration. This feature of the algorithm is incompatible with achieving any envy-based guarantees. We note that this feature is common to many other algorithms achieving share-based guarantees in the fair division literature. We overcome this barrier and develop a novel algorithm (Algorithm 2) that removes the *myopic* nature of Algorithm 1 and also looks into the future and modifies the already-allocated bundles if needed. Interestingly enough, the share-based guarantee that we maintain for a subset of agents (whose size keep growing) throughout the execution of Algorithm 2 helps us to prove envy-based guarantees as well.

Our first result improves the guarantees shown by Chaudhury et al. (2021) where they develop a pseudo-polynomial time algorithm to compute a partial allocation that is both $1/2$-MMS and EFX. Also, Amanatidis, Markakis, and Ntokos (2020) develop an efficient algorithm to compute a complete allocation that is simultaneously 0.553-MMS and 0.618-EFX; note that, this is incomparable to the guarantees that we develop in this work. On the other hand, the best known approximation factors, prior to our work, for simultaneous guarantees on MMS and EF1 was by Amanatidis, Markakis, and Ntokos (2020) where they efficiently find allocations that are $4/7$-MMS and EF1.

Finally, we also exhibit a constructive proof of

3. the existence of a (partial) allocation that is both α-MMS and EFX for $\alpha = \max(2/3, \frac{1}{2-p/n})$, where $p < n$ goods are unallocated and given to charity such that no agent envies the charity [Theorem 4.6].

Here, we improve the result of Chaudhury et al. (2021) where they prove the same existential result except that $\alpha = \frac{1}{2-p/n}$. We note that their (and our) algorithm has no power on what p will be except that it cannot be larger than $n-1$. Hence, their result does not prove any existential result on the simultaneous guarantees for EFX and δ-MMS for any constant $\delta > 1/2$.

1.2 Further Related Work

For the MMS problem, (Kurokawa, Procaccia, and Wang 2018) showed the existence of $2/3$-MMS allocations, while Barman and Krishnamurthy established its tractability for instances with additive valuations. Many follow-up works are filled with extensive studies to improve the approximation factor for MMS allocations e.g., see (Amanatidis et al. 2017b; Kurokawa, Procaccia, and Wang 2018; Ghodsi et al. 2018; Barman and Krishnamurthy 2020; Garg and Taki 2020; Feige, Sapir, and Tauber 2021; Akrami et al. 2023b; Akrami and Garg 2024) for additive, (Barman and Krishnamurthy 2020; Ghodsi et al. 2018; Uziahu and Feige 2023) for submodular, (Ghodsi et al. 2018; Seddighin and Seddighin 2022; Akrami et al. 2023c) for XOS, and (Ghodsi et al. 2018; Seddighin and Seddighin 2022) for subadditive valuations.

For EFX allocations, (Plaut and Roughgarden 2020) proved its existence for two agents with monotone valuations. A breakthrough result by Chaudhury, Garg, and Mehlhorn proved the existence of EFX allocations for instances with three agents having additive valuations. Many follow-up works strengthened this result with more general valuations (Chaudhury, Garg, and Mehlhorn 2020; Berger et al. 2022; Akrami et al. 2023a). EFX allocations exist when agents have identical (Plaut and Roughgarden 2020), binary (Halpern et al. 2020), or bi-valued (Amanatidis et al. 2021) valuations. Several approximations (Chaudhury et al. 2021; Amanatidis, Markakis, and Ntokos 2020; Chan et al. 2019; Farhadi et al. 2021) and relaxations (Amanatidis et al. 2021; Caragiannis, Gravin, and Huang 2019; Berger et al. 2022; Mahara 2021; Jahan et al. 2023; Berendsohn, Boyadzhiyska, and Kozma 2022; Akrami, Rezvan, and Seddighin 2022) of EFX have become an important line of research in discrete fair division. Inspired by the work of Aziz et al. (2018), a recent work by Caragiannis et al. (2023) has defined an interesting and useful relaxation of EFX, called *epistemic* EFX (EEFX). EEFX allocations are guaranteed to exist for instances with monotone valuations (Akrami and Rathi 2024) and can be computed in polynomial time for instances with additive valuations (Caragiannis et al. 2023).

Some of these fairness criteria have also been studied in combination with other objectives, such as Pareto optimality (Barman, Krishnamurthy, and Vaish 2018), truthfulness (Amanatidis, Birmpas, and Markakis 2016; Amanatidis et al. 2017a) or maximizing the Nash welfare (Caragiannis et al. 2016; Caragiannis, Gravin, and Huang 2019; Chaudhury et al. 2021).

An excellent recent survey by Amanatidis et al. (2022) discusses the above fairness concepts and many more. Another aspect of discrete fair division which has garnered an extensive research is when the items that needs to be divided are *chores*. We refer the readers to the survey by Guo, Li, and Deng (2023) for a comprehensive discussion.

2 Definitions and Notation

For any positive integer k, we use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. We write $\mathcal{N} = [n]$ to denote the set of n agents and $\mathcal{M} = \{g_1, \ldots, g_m\}$ to denote the set of m indivisible items. For an agent $i \in \mathcal{N}$, the valuation function $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ represents her value over the set of items. For simplicity, we will often write g instead of $\{g\}$ for an item $g \in \mathcal{M}$. In this work, we assume that valuation functions v_i's are additive i.e., for any agent $i \in \mathcal{N}$, $v_i(S) = \sum_{g \in S} v_i(g)$ for any subset $S \subseteq \mathcal{M}$. We denote a fair division instance by $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$, where $\mathcal{V} = (v_1, v_2, \ldots, v_n)$. When we say 'fair division instance with additive valuations', we mean an instance with every agent having an additive valuation.

An allocation $X = (X_1, X_2, \ldots, X_n)$ is a partition of a subset of $\mathcal{M}$ into n bundles, such that X_i is the bundle allocated to agent $i \in [n]$ and $P(X) = M \setminus \bigcup_{i \in [n]} X_i$ is the set (pool) of unallocated goods. If $P(X) = \emptyset$, then we say X is a *complete* allocation, otherwise, we say X is a *partial* allocation. Also, we write Π_n to denote the set of all partitions of $\mathcal{M}$ into n bundles, i.e., the set of all n-partitions of the set $\mathcal{M}$.

We study the approximations and relaxations of classic fairness notions of envy-freeness and proportionality. We begin by defining the concept of *strong envy* to state the fairness notions of EFX and EF1.

Definition 2.1 (Strong Envy)**.** Upon receiving bundle A, we say that agent i *strongly envies* a bundle B, if there exists an item $g \in B$ such that $v_i(A) < v_i(B \setminus g)$. Given an allocation X, agent i *strongly envies* agent j if there exists an item $g \in X_j$ such that $v_i(X_i) < v_i(X_j \setminus g)$.

Definition 2.2 (Envy-freeness up to any item (EFX))**.** An allocation X is EFX if no agent strongly envies any other agent. In other words, for all $i, j \in \mathcal{N}$ and all $g \in X_j$, we have $v_i(X_i) \geq v_i(X_j \setminus g)$.

For any $\alpha \geq 0$, an allocation X is α-EFX, if for all $i, j \in \mathcal{N}$ and all $g \in X_j$, we have $v_i(X_i) \geq \alpha \cdot v_i(X_j \setminus g)$.

Definition 2.3 (Envy-freeness up to one item (EF1))**.** An allocation X is EF1, if for all $i, j \in \mathcal{N}$, we either have $v_i(X_i) \geq v_i(X_j)$, or there exists $g \in X_j$ such that $v_i(X_i) \geq v_i(X_j \setminus g)$.

Similarly, for any $\alpha \geq 0$, an allocation X is α-EF1, if for all $i, j \in \mathcal{N}$, we either have $v_i(X_i) \geq \alpha \cdot v_i(X_j)$ or there exists $g \in X_j$ such that $v_i(X_i) \geq \alpha \cdot v_i(X_j \setminus g)$.

We now define the concept of *most envious agent* for a bundle, which come useful in developing our algorithms.

Definition 2.4 (Most envious agent)**.** Given a bundle B and a (partial) allocation X, an agent $i \in \mathcal{N}$ is a *most envious agent* of bundle B, if there exists a proper subset $B' \subsetneq B$ such that $v_i(B') > v_i(X_i)$ and no other agent $j \in \mathcal{N}$ such that $j \neq i$ strongly envies B'.

Observation 2.5. Given a fair division instance with additive valuations, consider a bundle B and a (partial) allocation X. If there exists an agent i who strongly envies B, then there exists an agent who is a most envious agent of B, and she can be identified in polynomial time.

We next discuss the share-based fairness notion of *maximin share* (MMS). We define the maximin share value of an agent $i \in [n]$ as the maximum value she can guarantee for herself, if she partitions the goods into n bundles and receives a bundle with minimum value (to her). Then, for any agent $i \in [n]$, we write her maximin share value as,

$$\mathsf{MMS}_i^n(\mathcal{M}) := \max_{(A_1,\dots,A_n)\in\Pi_n} \min_{A_j} v_i(A_j). \tag{1}$$

where, Π_n is the set of all partitions of $\mathcal{M}$ into n bundles. When n and $\mathcal{M}$ are clear from the context, we write MMS_i instead of $\mathsf{MMS}_i^n(\mathcal{M})$.

Definition 2.6 (α-MMS Allocation)**.** For any $\alpha \in [0, 1]$, allocation X is α-MMS, if for all agents $i \in \mathcal{N}$, we have $v_i(X_i) \geq \alpha \cdot \mathsf{MMS}_i$. We say an allocation X is MMS, if it is 1-MMS.

Note that, the definition of MMS dictates that for all $i \in \mathcal{N}$, there exists a partition $(A_1, \dots, A_n)$ of $\mathcal{M}$ such that $v_i(A_j) \geq \mathsf{MMS}_i^n(\mathcal{M})$ for all $j \in [n]$. We call such a partition as an MMS-*partition of agent* i. Similarly, a partition $(A_1, \dots, A_n)$ of $\mathcal{M}$ such that $v_i(A_j) \geq \alpha\mathsf{MMS}_i^n(\mathcal{M})$ for all $j \in [n]$ is an α-MMS-*partition of agent* i.

Proposition 2.7 (Woeginger (1997))**.** *Given any fair division instance with additive valuations, there exists a PTAS to compute an* MMS-*partition of any agent* $i \in \mathcal{N}$*, and hence her* MMS_i *value as well.*

Lastly, we define two graphs inspired by share-based and envy-based fairness notions, that will prove useful in our algorithms.

Definition 2.8 (Threshold-Graph)**.** Given a partition $Y = (Y_1, \dots, Y_n)$ of $\mathcal{M}$ into n bundles and given a vector $t = (t_1, \dots, t_n) \in \mathbb{R}^n_{\geq 0}$, we define the *threshold-graph* as an undirected bipartite graph $T_{\langle Y,t\rangle} = (V, E)$, where V has one part consisting of n nodes corresponding to the agents and another part with n nodes corresponding to the bundles $Y_1, \dots, Y_n$. There exists an edge (i, j) between (the node corresponding to) agent i and (the node corresponding to) bundle Y_j if and only if $v_i(Y_j) \geq t_i$. For all $i \in [n]$, we call t_i, the threshold share value of agent i.

For a subset S of the nodes, we write $N(S)$ to denote the set of neighbours of the nodes in S in the threshold graph.

Definition 2.9 (Envy-Graph)**.** Given an allocation X, we define the *envy-graph* of X as a directed graph $G_X = (V, E)$ where V is a set of n nodes corresponding to agents, and there exists an edge from (the node corresponding to) agent i to (the node corresponding to) agent j, if and only if agent i envies agent j, i.e., $v_i(X_j) > v_i(X_i)$.

Now we briefly discuss the guarantees EFX/EF1 allocations can provide for MMS and vice versa. Amanatidis, Birmpas, and Markakis (2018) gave a comprehensive comparison between these notions of fairness. Here we mention a few.

Proposition 2.10 (Amanatidis, Birmpas, and Markakis (2018))**.** *For arbitrary* $n \geq 1$*, any* EFX *allocation is also a* $4/7$-MMS *allocation. On the other hand, an* EFX *allocation is not necessarily an* α-MMS *allocation for* $\alpha > 0.5914$ *and large enough* n*.*

Proposition 2.11 (Amanatidis, Birmpas, and Markakis (2018))**.** *An* EF1 *allocation is not necessarily an* α-MMS *allocation for any* $\alpha > 1/n$*.*

Proposition 2.12. *For* $n \geq 3$ *and any* $\alpha > 0$*, an* α-MMS *allocation is not necessarily* β-EF1 *for any* $\beta > 0$*.*

Therefore, we can conclude that, by guaranteeing one of approximate MMS or approximate EFX/EF1, one cannot obtain a good guarantee for the other notion of fairness for free.

3 Guaranteeing $\frac{2}{3}$-MMS

In this section, we describe and analyze the algorithm developed by Amanatidis et al. (2017b); Procaccia and Wang (2014) to compute $2/3$-MMS allocations for fair division instances with additive valuations. We rewrite it and analyze it in our own words (in Algorithm 1) since we use it to develop our main algorithm (Algorithm 2) to compute allocations that are both $2/3$-MMS and EFX.

Budish, while introducing maximin share, also showed that it is scale-invariant. That is, we can assume $\mathsf{MMS}_i = 1$ for all agents $i \in \mathcal{N}$.

Surprisingly, Algorithm 1 does *not* rely on the two most commonly used tools for computing approximate MMS allocations, namely *ordered instances* and *valid reductions*. We refer the reader to (Akrami and Garg 2024) for a description of these tools. Unfortunately, none of these tools can be used when dealing with envy-based notions of fairness. And hence, most of the previous works that achieve approximate MMS guarantees do not obtain any envy-based criteria results. On the other hand, most of the previous work that achieve simultaneous guarantees for MMS and EFX/EF1 are obtained by manipulating algorithms that provide EFX/EF1 guarantees so that some approximation for MMS can also be achieved (Chaudhury et al. 2021; Amanatidis, Markakis, and Ntokos 2020). However, so far, the envy-based algorithmic techniques have not been strong enough to also attain $2/3$-MMS guarantee.

Overview of Algorithm 1: Algorithm 1 successively allocates a bundle of items to some selected agents in each step and removes them from consideration. In particular, in each round of Algorithm 1 with n' remaining agents, we ask a remaining agent i to divide the remaining items into n' bundles $X_1, \dots, X_{n'}$, each of value at least $2/3$ to her. By

Algorithm 1: $\texttt{approxMMS}(\mathcal{I})$

Input: A fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$ with additive valuations
Output: An allocation X

```
Let MMS_i = MMS_i^n(M) for all i ∈ [n]
while N ≠ ∅ do
    n ← |N|
    Let i ∈ N
    Let (X_1, ..., X_n) be a partition of M such that v_i(X_j) ≥ (2/3)MMS_i
    Let T_⟨X,t⟩ be the threshold-graph with X = (X_1, ..., X_n) and t = (2/3)(MMS_1, ..., MMS_n) for agents in [n]
    Let M = {(k + 1, X_{k+1}), ..., (n, X_n)} be a matching of size at least 1 such that N({X_{k+1}, ..., X_n}) = {k + 1, ..., n} and X_j is matched to j for all j ∈ [n] \ [k];
    N ← [k];
    M ← M \ ⋃_{ℓ∈[n]\[k]} X_ℓ;
end while
return (X_1, X_2, ..., X_n);
```

Lemma 3.1, the above is always possible at every step of the algorithm. Then, we consider the threshold graph $T_{\langle X,t\rangle}$ with $X = (X_1, \dots, X_{n'})$ and $t = (\frac{2}{3}, \dots, \frac{2}{3})$ and find a matching between the bundles and the agents such that (i) every matched agent has a value of at least $2/3$ for the bundle matched to her and (ii) every unmatched agent values any of the matched bundles at less than $2/3$. We allocate according to this matching, and remove the matched agents with their matched bundles. As long as there is any remaining agent, we repeat the above process. See Algorithm 1 for the pseudo-code of this algorithm. A similar technique is also used in (Steinhaus 1948; Kuhn 1967; Aigner-Horev and Segal-Halevi 2022; Hummel 2024).

Lemma 3.1 (Kurokawa, Procaccia, and Wang (2018))**.** *Fix an agent $i \in \mathcal{N}$ and some $k < n$. Consider k bundles $A_1, \dots, A_k \subseteq \mathcal{M}$ such that for all $j \in [k]$, we have $v_i(A_j) < \frac{2}{3} \cdot \mathsf{MMS}_i^n(\mathcal{M})$ for agent i. Then, there exists a partition $(B_1, \dots, B_{n-k})$ of the remaining items in $\mathcal{M} \setminus \cup_{j\in[k]} A_j$ into $n - k$ bundles such that $v_i(B_j) \geq \frac{2}{3} \cdot \mathsf{MMS}_i^n(\mathcal{M})$ for all $j \in [n - k]$.*

Theorem 3.2 (Amanatidis et al. (2017b))**.** *For fair division instances with additive valuations, Algorithm 1 returns a $\frac{2}{3}$-*MMS *allocation.*

4 $\frac{2}{3}$-MMS Together with EFX

In this section, we modify Algorithm 1 such that the output is a (partial) allocation which is still $2/3$-MMS and now becomes EFX as well. Note that, in Algorithm 1 and generally in the algorithmic technique of Amanatidis et al. (2017b), once an agent receives a bundle X_i, X_i does become her bundle in the final output allocation. So, once agent i receives the bundle X_i, she is out of the consideration. This guarantees that agent i will have the same utility $v_i(X_i)$ in the end of the algorithm but it does not guarantee anything about how much i values other bundles formed once she is removed from consideration. And, therefore, it cannot guarantee EFX (or even EF1) property.

We overcome this barrier by developing Algorithm 2 in this section. Here, we again allocate a bundle of items to some selected agents in each step, but we modify them carefully in a later stage. As we will describe next, this feature of our algorithm removes the *myopic* nature of Algorithm 1 and lets us achieve envy-based fairness guarantees, while maintaining $2/3$-MMS guarantees.

Overview of Algorithm 2: In each round of Algorithm 2 with $n' \leq n$ remaining agents, we ask a remaining agent i to partition the remaining goods into n' bundles $X_1, \dots, X_{n'}$ of value at least $(2/3)\mathsf{MMS}_i$. We prove, in Lemma 3.1, that it is always feasible to perform the above process at every step of the algorithm. We then shrink these bundles to guarantee that every remaining agent values each strict subset of these bundles less than $2/3$ fraction of their MMS value. For simplicity, we rename the shrinked bundles again as $X_1, \dots, X_{n'}$.

Now, let us assume that, after the process of shrinking, we still have an agent j who was allocated a bundle in previous iterations and who strongly envies one of $X_1, \dots, X_{n'}$, say, for instance, X_j. Let us denote a^* to be the most envious agent of X_j. We allocate, to a^*, a subset of X_j which a^* envies but no agent strongly envies. In this way, we guarantee two things at each point during the algorithm, the current (partial) allocation among the agents who received a bundle so far is (a) EFX and (b) all these agents receive $2/3$ fraction of their MMS value. See Algorithm 2 for the pseudocode.

To the best of our knowledge, none of the previous algorithms computing an EFX allocation allocates a bundle to some of the agents and nothing to the rest in an intermediate step. It might also seem counter-intuitive to do so, since we need to guarantee that there are enough items left to satisfy the agents who have received nothing so far. We are able to make it possible in Algorithm 2, since we know that all the remaining agents (who have not yet received anything) value all the already allocated bundles less than $2/3$ fraction of their MMS value. Interestingly enough, the share-based guarantee that we are maintaining helps us to prove envy-based guarantees as well.

Lemma 4.1. *For a given partition $X \in \Pi_n$ of $\mathcal{M}$ and a threshold vector $t = (t_1, \dots, t_n)$, assume for all $j \in [n]$, there is an agent i such that $v_i(X_j) \geq t_i$. Then $T_{\langle X,t\rangle}$ has a non-empty matching $M = \{(i_1, X_{j_1}), \dots, (i_k, X_{j_k})\}$ such that $N(\{X_{j_1}, \dots, X_{j_k}\}) = \{i_1, \dots, i_k\}$. Moreover, M can be computed in polynomial time.*

Theorem 4.2. *For any fair division instance with additive valuations, Algorithm 2 returns a (partial) allocation that is both* EFX *and* $2/3$-MMS.

Proof. We will begin by proving the correctness of Algorithm 2, and then prove that it always terminates. First note that by Lemma 3.1, at the beginning of each iteration of the while-loop, any agent $i \in \mathcal{N}'$ can partition the remaining

Algorithm 2: $\texttt{approxMMSandEFX}(\mathcal{I})$

Input: A fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$ with additive valuations
Output: An allocation X

1: Let $\mathsf{MMS}_i = \mathsf{MMS}_i^n(\mathcal{M})$ for all $i \in [n]$
2: $\mathcal{N}' \leftarrow [n]$
3: **while** $\mathcal{N}' \neq \emptyset$ **do**
4: $\quad n' \leftarrow |\mathcal{N}'|$
5: $\quad$ Let $i \in \mathcal{N}'$
6: $\quad$ Let $(X_1, \dots, X_{n'})$ be a partition of $\mathcal{M}$ such that $v_i(X_j) \geq \frac{2}{3}\mathsf{MMS}_i$
7: $\quad$ **for** $j \in [n']$ **do**
8: $\quad\quad X_j \leftarrow$ minimal subset of $X'_j \subseteq X_j$ such that $\exists i' \in [n']$ with $v_{i'}(X'_j) \geq \frac{2}{3}\mathsf{MMS}_{i'}$
9: $\quad\quad$ **if** $\exists a \in [n] \setminus [n']$ such that a strongly envies X_j **then**
10: $\quad\quad\quad$ Let $a^* \in [n] \setminus [n']$ be a most envious agent of X_j
11: $\quad\quad\quad$ Let $X'_j \subseteq X_j$ be minimal such that $v_{a^*}(X'_j) > v_{a^*}(X_{a^*})$ and no agent strongly envies X'_j
12: $\quad\quad\quad \mathcal{M} \leftarrow \mathcal{M} \cup X_{a^*} \setminus X'_j$
13: $\quad\quad\quad X_{a^*} \leftarrow X'_j$
14: $\quad\quad\quad$ Go to Line 3
15: $\quad\quad$ **end if**
16: $\quad$ **end for**
17: $\quad$ Let $T_{\langle X,t \rangle}$ be the threshold-graph with $X = (X_1, \dots, X_{n'})$ and $t = \frac{2}{3}(\mathsf{MMS}_1, \dots, \mathsf{MMS}_{n'})$ for agents in $[n']$
18: $\quad$ Let $M = \{(k+1, X_{k+1}), \dots, (n', X_{n'})\}$ be a matching of size at least 1 such that $N(\{X_{k+1}, \dots, X_n\}) = \{k+1, \dots, n\}$ and X_j is matched to j for all $j \in [n] \setminus [k]$;
19: $\quad \mathcal{N}' \leftarrow [k]$;
20: $\quad \mathcal{M} \leftarrow \mathcal{M} \setminus \bigcup_{\ell \in [n] \setminus [k]} X_\ell$;
21: **end while**
22: **return** $(X_1, X_2, \dots, X_n)$;

goods into $|\mathcal{N}'|$ many sets of value at least MMS_i (to her). The reason is that although the previously assigned bundles can change, their value for agent i (in fact any agent) can only decrease and the condition of lemma 3.1 holds.

Consider any arbitrary iteration of the while-loop during the execution of Algorithm 2. Let us assume there are n' remaining agents at the start of this iteration. Without loss of generality, we can rename these remaining agents as $1, 2, \dots, n'$. This means that every agent $i \in [n] \setminus [n']$, has been assigned some bundle, say X_i. We begin by proving that $v_i(X_i) \geq \frac{2}{3}\mathsf{MMS}_i$ and that agent i does not strongly envy any agent $j \in [n] \setminus [n']$.

We establish the above claim by induction. Since, initially no agent is assigned any bundle, the claim holds. Now, as the induction hypothesis, we assume that agents in $[n] \setminus [n']$ are already assigned a bundle and the (partial) allocation restricted to them is $2/3$-MMS and EFX. Any change in the bundles as a result of the current while-loop can be examined by the following two cases: either the if-condition in Line 9 is satisfied, and hence, only the bundle of agent a^* changes in this iteration, or the if-condition in Line 9 is not satisfied.

Case 1: The if-condition in Line 9 is satisfied. First, note that, only the bundle of agent a^* changes in this case. Let X_{a^*} and X'_{a^*} be the bundle of agent a^* before and after this iteration of the while-loop respectively. By Line 11, we know that $v_{a^*}(X'_{a^*}) > v_{a^*}(X_{a^*}) \geq \frac{2}{3}\mathsf{MMS}_{a^*}$, and hence, the allocation restricted to $[n] \setminus [n']$ is still $2/3$-MMS. Moreover, by the choice of X'_{a^*} in Line 10, no agent in $[n] \setminus [n']$ strongly envies X'_{a^*}. Since a^* did not strongly envy anyone while owning X_{a^*}, she still does not strongly envy anyone while owning X'_{a^*}. Hence, the allocation restricted to the set of agents in $[n] \setminus [n']$ is EFX and $2/3$-MMS.

Case 2: The if-condition in Line 9 is not satisfied. Using Lemma 4.1, we know that the threshold-graph considered in Line 17 contains a matching M of size at least one, such that, no unmatched agent has an edge to a matched bundle.

Now, without loss of generality, we rename the agents and bundles such that $M = \{(k+1, X_{k+1}), \dots, (n', X_{n'})\}$. Therefore, agents in the set $[n] \setminus [k]$ hold some non-empty bundle. Note that, by induction hypothesis and by the definition of the threshold-graph, we know that for all agents $i \in [n] \setminus [k]$, we have $v_i(X_i) \geq \frac{2}{3}\mathsf{MMS}_i$.

Therefore, it remains to prove that the allocation restricted to agents in $[n] \setminus [k]$ is EFX as well. We split these agents into the set $[n] \setminus [n']$ and $[n'] \setminus [k]$. By induction hypothesis, we already know the allocation restricted to $[n] \setminus [n']$ is EFX. Since the if-condition in Line 9 is not satisfied, no agent in $[n] \setminus [n']$ strongly envies any agent in $[n'] \setminus [k]$. For all $i \in [n'] \setminus [k]$, we have $v_i(X_i) \geq \frac{2}{3}\mathsf{MMS}_i$ and $v_i(X_j) < \frac{2}{3}\mathsf{MMS}_i$ for all $j \in [n] \setminus [n']$. Hence, no agent in $[n'] \setminus [k]$ envies any agent in $[n] \setminus [n']$. Also, for all $i, j \in [n'] \setminus [k]$ and all $X'_j \subsetneq X_j$, we have $v_i(X'_j) < \frac{2}{3}\mathsf{MMS}_i$ (see Line 8). Since $v_i(X_i) \geq \frac{2}{3}\mathsf{MMS}_i$, i does not strongly envy j. Finally, we prove that Algorithm 2 terminates and allocates a non-empty bundle to all agents. Let us write A to denote the set of agents who are allocated a non-empty bundle at any point during the execution of Algorithm 2. We will prove that after each iteration of the while-loop, the vector $(\sum_{i \in A} v_i(X_i), |A|)$ increases lexicographically, and hence, the algorithm must terminate. In Case 1, the utility of a^* increases while the utility of all other agents in A does not change and also $|A|$ does not change. In Case 2, since the matching M found in Line 18 is of size at least one, at least one more agent is added to the set A and thus $|A|$ increases. Since, all agents who were previously in A, remain in A and their utilities do not change, the claim follows. $\square$

Note that, the vector $(\sum_{i \in A} v_i(X_i), |A|)$ can take pseudo-polynomially many values, and the only steps in Algorithm 2 that cannot be executed in polynomial time are related to computing the exact MMS values of agents and the construction of the bundles $X = (X_1, \dots, X_{n'})$ such that $v_i(X_j) \geq (2/3)\mathsf{MMS}_i$ in Line 6 of the while-loop. However, if we replace the MMS bound $2/3$ with $2/3 - \varepsilon$ for any constant $\epsilon > 0$, these steps can be executed in polynomial time. Therefore, we obtain the following result.

Theorem 4.3. *For fair division instances with additive valuations and any constant $\varepsilon > 0$, a (partial) allocation that is both* EFX *and* $(2/3 - \varepsilon)$-MMS *can be computed in pseudo-polynomial time.*

The only reason why the algorithm runs pseudo-polynomial time and not polynomial time, is that $\sum_{i \in A} v_i(X_i)$ in $(\sum_{i \in A} v_i(X_i),\ |A|)$ can take pseudo-polynomially many values. By relaxing the notion of exact EFX to $(1-\delta)$-EFX for any constant δ, we make sure that $v_i(X_i)$ can improve $\log_{1/(1-\delta)}(v_i(\mathcal{M}))$ many times which bounds the total number of rounds polynomially.

Theorem 4.4. *For fair division instances with additive valuations and any constant $\delta > 0$ and $\varepsilon > 0$, a (partial) allocation that is both* $(1-\delta)$-EFX *and* $(2/3 - \varepsilon)$-MMS *can be computed in polynomial time.*

4.1 Ensuring $2/3$-MMS and EFX with Charity

In this section, we show that we can bound the number and the value of items that go unallocated in Algorithm 2. We do so by using the algorithm `EFXwithCharity` developed by Chaudhury et al. (2021) which takes a partial allocation Y as input and outputs a (partial) EFX allocation X with the properties mentioned in Theorem 4.5.

Theorem 4.5. *(Chaudhury et al. 2021) Given a (partial)* EFX *allocation Y, there exists a (partial)* EFX *allocation $X = (X_1, \ldots, X_n)$, such that for all $i \in [n]$*

1. *X is $\frac{1}{2-|P(X)|/n}$-MMS, and*
2. *$v_i(X_i) \geq v_i(Y_i)$, and*
3. *$v_i(X_i) \geq v_i(P(X))$, and*
4. *$|P(X)| < s$,*

where s is the number of sources in the envy-graph of X.

Therefore, if we run `EFXwithCharity` on the output of Algorithm 2 (which is EFX and $2/3$-MMS), we end up with a (partial) EFX allocation which is still $2/3$-MMS but also has all the properties that `EFXwithCharity` guarantees.

Theorem 4.6. *For any fair division instance with additive valuations, there exists a (partial)* EFX *allocation $X = (X_1, \ldots, X_n)$ such that*

1. *X is $\max(2/3, \frac{1}{2-p/n})$-MMS, and*
2. *for all $i \in [n]$, $v_i(X_i) \geq v_i(P(X))$, and*
3. *$|P(X)| < s$,*

where s is the number of sources in the envy-graph of X.

5 $\frac{2}{3}$-MMS Together with EF1

In this section, we show that we can compute a complete allocation that is both $2/3$-MMS and EF1. Starting from the output of Algorithm 2, we run the well-known *envy-cycle elimination procedure* (Lipton et al. 2004) on the remaining items to obtain an EF1 allocation which is $2/3$-MMS as well; see Algorithm 3. We note that our result improves upon the previously best known approximation factor by Amanatidis, Markakis, and Ntokos (2020) where they efficiently find allocations that are $4/7$-MMS and EF1.

The following lemma follows from the work of (Lipton et al. 2004).

Algorithm 3: `approxMMSandEF1`$(\mathcal{I})$

Input: A fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$
Output: A complete allocation X

```
X ← approxMMSandEFX(I)
X ← envyCycleElimination(I, X)
return X;
```

Lemma 5.1. *Given an instance $\mathcal{I}$, if X is a partial* EF1 *allocation, then* `envyCycleElimination`$(\mathcal{I}, X)$ *returns a complete* EF1 *allocation Y in polynomial time such that $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$.*

We now prove our next result that deals with the compatibility of EF1 allocations with MMS guarantees.

Theorem 5.2. *For fair division instances with additive valuations, Algorithm 3 returns a complete allocation which is* EF1 *and* $2/3$-MMS.

Proof. For a given fair division instance, Algorithm 3 begins by running Algorithm 2 as a subroutine. By Theorem 4.2, we know that `approxMMSandEFX`$(\mathcal{I})$ returns a partial allocation X which is $2/3$-MMS and EFX and thus EF1. Then, it runs envy-cycle elimination with the remaining items. And, by Lemma 5.1, we know that `envyCycleElimination`$(\mathcal{I}, X)$ returns a complete allocation Y which is EF1. Moreover, Lemma 5.1 shows that $v_i(Y_i) \geq v_i(X_i)$ for all agents i. Since X is a $2/3$-MMS allocation, Y continues to be a $2/3$-MMS allocation as well. This completes our proof. ☐

Note that, the envy-cycle elimination procedure runs in polynomial time. For any constant $\varepsilon > 0$ and $\delta > 0$, by Theorem 4.3, we can compute a complete a $(2/3-\varepsilon)$-MMS and EF1 allocation in pseudo-polynomial and by Theorem 4.4, we can compute a $(2/3 - \varepsilon)$-MMS and $(1-\delta)$-EF1 allocation in polynomial time.

Theorem 5.3. *For fair division instances with additive valuations and any constants $\varepsilon > 0$ and $\delta > 0$, a complete allocation that is both* EF1 *and* $(2/3-\varepsilon)$-MMS *can be computed in pseudo-polynomial time and a complete allocation that is both* $(1-\delta)$-EF1 *and* $(2/3 - \varepsilon)$-MMS *can be computed in polynomial time.*

6 Conclusion

In this work, we embark upon pushing our understanding of achieving guarantees for MMS with EFX/EF1 notions of fairness. We improve the approximation guarantees for the above by developing pseudo-polynomial time algorithms to compute, for any constant $\varepsilon > 0$, (i) a partial allocation that is both $(2/3 - \varepsilon)$-MMS and EFX, and (ii) a complete allocation that is both $(2/3 - \varepsilon)$-MMS and EF1.

While enhancing the above fairness guarantees, we develop a new technique, via Algorithm 2, for finding desired partial EFX allocations, in particular, where we have a provable good bound on the amount of value each agent receives. An important line for future work is to further improve the simultaneous guarantees for achieving fairness notions of MMS with EFX/EF1.

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