Mixed Fair Division: A Survey

Shengxin Liu[1], Xinhang Lu[2], Mashbat Suzuki[2], Toby Walsh[2]

[1]Harbin Institute of Technology, Shenzhen, China
[2]UNSW Sydney, Australia
sxliu@hit.edu.cn, {xinhang.lu, mashbat.suzuki, t.walsh}@unsw.edu.au

Abstract

The fair allocation of resources to agents is a fundamental problem in society and has received significant attention and rapid developments from the game theory and artificial intelligence communities in recent years. The majority of the fair division literature can be divided along at least two orthogonal directions: goods versus chores, and divisible versus indivisible resources. In this survey, besides describing the state of the art, we outline a number of interesting open questions in three *mixed* fair division settings: (i) indivisible goods and chores, (ii) divisible and indivisible goods (i.e., mixed goods), and (iii) fair division of indivisible goods with subsidy.

1 Introduction

Fair division concerns the problem of *fairly* allocating resources among agents with heterogeneous preferences over the resources, a fundamental research topic in computational social choice (Rothe 2016; Brandt et al. 2016; Endriss 2017). It has a long, rich history dating back to (Steinhaus 1948), and has attracted ongoing interest from mathematicians, economists, and computer scientists (Brams and Taylor 1996; Robertson and Webb 1998; Moulin 2019; Aziz 2020; Walsh 2020; Suksompong 2021; Amanatidis et al. 2023; Nguyen and Rothe 2023). Moreover, fair division methods have gradually been deployed in practice and made publicly available (e.g., Goldman and Procaccia 2015; Budish et al. 2017; Igarashi and Yokoyama 2023).

The vast majority of fair division literature can be divided along two orthogonal directions according to:

- the (in)divisibility of the resources, and
- agents' valuations over the resources.

Specifically, in the former case, the resource is either *divisible* or *indivisible*, and in the latter case, the resource consists of either *goods* (positively valued) or *chores* (negatively valued). In many real-world scenarios, however, the resource to be allocated may be a mixture of different types. Our first example demonstrates a mixture of (indivisible) *goods and chores*: when distributing household tasks, some family member may enjoy cooking while others may find it torturous. The next example touches on a mixture of *divisible and indivisible* goods: in divorce settlement, we usually have divisible goods like money or shares, as well as indivisible goods like houses, cars, paintings, etc. Alternatively, monetary compensation (a.k.a. subsidies) could help circumvent unfair allocations of indivisible inheritances.

In this survey, we discuss fair division with mixed types of resources, capturing the aforementioned real-world applications. We mainly focus on the following three mixed fair division domains, which has received growing attention in recent years: Section 3 considers fair division of indivisible goods and chores, in which each agent may have non-negative or non-positive valuation over each item; Section 4 focuses on fair division of mixed divisible and indivisible goods (*mixed goods*); Section 5 focuses on fair division of indivisible goods with subsidy. Clearly, the first two domains relax one of the two orthogonal directions mentioned earlier. For the last two domains, they share some similarity in the sense that subsidy could be viewed as a divisible good. The key difference lies in how they approach fairness. In Section 4, both the divisible and indivisible goods are *fixed* in advance and we find *approximately* fair allocations. In Section 5, we allocate indivisible goods but introduce some *additional* amount of money in order to satisfy *exact* fairness.

This survey outlines new fairness notions and related theoretical results that are addressed in the above mixed fair division settings as well as highlights a number of open questions and interesting directions for future research.[1] While we only focus on fair division, the idea of combining mixed types of resources has been investigated in a *collective choice* context (Lu et al. 2023), where all agents share a selected subset of the resources. Extending the idea further to *participatory budgeting* (Aziz and Shah 2021; Rey and Maly 2023) or *public decision making* (Conitzer, Freeman, and Shah 2017) is an interesting and practical direction.

2 Preliminaries

For each $k \in \mathbb{N}$, let $[k] := \{1, 2, \dots, k\}$. Denote by $N = [n]$ the set of agents to whom we allocate some resource R, which may, e.g., consists of indivisible goods and chores (Section 2.2) or be a mix of divisible and indivisible goods (Section 2.3). An allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ is a partition of resource R into n bundles such that A_i is the bundle

[1]See (Liu et al. 2023) for the full version of this survey.

allocated to agent $i \in N$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$; note that A_i can be empty. An allocation is said to be *complete* if the entire resource is allocated, i.e., $\bigcup_{i \in N} A_i = R$, and *partial* otherwise. Unless specified otherwise, we assume allocations considered in this survey are complete.

2.1 Cake Cutting

When resource R is heterogeneous and infinitely *divisible*, the corresponding problem is commonly known as *cake cutting* (Procaccia 2016; Lindner and Rothe 2016). We will use cake and divisible goods interchangeably. The cake, denoted by D, is represented by the normalized interval $[0, 1]$. A *piece of cake* is a union of finitely many disjoint (closed) intervals. Each agent $i \in N$ is endowed with an integrable *density function* $f_i \colon [0, 1] \to \mathbb{R}_{\geq 0}$, capturing how the agent values each part of the cake. Given a piece of cake $S \subseteq [0, 1]$, agent i's utility over S is defined as $u_i(S) \coloneqq \int_{x \in S} f_i \,\mathrm{d}x$. Denote by $(D_1, D_2, \dots, D_n)$ the allocation of cake D. In order to access agents' density functions, the cake-cutting literature usually adopts the *RW model* (Robertson and Webb 1998), which allows an algorithm to interact with the agents via the following queries:

- $\textsc{Eval}_i(x, y)$ returns $u_i([x, y])$;
- $\textsc{Cut}_i(x, \alpha)$ asks agent i to return the leftmost point y such that $u_i([x, y]) = \alpha$, or state that no such y exists.

2.2 Mixed Indivisible Goods and Chores

We present here a general model where an agent may have positive or negative utility for each indivisible item. Denote by $O = [m]$ the set of indivisible items, and $\mathcal{O} = (O_1, O_2, \dots, O_n)$ the allocation of items O, in which agent i gets *bundle* O_i (i.e., a subset of O). Each agent $i \in N$ is endowed with a *utility function* $u_i \colon 2^O \to \mathbb{R}$ such that $u_i(\emptyset) = 0$, capturing how the agent values each bundle of the items. We will write $u_i(o)$ instead of $u_i(\{o\})$ for simplicity. A utility function u is said to be *additive* if $u(O') = \sum_{o \in O'} u(o)$ for any $O' \subseteq O$. Unless specified otherwise, we assume agents have additive utilities. We say that an item $o \in O$ is a *good* (resp., *chore*) for agent i if $u_i(o) \geq 0$ (resp., $u_i(o) \leq 0$), and let G_i (resp., C_i) be the set of goods (resp., chores) for agent i. It is worth noting that here, each agent $i \in N$ has her own *subjective* opinion on whether an item $o \in O$ is a good (i.e., $u_i(o) \geq 0$) or a chore (i.e., $u_i(o) \leq 0$). The model comprises scenarios when items are *objective* goods (resp., chores) for all agents, in which case we will specifically mention that we consider indivisible-goods (resp., indivisible-chores) setting.

(Doubly-)Monotonic Utilities While we mostly focus on additive utilities, we will identify some results that still hold with a larger class of utility functions. The utility function u_i of agent $i \in N$ is said to be *doubly-monotonic* if agent i can partition the items as $O = G_i \sqcup C_i$ such that for any item $o \in O$ and for any bundle $O' \subseteq O \setminus \{o\}$,

- $u_i(O' \cup \{o\}) \geq u_i(O')$ if $o \in G_i$, and
- $u_i(O' \cup \{o\}) \leq u_i(O')$ if $o \in C_i$.

In the indivisible-goods (resp., indivisible-chores) setting, all agents $i \in N$ have *monotonically non-decreasing (resp., non-increasing)* utility functions, that is, $u_i(S) \leq u_i(T)$ (resp., $u_i(S) \geq u_i(T)$) for any bundles $S \subseteq T \subseteq O$.

2.3 Mixed Divisible and Indivisible Goods

In the fair division model with both divisible *and* indivisible goods (henceforth *mixed goods*), resource R consists of a cake $D = [0, 1]$ and a set of indivisible *goods* $O = [m]$. Each agent $i \in N$ has a density function f_i over the cake as defined in Section 2.1 and an additive utility function u_i over indivisible goods O. Denote by $\mathcal{A} = (A_1, A_2, \dots, A_n)$ the allocation of mixed goods, where $A_i = D_i \cup O_i$ is the *bundle* allocated to agent i. Agent i's utility is defined as $u_i(A_i) \coloneqq u_i(D_i) + u_i(O_i)$. Further discussions about the model, e.g., fairness notions, extensions, are provided in Section 4.

2.4 Solution Concepts

We start with an economic efficiency notion that is fundamental in the context of fair division. Given an allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$, another allocation $\mathcal{A}' = (A'_1, A'_2, \dots, A'_n)$ is said to be a *Pareto improvement* if $u_i(A'_i) \geq u_i(A_i)$ for all $i \in N$ and $u_j(A'_j) > u_j(A_j)$ for some $j \in N$. Alternatively, we say that $\mathcal{A}$ is *Pareto dominated* by $\mathcal{A}'$. An allocation is said to satisfy *Pareto optimality (PO)* if it does not admit a Pareto improvement.

We now proceed to introduce comparison-based fairness notions, followed by fair-share-based notions.

(Approximate) Envy-Freeness *Envy-freeness*—the epitome of fairness, as Procaccia (2020) put it—requires that every agent likes her own bundle at least as much as the bundle given to any other agent.

Definition 2.1 (EF (Tinbergen 1930; Foley 1967; Varian 1974)[2])**.** An allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ is *envy-free (EF)* if for any pair of agents $i, j \in N$, $u_i(A_i) \geq u_i(A_j)$.

While envy-freeness can always be satisfied in cake cutting (Alon 1987), this is not the case with indivisible items. To circumvent this issue, relaxations of envy-freeness have been proposed and studied.

Definition 2.2 (EF1 (Lipton et al. 2004; Budish 2011; Aziz et al. 2022))**.** An indivisible allocation $\mathcal{O} = (O_1, O_2, \dots, O_n)$ is said to satisfy *envy-freeness up to one item (EF1)* if for every pair of agents $i, j \in N$ such that $O_i \cup O_j \neq \emptyset$, there exists some item $o \in O_i \cup O_j$ such that $u_i(O_i \setminus \{o\}) \geq u_i(O_j \setminus \{o\})$.

Intuitively, EF1 requires that when i envies j, the envy can be eliminated by removing i's most preferred good from j's bundle *or* i's least valuable chore from her own bundle. We strengthen EF1 based on the idea that any envy should be eliminated by removing *any* non-zero valued item.

Definition 2.3 (EFX for indivisible goods and chores (Aziz et al. 2022; Aziz and Rey 2020; Hosseini et al. 2023))**.** An indivisible allocation $\mathcal{O} = (O_1, O_2, \dots, O_n)$ is *envy-free up to any item with non-zero value (EFX)* if for any pair of agents $i, j \in N$ and any item $o \in (C_i \cap O_i) \cup (G_i \cap O_j)$ such that $u_i(o) \neq 0$, we have $u_i(O_i \setminus \{o\}) \geq u_i(O_j \setminus \{o\})$.

[2]We refer the interested readers to the paper of Heilmann and Wintein (2021) for more discussion on (Tinbergen 1930).

We defer our discussion on relaxations of envy-freeness in the mixed-goods model to Section 4. Spoiler alert – Bei et al. (2021a) proposed a notion that naturally combines envy-freeness and EF1 together and is guaranteed to be satisfiable.

Envy Graph The envy relations between the agents in an allocation is commonly captured by the *envy graph*, in which the vertices correspond to the agents and there is directed edge from one agent to another if the former agent envies the latter (Lipton et al. 2004). Variants of the envy graph and additional techniques are introduced in many other papers (e.g., Halpern and Shah 2019; Bei et al. 2021a; Bhaskar, Sricharan, and Vaish 2021; Amanatidis et al. 2023).

Maximin Share Guarantee We now introduce the maximin share (MMS) guarantee, and provide below a unified definition working for mixed fair division settings.[3]

Definition 2.4 (α-MMS (Budish 2011; Kulkarni, Mehta, and Taki 2021a; Bei et al. 2021b))**.** Given resource R, let $\Pi_n(R)$ be the set of n-partitions of R. Define the maximin share (MMS) of $i \in N$ as

$$\mathrm{MMS}_i = \max_{(P_1,\dots,P_n)\in\Pi_n(R)} \min_{j\in[n]} u_i(P_j).$$

An allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ of resource R is said to satisfy the *α-approximate maximin share guarantee (α-MMS)*, for some $\alpha \in [0, 1]$, if for every $i \in N$,

$$u_i(A_i) \geq \min\left\{\alpha \cdot \mathrm{MMS}_i(n, R), \frac{1}{\alpha} \cdot \mathrm{MMS}_i(n, R)\right\}.$$

That is, α-MMS requires that $u_i(A_i) \geq \alpha \cdot \mathrm{MMS}_i(n, R)$ when agent i has a non-negative maximin share (i.e., $\mathrm{MMS}_i(n, R) \geq 0$) and $u_i(A_i) \geq \frac{1}{\alpha} \cdot \mathrm{MMS}_i(n, R)$ when the agent has a negative maximin share (i.e., $\mathrm{MMS}_i(n, R) < 0$). When $\alpha = 1$, we simply refer to it as the MMS guarantee.

If an α-MMS allocation is guaranteed to exist, an α-MMS and PO allocation always exists, because an α-MMS allocation which does not admit a Pareto improvement is PO. Note that, however, it is coNP-hard to decide whether a given allocation is PO (de Keijzer et al. 2009; Aziz et al. 2019).

3 Mixed Indivisible Goods and Chores

3.1 Envy-freeness Relaxations

Chores might be viewed simply as "negative" goods. Ordinal methods for allocating goods can then be used directly by ordering chores after goods. Certain properties, however, are lost in such an approach. The fundamental problem is an asymmetry between goods and chores: an absence of goods is the worst possible outcome, but an absence of chores is the best possible outcome. We observe this (breakdown in) duality, e.g., when allocating goods using the *round-robin* algorithm, which works by arranging the agents in an arbitrary order, and letting each agent in the order choose her favourite item from the remaining items. With additive utilities, this is guaranteed to return an allocation that is EF1 (Caragiannis et al. 2019). This is not the case when we have *both goods and chores* (Aziz et al. 2022, Proposition 3). We can, however, modify the round-robin algorithm to ensure the allocation returned is EF1. At a high level, the *double round-robin algorithm* of Aziz et al. (2022) applies the round-robin algorithm twice as follows: Agents first pick *objective* chores in a round-robin fashion; we then *reverse* the picking order of the agents for the remaining items.

Theorem 3.1 (Aziz et al. (2022))**.** *For additive utilities, the double round-robin algorithm returns an EF1 allocation.*

In the indivisible-goods setting, another well-known method to compute an EF1 allocation (for agents with arbitrary monotonic utilities) is the *envy-cycle elimination algorithm* of Lipton et al. (2004), which works by iteratively allocating a good to an agent who is not envied by anyone else. We can always find such an agent by resolving *envy cycles* in the underlying envy graph of the partial allocation. As observed in (Bérczi et al. 2020; Bhaskar, Sricharan, and Vaish 2021), a naive extension of the method to the indivisible-chores setting (even for agents with additive utilities) could fail to find an EF1 allocation if envy cycles are resolved in an *arbitrary* way, let alone for mixed indivisible goods and chores. Intuitively speaking, it is because even if an agent gets a better bundle when we resolve an envy cycle, the bundle may not contain a large enough chore whose removal eliminates the envy. Nevertheless, Bhaskar, Sricharan, and Vaish (2021) introduced a key insight that we can always resolve the *top-trading envy cycle*, in which each agent only points to the agent she envies the most, and preserves EF1.

Theorem 3.2 (Bhaskar, Sricharan, and Vaish (2021))**.** *For doubly-monotonic utilities, the modified top-trading envy-cycle elimination algorithm (Bhaskar, Sricharan, and Vaish 2021, Algorithm 3) computes an EF1 allocation.*

With arbitrary utilities, an algorithm based on the envy graph finds an EF1 allocation for two agents and indivisible goods and chores in polynomial time (Bérczi et al. 2020). Can we extend this to three (or more) agents?

What about additionally demanding PO? The allocation returned by the above two methods may not be PO. In the context of allocating goods alone and additive utilities, an outcome that maximizes the Nash welfare is both EF1 and PO (Caragiannis et al. 2019).[4] The question regarding whether an EF1 and PO allocation always exists for chores alone remains unresolved. For indivisible goods and chores, Aziz et al. (2022) showed that an EF1 and PO allocation always exists for two agents using a discrete version of the well-known Adjusted Winner rule (Brams and Taylor 1996).

Open Question 1. For three (or more) agents and additive utilities, does an EF1 and PO allocation always exist? If so, can we compute the allocation in polynomial time?

A natural weakening of EF1 called *proportionality up to one item (PROP1)* has attracted interest (Conitzer, Freeman,

[3]MMS guarantee is considered as a natural relaxation of *proportionality*, which requires that every agent values her bundle at least $1/n$ of her value for the entire resource. In cake cutting (Section 2.1), a proportional allocation always exists (Steinhaus 1948).

[4]An maximum Nash welfare allocation deals with the drowning by zero problem by maximizing the number of agents receiving positive utilities, and then the product of these positive utilities.

and Shah 2017; Aziz et al. 2022). Loosely speaking, an allocation is PROP1 if each agent i receives her proportional share (i.e., $u_i(O)/n$) by obtaining an additional good or removing some chore from her bundle. The existence and computation of a PROP1 and PO allocation has been resolved, even if agents have asymmetric weights.

Theorem 3.3 (Aziz, Moulin, and Sandomirskiy (2020))**.** *For additive utilities, there exists a strongly polynomial-time algorithm that computes a PROP1 and PO allocation.*

Inspired by the concept of *group envy-freeness (GEF)* (Berliant, Thomson, and Dunz 1992)—a generalization of envy-freeness for equal-sized groups of agents, Aziz and Rey (2020) formalized relaxations of GEF (including the "up to one" relaxation abbreviated as *GEF1*) for the case of mixed indivisible goods and chores, which have both fairness and efficiency flavours. In addition to a clear taxonomy of the fairness concepts, Aziz and Rey devised polynomial-time algorithms to compute a GEF1 allocation for agents with identical utilities, or with ternary symmetric utilities of the form $\{-\alpha_i, 0, \alpha_i\}$ for a given $\alpha_i > 0$.

What if we consider a stronger fairness property like EFX, envy-freeness up to any good? With additive utilities, EFX allocations may not exist (e.g., *lexicographic preferences* over *objective* goods and chores (Hosseini et al. 2023)). However, in special cases, polynomial-time algorithms exist that always return an EFX and PO allocation (Aleksandrov and Walsh 2020; Hosseini, Mammadov, and Wąs 2023).

3.2 MMS

With indivisible goods and chores, it is NP-hard to approximate MMS values within *any* constant factor even for just two agents with an identical valuation (Kulkarni, Mehta, and Taki 2021a). Intuitively speaking, the bottleneck is that the absolute MMS value can be arbitrarily small. Kulkarni, Mehta, and Taki (2021b) gave a PTAS to compute an agent's MMS value when its absolute value is at least $1/p$ times either the total value of all the goods or total cost of all the chores, for some constant p greater than 1.

Regarding approximate-MMS allocations, with indivisible goods and chores, for any fixed $\alpha \in (0, 1]$, an α-MMS allocation may not exist (Kulkarni, Mehta, and Taki 2021a), in sharp contrast to the indivisible-goods and indivisible-chores settings. For the special case of a constant number of agents where the total value of goods is some factor away of the total absolute value of chores, Kulkarni, Mehta, and Taki gave a PTAS to find an $(\alpha - \varepsilon)$-MMS and γ-PO allocation when given $\varepsilon, \gamma > 0$, for the highest possible $\alpha \in (0, 1]$. It thus motivates the study of existence and computation of (approximate-)MMS (and possibly with PO) if agents' preferences are more restricted. To this end, given lexicographic preferences over mixed indivisible goods and chores, an MMS and PO allocation always exists and can be computed in polynomial time (Hosseini et al. 2023; Hosseini, Mammadov, and Wąs 2023).

Starting in (Bogomolnaia et al. 2017), an interesting line of work has addressed fair division of *divisible* goods and chores (Garg and McGlaughlin 2020; Garg et al. 2021; Chaudhury et al. 2023). Such divisible allocations might be adapted into randomized algorithms for indivisible goods and chores. This then naturally suggests another interesting direction for future study:

Open Question 2. Can we obtain a randomized allocation of indivisible goods and chores which has good (exact) fairness ex ante from which we can construct integral allocations with good (approximate) fairness ex post?

Such a "best-of-both-worlds" perspective has recently been receiving attention with indivisible goods (e.g., Babaioff, Ezra, and Feige 2022; Aziz et al. 2023a; Hoefer, Schmalhofer, and Varricchio 2023; Feldman et al. 2023).

4 Mixed Divisible and Indivisible Goods

When allocating mixed goods, Bei et al. (2021a) proposed the following fairness concept called *envy-freeness for mixed goods* that naturally generalizes envy-freeness and EF1 to the mixed-goods model and is guaranteed to exist.

Definition 4.1 (EFM_0 (Bei et al. 2021a, Definition 2.3))**.** An allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ of mixed goods $R = D \cup O$ is said to satisfy *envy-freeness for mixed goods (EFM_0)* if for any pair of $i, j \in N$,

- if j's bundle consists of only indivisible goods, there exists some $g \in A_j$ such that $u_i(A_i) \geq u_i(A_j \setminus \{g\})$;
- otherwise, $u_i(A_i) \geq u_i(A_j)$.

With only divisible (resp., indivisible) goods, EFM_0 reduces to envy-freeness (resp., EF1). Moreover, EFM_0 remains guaranteed to exist with mixed goods.

Theorem 4.2 (Bei et al. (2021a))**.** *An EFM_0 allocation of mixed goods always exists for any number of agents and can be found in polynomial time with polynomially many Robertson-Webb queries and calls to an oracle which could return a perfect partition of a cake.*

Despite the strong fairness guarantee provided by EFM_0, the notion is incompatible with PO (Bei et al. 2021a, Example 6.3). The counter-example hinges on the fact that in an EFM_0 allocation, agent i should not envy agent j if j's bundle contains any positive *amount* of the cake, although i may value the piece of cake at zero. The fairness criterion is called EFM in (Bei et al. 2021a); we rename it by following the nomenclature of "EFX_0 and EFX" from Kyropoulou, Suksompong, and Voudouris (2020). We let EFM be a shorthand for the following natural variant.

Definition 4.3 (EFM (Bei et al. 2021a, Definition 6.4))**.** An allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ of mixed goods $R = D \cup O$ is said to satisfy *weak envy-freeness for mixed goods (EFM)* if for any pair of $i, j \in N$,

- if j's bundle consists of indivisible goods with either no divisible good or divisible good that yields value zero to agent i (i.e., $u_i(D_j) = 0$), there exists an indivisible good $g \in A_j$ such that $u_i(A_i) \geq u_i(A_j \setminus \{g\})$;
- otherwise, $u_i(A_i) \geq u_i(A_j)$.

Open Question 3. Are EFM and PO compatible?

Other questions concerning simultaneously fairness and economic efficiency, such as maximizing social welfare within fair allocations (Aziz et al. 2023b; Sun, Chen, and

Doan 2023; Bu et al. 2023), analyzing the *price of fairness*—the worst-case loss of social welfare due to fairness constraints (Bertsimas, Farias, and Trichakis 2011; Caragiannis et al. 2012; Bei et al. 2021c; Barman, Bhaskar, and Shah 2020), etc., are equally relevant and worthy of exploration in mixed fair division settings.

While Theorem 4.2 was presented in the context of additive utilities, neither the algorithm of Bei et al. (2021a) nor its analysis hinges on the assumption of the utilities over indivisible goods being additive. As a matter of fact, EFM_0 (and hence EFM) can still always be satisfied even if agents have monotonic utilities over the indivisible goods, as long as utilities over the divisible goods are additive. An EFM allocation, however, may not exist if agents' utilities over divisible goods are not additive (Liu et al. 2023).

The mixed-goods model was later further extended to the *mixed-resources* model (Bhaskar, Sricharan, and Vaish 2021), in which the resource R consists of a set $O = [m]$ of indivisible *items* as defined in Section 2.2 and a divisible resource $[0, 1]$ which is either an objective divisible good (i.e., $\forall i \in N$, $f_i \colon [0, 1] \to \mathbb{R}_{\geq 0}$) or an objective divisible chore (i.e., $\forall i \in N$, $f_i \colon [0, 1] \to \mathbb{R}_{\leq 0}$), referred to as a "bad cake". An allocation of the mixed resources and agents' utilities in the allocation is defined the same way as in Section 2.3. With mixed resources, an allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ is said to satisfy *envy-freeness for mixed resources (EFM)* if for any pair of $i, j \in N$, either i does not envy j, that is, $u_i(A_i) \geq u_i(A_j)$, or all of the following hold:

- $u_i(D_i) \geq 0$, i.e., i does not have any bad cake,
- $u_i(D_j) \leq 0$, i.e., j does not have any cake, and
- $\exists o \in O_i \cup O_j$ such that $u_i(A_i \setminus \{o\}) \geq u_i(A_j \setminus \{o\})$.

Theorem 4.4 (Bhaskar, Sricharan, and Vaish (2021))**.** *An EFM allocation always exists when allocating mixed resources consisting of doubly-monotonic indivisible items and a divisible chore.*

The high-level algorithmic idea to compute an EFM allocation, albeit being deployed in different settings (Bei et al. 2021a; Bhaskar, Sricharan, and Vaish 2021), is as follows:

- We start with an EF1 allocation of the indivisible items. The partial allocation is therefore EFM.
- Next, we identify a subset of agents (referred to as an *addable set*) among whom we divide some divisible resources using a perfect allocation—we ensure that the EFM property is still preserved. Along the way, in order to identify an addable set, we may need to rotate bundles of the agents involved in an envy cycle. This step is repeated until we allocate all divisible resources.

The perfect allocation cannot be implemented with a finite number of queries in the RW model, even if there are only two agents (Robertson and Webb 1998). An EFM_0 (and hence EFM) allocation can be computed efficiently for two agents with general additive valuations, and for n agents with structured utilities over the cake (Bei et al. 2021a).

Open Question 4. Does there exist a bounded or even *finite* protocol in the RW model to compute an EFM allocation?

It is also challenging to integrate the algorithmic framework into the setting with indivisible chores and a cake. The EFM existence is only proved for special cases where agents have identical rankings of the indivisible chores or $m \leq n + 1$ (Bhaskar, Sricharan, and Vaish 2021).

Open Question 5. Does there always exist an EFM allocation when allocating indivisible chores and a cake?

An affirmative answer to the above question may pave the way for solving the existence of EFM in a more general setting where resource R consists of divisible and indivisible items, and each item, either divisible or indivisible, may be a good to some agents but a chore for others.

As valuations over the resources are elicited from the agents, we may want to explore the power and limitations of *truthful* mechanisms in addition to being fair. Li et al. (2023) modelled the mixed goods as a set of indivisible goods together with a set of *homogeneous* divisible goods. While truthfulness and EFM are incompatible even if there are only two agents having additive utilities over a single indivisible goods and a single divisible goods, Li et al. designed truthful and EFM mechanisms in several special cases where the expressiveness of agents' utilities are further restricted. An intriguing question left open in their paper is to show the (in)compatibility between truthfulness and EFM when agents have binary additive utilities over all goods.

Other fairness notions have also been defined and studied when allocating mixed goods. Bei et al. (2021b) extended the study of (approximate) MMS guarantee to the mixed-goods model. Nishimura and Sumita (2023) established the connection of *maximum Nash welfare* with a stronger variant of EFM when agents' utilities are binary and linear for each good, and a weaker variant of EFM when agents' utilities are general additive. Kawase, Nishimura, and Sumita (2023) studied fair mixed-goods allocations whose utility vectors minimizes a symmetric convex function.

5 Indivisible Goods with Subsidy

We now discuss how to allocate indivisible goods fairly through monetary compensation. As money can be thought of as a homogeneous divisible good, this setting fits into the framework of mixed goods studied in Section 4. The key difference is that in this section, we consider money as a tool to achieve envy-freeness rather than an exogenously given resource to be divided fairly. Earlier works in this line of research have mainly focused on the unit demand setting (Maskin 1987; Klijn 2000). The setting with an arbitrary number of goods under general additive valuations was considered only recently by Halpern and Shah (2019).

Let us first discuss what it means to be fair in the presence of monetary compensations (also called subsidy payments). Denote by $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n_{\geq 0}$ the vector of subsidy payments, where p_i denotes the amount given to agent i.

Definition 5.1. An allocation with payments $(\mathcal{O}, p)$ is *envy-free* if for any pair of $i, j \in N$, $u_i(O_i) + p_i \geq u_i(O_j) + p_j$.

In other words, an allocation with payments is envy-free if every agent prefers their own bundle plus payment to the bundle plus payment of any other agent. Note that not all allocations can be made envy-free by introducing payments.

For example, consider an instance with two agents, a single good g, and $u_1(g) > u_2(g)$. If the good is allocated to agent 2, then no subsidy payments exists so that the resulting allocation with payments is envy-free. An allocation is *envy-freeable* if it can be made envy-free by introducing payments; see also the following characterization.

Theorem 5.2 (Halpern and Shah (2019)). *For an allocation $\mathcal{O}$, the following statements are equivalent:*

- *$\mathcal{O}$ is envy-freeable.*
- *$\mathcal{O}$ maximizes utilitarian welfare among all reassignments of the bundles, i.e., for every permutation σ of the agents, $\sum_{i=1}^{n} u_i(O_i) \geq \sum_{i=1}^{n} u_i(O_{\sigma(i)})$.*
- *The envy graph $G_{\mathcal{O}}$ contains no positive-weight cycle.*[5]

It follows from Theorem 5.2 that any allocation can be made envy-freeable by reassigning the bundles. Furthermore, Halpern and Shah (2019) showed that for a fixed envy-freeable allocation $\mathcal{O}$, setting $p_i = \ell_{G_{\mathcal{O}}}(i)$, where $\ell_{G_{\mathcal{O}}}(i)$ denotes the maximum weight of any path starting from agent i in $G_{\mathcal{O}}$, not only makes $(\mathcal{O}, p)$ envy-free but also minimizes the total subsidy required for doing so.

Considering budgetary limitations of the mechanism designer, it is natural to study how much subsidy payment is required to guarantee envy-freeness. Halpern and Shah (2019) conjectured that under additive valuations, subsidy of $n-1$ always suffices.[6] Brustle et al. (2020) affirmatively settled this conjecture, and showed an stronger result as follows.

Theorem 5.3 (Brustle et al. (2020)). *For additive utilities, there exists a polynomial-time algorithm which outputs an envy-free allocation with subsidy $(\mathcal{O}, p)$ such that:*

- *Subsidy to each agent is at most one, i.e., $p_i \leq 1$.*
- *$\mathcal{O}$ is EF1 and balanced (i.e., $||O_i| - |O_j|| \leq 1 \forall i, j \in N$).*

Theorem 5.3 implies that the total subsidy needed is at most $n-1$, because if a subsidy payment eliminates envy, then these payments can be uniformly lowered while maintaining envy-freeness, and there is at least one agent who gets zero subsidy.[7] Furthermore, the bound of $n-1$ is tight. Consider an instance with a single good and n agents who all value the good at 1. Any envy-free allocation with subsidy of the instance must have a total subsidy of at least $n-1$.

The subsidy needed to guarantee envy-freeness is much less understood for valuation classes that are beyond additive. For monotonic utilities, Kawase et al. (2023) improved upon a result of Brustle et al. (2020) by showing that given an EF1 allocation, it can be computed in polynomial time an envy-free allocation with a subsidy of at most $n-1$ per agent and a total subsidy of at most $\frac{n(n-1)}{2}$.[8] As there are no lower bounds known beyond the aforementioned $n-1$ bound, this leads to a natural question.

[5] In (Halpern and Shah 2019), given an allocation $\mathcal{O}$, its envy graph $G_{\mathcal{O}}$ is the complete weighted directed graph in which for each pair of agents $i, j \in N$, directed edge (i, j) has weight $w(i, j) = u_i(O_j) - u_i(O_i)$.

[6] Each good is worth at most 1 for every agent. This is achieved without loss of generality through a scaling argument.

[7] On the other hand, Wu, Zhang, and Zhou (2023) recently showed that a total subsidy of $\frac{n}{4}$ ensures proportionality.

[8] Kawase et al.'s result works for doubly-monotonic utilities.

Open Question 6. For monotonic utilities, does there exists an allocation with total subsidy bound of $O(n^{2-\epsilon})$ for some $\epsilon > 0$?

There has been progress made towards above problem in restricted domains. Goko et al. (2022) showed that when the utilities are submodular functions with binary marginals, a total subsidy payment of $n-1$ suffices. Subsequently, Barman et al. (2022) showed that for general set valuations with binary marginals total subsidy payment of $n-1$ suffices.

A closely related direction is to study the optimization problem of computing an allocation using minimum total subsidy that achieves envy-freeness. However, approximating the minimum subsidy to any multiplicative factor is NP-hard. As a result, existing works have focused on additive approximation algorithms. Caragiannis and Ioannidis (2021) showed that for constant number of agents, an ε additive approximation algorithm can be computed in time polynomial in the number of goods and $1/\varepsilon$. Furthermore, they showed that the problem is hard to approximate to within an additive factor of $c \sum_{i \in N} u_i(O)$ for some small constant c.

For a mechanism to utilize subsidy payments, it is necessary to possess sufficient funds to disburse such subsidies. In many settings, however, the mechanism may not have access to adequate funds, making it difficult to implement. Such an issue can be circumvented if we allow for negative payments and additionally require $\sum_{i \in N} p_i = 0$. These types of payments are referred to as *transfer payments*. It can be seen that subsidy payments and transfer payments are interchangeable since whenever there is an envy-free allocation with subsidies, subtracting the average subsidy from each agent's individual payment results in payments that sum to zero. Thus, many of the results on subsidy payments carry over easily to the setting with transfer payments. Narayan, Suzuki, and Vetta (2021) studied whether transfer payments can be used to achieve both fairness and efficiency.[9] They showed that, for general monotone valuations, there exists an envy-free allocation with transfer payments whose Nash welfare is at least $e^{-\frac{1}{e}}$-fraction of the optimal Nash welfare. As for utilitarian welfare, they give algorithms to compute an envy-free allocation with transfers that achieves a prescribed target welfare with a near-optimal bound on the amount of total transfer payments needed. In a related work, Aziz (2021) showed that transfer payments can be used to give an allocation that is both envy-free and equitable provided that the valuation function is supermodular.

6 Conclusion

In this survey, we have discussed several mixed fair division settings that generalize classic models in different ways. As seen in Sections 4 and 5, divisible resources to some extent help achieve stronger fairness properties. Similarly, Sections 3 and 4 demonstrate that approximate fairness can still be achieved with mixed types of resources. However, simultaneously achieving approximate envy-freeness *and* PO is a challenging problem in both mixed fair division settings, in contrast to, e.g., the classic setting with indivisible goods.

[9] Transfer payments are better suited for studying welfare notions because they do not alter the social welfare of an allocation.

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