Decomposing Constraint Networks for Calculating c-Representations

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Abstract

It is well-known from probability theory that network-based methods like Bayesian networks constitute remarkable frameworks for efficient probabilistic reasoning. In this paper, we focus on qualitative default reasoning based on Spohn's ranking functions for which network-based methods have not yet been studied satisfactorily. With constraint networks, we develop a framework for iterative calculations of c-representations, a family of ranking models of conditional belief bases which show outstanding properties from a commonsense and formal point of view, that are characterized by assigning possible worlds a degree of implausibility via penalizing the falsification of conditionals. Constraint networks unveil the dependencies among these penalty points (and hence among the conditionals) and make it possible to compute the penalty points locally on so-called safe sub-bases. As an application of our framework, we show that skeptical c-inferences can be drawn locally from safe sub-bases without losing validity.

Introduction

Nonmonotonic reasoning is a subdiscipline of *knowledge representation and reasoning* in AI which introduces defeasibility into formal logics. *Conditionals* $(B|A)$ defined over a background theory $\mathcal{T}$ with $A, B \in \mathcal{T}$ constitute an expedient formalization of defeasible statements of the form "if A holds, then usually B follows." Thus, conditional belief bases serve as a valuable starting point for logic-based inductive reasoning under defeasibility. The formal semantics of conditionals is usually given by preference relations over *possible worlds* (Lewis 1986). In this paper, we express preferences by *ranking functions* (Spohn 2014) which assign to possible worlds and, based on that, to formulas a degree of implausibility. A ranking function κ *accepts* a conditional $(B|A)$ if the *verification* of the conditional is more plausible than its *falsification*, in symbols if $\kappa(A \wedge B) < \kappa(A \wedge \neg B)$. Reasoning in such semantic-based formalisms, however, is typically complex and computationally expensive such that a general objective in this research area is to reduce complexity by *modularizing* resp. *localizing* reasoning, and to focus on relevant information in this way.

A common strategy for localizing reasoning based on a conditional belief base Δ is to split Δ into pairwise disjoint sub-bases. Such a splitting can be seen as a success if the (global) ranking models of Δ can be assembled from (local) ranking models of the sub-bases. A prominent splitting technique is *syntax splitting* (Parikh 1999; Kern-Isberner, Beierle, and Brewka 2020), for instance, which splits Δ into sub-bases that are formulated over pairwise disjoint background languages. In many practical applications this requirement is too strong, though. Here, we consider the more general case where a splitting into independent sub-bases is not presumed and ask under which alternative circumstances the localization of reasoning is still successful.

In this paper, we investigate a specific family of ranking models, so-called *c-representations* κ_{η_Δ} (Kern-Isberner 2004), which are characterized by a mapping $\eta_\Delta \colon \Delta \rightarrow \mathbb{N}_0$ that assigns natural numbers to the conditionals in Δ. These numbers are used to penalize possible worlds for falsifying the conditionals and are determined by a constraint satisfaction problem $\mathsf{CSP}(\Delta)$. When Δ syntactically splits into sub-bases $\Delta_1, \ldots, \Delta_m$, penalty point mappings η_{Δ_i} for the sub-bases Δ_i can simply be expanded to a penalty point mapping for Δ (Kern-Isberner, Beierle, and Brewka 2020). In this case, localizing reasoning resp. the computation of a global c-representation of Δ is easy. This no longer holds when the sub-bases are not pairwise disjoint but only form a cover of Δ with overlapping sub-bases. Expanding the local penalty point mappings to a global one is possible only if $\eta_{\Delta_i}(\delta) = \eta_{\Delta_j}(\delta)$ for $\delta \in \Delta_i \cap \Delta_j$ is satisfied. This condition, which we call *c-consistency condition*, is necessary for such an expansion but in general not sufficient, though.

The main contribution of this paper is the conceptualization of so-called *safe covers* $\mathcal{O}(\Delta)$ which constitute a specific class of covers of belief bases Δ and based on which we develop *constraint networks* that hierarchically organize the constraints in $\mathsf{CSP}(\Delta)$. With constraint networks it is possible to solve the constraints referring to the sub-bases in $\mathcal{O}(\Delta)$ locally and propagate calculated penalty points through the network in order to comply with the c-consistency condition. Constraint networks generalize the so-called *constraint splittings* (Beierle, Haldimann, and Kern-Isberner 2021) and, therewith, subsume all common splitting techniques for conditional belief bases such as the already addressed syntax splitting. Besides the computation of c-representations, we show that drawing *skeptical c-inferences* can be localized based on *safe sub-bases*.

ω	$\mathsf{ver}_{\Delta_{\mathsf{ex}}}(\omega)$	$\mathsf{fal}_{\Delta_{\mathsf{ex}}}(\omega)$	$\kappa_{\vec{\eta}_{\Delta_{\mathsf{ex}}}}(\omega)$
abc	$\{\delta_2\}$	$\emptyset$	0
$ab\bar{c}$	$\{\delta_2, \delta_3\}$	$\emptyset$	0
$a\bar{b}c$	$\{\delta_3\}$	$\emptyset$	0
$a\bar{b}\bar{c}$	$\{\delta_3\}$	$\{\delta_4\}$	3
$\bar{a}bc$	$\emptyset$	$\{\delta_1, \delta_2\}$	2
$\bar{a}b\bar{c}$	$\emptyset$	$\{\delta_1, \delta_2, \delta_3\}$	3
$\bar{a}\bar{b}c$	$\{\delta_1, \delta_5\}$	$\{\delta_3\}$	1
$\bar{a}\bar{b}\bar{c}$	$\{\delta_1, \delta_4\}$	$\{\delta_3, \delta_5\}$	2

Table 1: Verified and falsified conditionals from Δ_{ex} and the c-representation $\kappa_{\vec{\eta}_{\Delta_{\mathsf{ex}}}}$ of Δ_{ex} wrt. $\vec{\eta}_{\Delta_{\mathsf{ex}}} = (1, 1, 1, 3, 1)$.

The paper is organized as follows. First, we briefly recall some basics on conditional reasoning in general and c-representations in particular. After discussing related work and the possibility to simplify the constraint satisfaction problem $\mathsf{CSP}(\Delta)$, we present our approach based on safe covers and constraint networks. We conclude by highlighting the main results of the paper and point out future work.

Preliminaries

We consider a *propositional language* $\mathcal{L}(\Sigma)$ over a finite *signature* Σ which is defined as usual by using the common connectives $\wedge$, $\vee$, and $\neg$. To shorten expressions, we write AB for $A \wedge B$, $\overline{A}$ for $\neg A$, and $\top$ for $A \vee \overline{A}$.

A *conditional* $(B|A)$ where $A, B \in \mathcal{L}(\Sigma)$ is a formal representation of the defeasible statement "if A holds, then usually B follows" and calls for a trivalent evaluation based on *possible worlds*. Here, a *possible world* ω is a propositional interpretation represented as a complete conjunction of literals, i.e., each atom from Σ occurs in ω once, either positive or negated. Then, a conditional $(B|A)$ is *verified* in ω if $\omega \models AB$, it is *falsified* if $\omega \models A\overline{B}$, and it is *not applicable* if $\omega \models \overline{A}$. The set of all possible worlds over Σ is denoted with $\Omega(\Sigma)$. We call finite sets of conditionals *belief base* and subsets of belief bases *sub-base*. Further, we denote with

$$\mathsf{ver}_\Delta(\omega) = \{(B|A) \in \Delta \mid \omega \models AB\},$$
$$\mathsf{fal}_\Delta(\omega) = \{(B|A) \in \Delta \mid \omega \models A\overline{B}\},$$

the sets of the conditionals from a belief base Δ which are verified resp. falsified in the possible world ω. Analogously, the sets of possible worlds which verify resp. falsify the conditional $\delta = (B|A)$ are denoted with

$$\mathsf{ver}(\delta) = \{\omega \in \Omega(\Sigma) \mid \omega \models AB\},$$
$$\mathsf{fal}(\delta) = \{\omega \in \Omega(\Sigma) \mid \omega \models A\overline{B}\}.$$

In examples, we enumerate the conditionals in belief bases and write $\Delta = \{\delta_1, \dots, \delta_n\}$ with $\delta_i = (A_i|B_i)$ for $i \in \{1, \dots, n\}$ such that we can refer to the conditionals by their indices. We use the abbreviation $[n]$ for $\{1, \dots, n\}$.

Example 1. *As a running example in this paper, we consider the belief base* $\Delta_{\mathsf{ex}} = \{\delta_i \mid i \in [5]\}$ *with*

$$\delta_1 = (\bar{b}|\bar{a}), \qquad \delta_2 = (a|b), \qquad \delta_3 = (a|\bar{b} \vee \bar{c}),$$
$$\delta_4 = (\bar{a}|\bar{b}\bar{c}), \qquad \delta_5 = (c|\bar{a}\bar{b}),$$

over the signature $\Sigma_{\mathsf{ex}} = \{a, b, c\}$. *For the possible worlds* $\omega \in \Omega(\Sigma_{\mathsf{ex}})$, $\mathsf{ver}_{\Delta_{\mathsf{ex}}}(\omega)$ *and* $\mathsf{fal}_{\Delta_{\mathsf{ex}}}(\omega)$ *are shown in Table 1.*

The semantics of conditionals is based on *ranking functions* $\kappa\colon \Omega(\Sigma) \to \mathbb{N}_0^\infty$ here, which map possible worlds to a degree of implausibility while satisfying the normalization condition $\kappa^{-1}(0) \neq \emptyset$ (Spohn 2014). Ranking functions are extended to formulas by $\kappa(A) = \min_{\omega \models A} \kappa(\omega)$.[1] A ranking function κ *accepts* a conditional $(B|A)$ if $\kappa(AB) < \kappa(A\overline{B})$, written $\kappa \models (B|A)$. If κ accepts every conditional in a belief base Δ, then κ is a *ranking model* of Δ. A belief base is *consistent* if it has a ranking model. It is a straightforward result that sub-bases of consistent belief bases are consistent, too.

c-Representations (Kern-Isberner 2004) constitute a special family of ranking models which can be written as

$$\kappa_{\eta_\Delta}(\omega) = \sum\nolimits_{\delta \in \mathsf{fal}_\Delta(\omega)} \eta_\Delta(\delta), \tag{1}$$

where $\eta_\Delta\colon \Delta \to \mathbb{N}_0$ maps the conditionals from Δ to a natural number such that the constraint satisfaction problem $\mathsf{CSP}(\Delta) = \{C_\Delta(\delta) \mid \delta \in \Delta\}$ with $C_\Delta(\delta)$ defined by

$$\eta_\Delta(\delta) > \min_{\omega \in \mathsf{ver}(\delta)} \{\sum\nolimits_{\delta' \in \Delta \setminus \{\delta\}\colon \omega \in \mathsf{fal}(\delta')} \eta_\Delta(\delta')\} - \min_{\omega \in \mathsf{fal}(\delta)} \{\sum\nolimits_{\delta' \in \Delta \setminus \{\delta\}\colon \omega \in \mathsf{fal}(\delta')} \eta_\Delta(\delta')\} \tag{2}$$

is satisfied. The *constraint* $C_\Delta(\delta)$ is nothing else than the claim that κ_{η_Δ} accepts δ, and $\eta_\Delta(\delta)$ can be understood as a *penalty point* for falsifying δ. Every solution of $\mathsf{CSP}(\Delta)$ leads to a c-representation of the form (1), and every consistent belief base has such a c-representation. If the conditionals in Δ are enumerated, we also use a vector notation for the penalty points and write $\vec{\eta}_\Delta = (\eta_1, \dots, \eta_n)$ where $n = |\Delta|$ and $\eta_i = \eta_\Delta(\delta_i)$ for $i \in [n]$.

Example 2. *As in the following examples, we continue Example 1.* $\mathsf{CSP}(\Delta_{\mathsf{ex}})$ *consists of*

$$C_{\Delta_{\mathsf{ex}}}(\delta_1)\colon \eta_1 > \min\{\eta_3, \eta_3 + \eta_5\} - \min\{\eta_2, \eta_2 + \eta_3\},$$
$$C_{\Delta_{\mathsf{ex}}}(\delta_2)\colon \eta_2 > \min\{0\} - \min\{\eta_1, \eta_1 + \eta_3\},$$
$$C_{\Delta_{\mathsf{ex}}}(\delta_3)\colon \eta_3 > \min\{0, \eta_4\} - \min\{0, \eta_1 + \eta_2, \eta_5\},$$
$$C_{\Delta_{\mathsf{ex}}}(\delta_4)\colon \eta_4 > \min\{\eta_3 + \eta_5\} - \min\{0\},$$
$$C_{\Delta_{\mathsf{ex}}}(\delta_5)\colon \eta_5 > \min\{\eta_3\} - \min\{\eta_3\}.$$

A solution of $\mathsf{CSP}(\Delta_{\mathsf{ex}})$ *is* $\vec{\eta}_{\Delta_{\mathsf{ex}}} = (1, 1, 1, 3, 1)$. *The corresponding c-representation of* Δ_{ex} *is shown in Table 1. In particular, the existence of this c-representation proves that* Δ_{ex} *is consistent. The penalty point vector* $(1, 1, 1, 3, 1)$ *is pareto-minimal, i.e., there is no other solution of* $\mathsf{CSP}(\Delta_{\mathsf{ex}})$ *all of its entries are componentwise equal or less than the entries of* $(1, 1, 1, 3, 1)$. $\mathsf{CSP}(\Delta_{\mathsf{ex}})$ *has two further pareto-minimal solutions, namely* $(2, 0, 1, 3, 1)$ *and* $(0, 2, 1, 3, 1)$.

c-Representations yield a nonmonotonic inference relation with particularly good inference behavior called *skeptical c-inference* (Beierle et al. 2021) which is given by

$$\Delta \mathrel{|\!\sim}_c (B|A) \text{ iff } \kappa_{\eta_\Delta} \models (B|A) \text{ or } \kappa_{\eta_\Delta}(A) = \infty \text{ for all c-representations } \kappa_{\eta_\Delta} \text{ of } \Delta. \tag{3}$$

[1] If A has no model, then $\kappa(A) = \infty$ applies.

We call c-representations of consistent belief bases Δ *global* and c-representations of sub-bases of Δ *local*. The main goal of this paper is to devise a network-based method which makes it possible to derive a global c-representation of Δ from a set of local ones and, therewith, *localize* reasoning, i.e., for instance, answering skeptical c-inference queries locally. Intuitively, it is crucial to find a proper decomposition of Δ into sub-bases for that purpose.

Related Work

Basically, there are only two approaches that are closely related to this paper. The first one is (Benferhat and Tabia 2010) which transfers the concept of *Bayesian networks* to ranking functions with conditional independence assumptions and local conditional ranking tables; here, the conditionals in the belief base have to follow an acyclic structure, i.e., no cycles like $(a|b), (b|a)$ are allowed. Moreover, (Kern-Isberner and Eichhorn 2015) revealed semantic problems with this approach. These two flaws can be overcome by the second approach (Eichhorn and Kern-Isberner 2014) that makes use of a hypergraph structure. That approach applies a specific, syntax-based covering hypertree-generation process to a consistent belief base Δ, decomposing it into subbases $\Delta_1, \dots, \Delta_m$. If local ranking models κ_i of Δ_i on the subsignatures Σ_i are provided, a global ranking model of Δ can be computed whose marginalizations coincide with the local κ_i's. The system $\langle \Sigma_i, \kappa_i \rangle_{i \in [m]}$ is called an *OCF-LEG-network* where for each $i \in [m]$, $\langle \Sigma_i, \kappa_i \rangle$ forms a *local event group (LEG)* as an adaptation from Lemmer's LEGs (Lemmer 1982, 1983) in probabilistics. There are two crucial preconditions that make this OCF-LEG-approach work: First, the *consistency condition*

$$\forall i, j \in [m] \colon \omega \in \Omega(\Sigma_i \cap \Sigma_j) \Rightarrow \kappa_i(\omega) = \kappa_j(\omega), \qquad (4)$$

and second, the *running intersection property (RIP)* (Pearl 1988; Kern-Isberner and Eichhorn 2015), claiming that the intersection of each Σ_i with all its predecessors (according to the hypertree) is included in one of the predecessors. The LEG-structure is not strong enough for c-representations, though. Even if the global ranking model is a c-representation, the local ones may be not, and the other way round.

In this paper, we develop a network-based approach that is especially tailored towards c-representations and makes use of a stronger consistency condition, called *c-consistency condition*, which can be ensured by iterative calculations of the local c-representations. This is possible by exploiting the characteristic structure of c-representations given by semantic-based constraint systems and decomposing belief bases into so-called *safe covers*. We also show that canonical safe covers satisfy a strong version of the RIP. A crucial difference to OCF-LEG-networks is that the decomposition of belief bases in OCF-LEG-networks is syntax-based while our approach is constraint driven and, thus, able to catch the complex interdependencies among conditionals better.

Constraint Reductions

Example 2 suggests to simplify the constraints in $\mathsf{CSP}(\Delta_{\mathsf{ex}})$ before solving it. For instance, $C_{\Delta_{\mathsf{ex}}}(\delta_3)$ reduces to $\eta_3 > 0$.

δ	$V_{\Delta_{\mathsf{ex}}}(\delta)$	$F_{\Delta_{\mathsf{ex}}}(\delta)$
$\delta_1 = (\bar{b}|\bar{a})$	$\{\{\delta_3\}, \{\delta_3, \delta_5\}\}$	$\{\{\delta_2\}, \{\delta_2, \delta_3\}\}$
$\delta_2 = (a|b)$	$\{\emptyset\}$	$\{\{\delta_1\}, \{\delta_1, \delta_3\}\}$
$\delta_3 = (a|\bar{b} \vee \bar{c})$	$\{\emptyset, \{\delta_4\}\}$	$\{\emptyset, \{\delta_1, \delta_2\}, \{\delta_5\}\}$
$\delta_4 = (\bar{a}|\bar{b}\bar{c})$	$\{\{\delta_3, \delta_5\}\}$	$\{\emptyset\}$
$\delta_5 = (c|\bar{a}\bar{b})$	$\{\{\delta_3\}\}$	$\{\{\delta_3\}\}$

Table 2: Constraint sets of the conditionals in Δ_{ex}.

We address this issue by following ideas from (Beierle, Haldimann, and Kern-Isberner 2021; Wilhelm et al. 2023) before we develop our framework on localizing the calculation of c-representations in the next sections. *Constraint reductions* are not mandatory for our further investigations but usually lead to much more fine-grained localizations.

In order to be able to define reduction rules for the constraints in $\mathsf{CSP}(\Delta)$, we reformulate $\mathsf{CSP}(\Delta)$. With

$$V_\Delta(\delta) = \{\{\delta' \in \Delta \setminus \{\delta\} \mid \omega \in \mathsf{fal}(\delta')\} \mid \omega \in \mathsf{ver}(\delta)\},$$
$$F_\Delta(\delta) = \{\{\delta' \in \Delta \setminus \{\delta\} \mid \omega \in \mathsf{fal}(\delta')\} \mid \omega \in \mathsf{fal}(\delta)\},$$

for conditionals δ, (2) can be equivalently rewritten to

$$\eta_\Delta(\delta) > \min_{S \in V_\Delta(\delta)} \{\sum_{\delta' \in S} \eta_\Delta(\delta')\} - \min_{S \in F_\Delta(\delta)} \{\sum_{\delta' \in S} \eta_\Delta(\delta')\}. \quad (5)$$

Example 3. *Table 2 shows the* constraint sets $V_{\Delta_{\mathsf{ex}}}(\delta)$ *and* $F_{\Delta_{\mathsf{ex}}}(\delta)$ *of the conditionals* $\delta \in \Delta_{\mathsf{ex}}$.

The reformulation of the constraints $C_\Delta(\delta)$ from (2) to the set-based notation (5) has the advantage that simplifications of the constraints can be performed by set manipulation. The goal is to find a mapping $\phi \colon \delta \mapsto (V^\phi_\Delta(\delta), F^\phi_\Delta(\delta))$ which reduces the constraint sets $V_\Delta(\delta)$ and $F_\Delta(\delta)$ to sets $V^\phi_\Delta(\delta)$ and $F^\phi_\Delta(\delta)$ such that the constraint satisfaction problem $\mathsf{CSP}^\phi(\Delta) = \{C^\phi_\Delta(\delta) \mid \delta \in \Delta\}$ with $C^\phi_\Delta(\delta)$ defined by

$$\eta_\Delta(\delta) > \min_{S \in V^\phi_\Delta(\delta)} \{\sum_{\delta' \in S} \eta_\Delta(\delta')\} - \min_{S \in F^\phi_\Delta(\delta)} \{\sum_{\delta' \in S} \eta_\Delta(\delta')\} \quad (6)$$

has the same solutions as $\mathsf{CSP}(\Delta)$. If this holds for all consistent belief bases Δ, then we call the mapping ϕ a *solution preserving constraint reduction*, or spcr for short, and $V^\phi_\Delta(\delta)$ and $F^\phi_\Delta(\delta)$ the *reduced constraint sets* of δ wrt. Δ and ϕ.

In (Beierle, Haldimann, and Kern-Isberner 2021; Wilhelm et al. 2023) the authors suggest rewriting rules for constraint sets which satisfy the requirements of spcrs. The following rules R1-R4 yield natural reductions of $\mathsf{CSP}(\Delta)$:

R1 If for $\delta \in \Delta$ there are $S, S' \in V_\Delta(\delta)$ with $S \subsetneq S'$, then

$$V_\Delta(\delta) \leftarrow V_\Delta(\delta) \setminus \{S'\}.$$

R2 If for $\delta \in \Delta$ there are $S, S' \in F_\Delta(\delta)$ with $S \subsetneq S'$, then

$$F_\Delta(\delta) \leftarrow F_\Delta(\delta) \setminus \{S'\}.$$

R3 If for $\delta \in \Delta$ one has $V_\Delta(\delta) \neq \emptyset \neq F_\Delta(\delta)$ and there is $\delta' \in \Delta$ with $\delta' \in S$ for all $S \in V_\Delta(\delta) \cup F_\Delta(\delta)$, then

$$V_\Delta(\delta) \leftarrow \{S \setminus \{\delta'\} \mid S \in V_\Delta(\delta)\},$$
$$F_\Delta(\delta) \leftarrow \{S \setminus \{\delta'\} \mid S \in F_\Delta(\delta)\}.$$

δ	$V^{\psi}_{\Delta_{\mathsf{ex}}}(\delta)$	$F^{\psi}_{\Delta_{\mathsf{ex}}}(\delta)$	$\Delta^{\psi}_{\Delta_{\mathsf{ex}}}(\delta)$
$\delta_1 = (\bar{b}\|\bar{a})$	$\{\{\delta_3\}\}$	$\{\{\delta_2\}\}$	$\{\delta_1, \delta_2, \delta_3\}$
$\delta_2 = (a\|b)$	$\{\emptyset\}$	$\{\{\delta_1\}\}$	$\{\delta_1, \delta_2\}$
$\delta_3 = (a\|\bar{b} \vee \bar{c})$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\delta_3\}$
$\delta_4 = (\bar{a}\|\bar{b}\bar{c})$	$\{\{\delta_3, \delta_5\}\}$	$\{\emptyset\}$	$\{\delta_3, \delta_4, \delta_5\}$
$\delta_5 = (c\|\bar{a}\bar{b})$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\delta_5\}$

Table 3: Reduced constraint sets and adjacent conditionals of the conditionals in Δ_{ex} wrt. ψ.

R4 If for $\delta \in \Delta$ one has $V_{\Delta}(\delta) = F_{\Delta}(\delta)$, then

$$V_{\Delta}(\delta) \leftarrow \emptyset \quad \text{and} \quad F_{\Delta}(\delta) \leftarrow \emptyset.$$

We call the spcr which is specified by exhaustively applying R1-R4 *natural* spcr and denote this specific spcr with ψ. The order in which the rewriting rules are applied is irrelevant.

Example 4. *For $\delta \in \Delta_{\mathsf{ex}}$, the reduced constraint sets $V^{\psi}_{\Delta_{\mathsf{ex}}}(\delta)$ and $F^{\psi}_{\Delta_{\mathsf{ex}}}(\delta)$ are shown in Table 3. The resulting constraint satisfaction problem $\mathsf{CSP}^{\psi}(\Delta_{\mathsf{ex}})$ consists of*

$$C^{\psi}_{\Delta_{\mathsf{ex}}}(\delta_1)\colon \eta_1 > \min\{\eta_3\} - \min\{\eta_2\} \qquad = \eta_3 - \eta_2,$$
$$C^{\psi}_{\Delta_{\mathsf{ex}}}(\delta_2)\colon \eta_2 > \min\{0\} - \min\{\eta_1\} \qquad = -\eta_1,$$
$$C^{\psi}_{\Delta_{\mathsf{ex}}}(\delta_3)\colon \eta_3 > \min\{0\} - \min\{0\} \qquad = 0,$$
$$C^{\psi}_{\Delta_{\mathsf{ex}}}(\delta_4)\colon \eta_4 > \min\{\eta_3 + \eta_5\} - \min\{0\} \qquad = \eta_3 + \eta_5,$$
$$C^{\psi}_{\Delta_{\mathsf{ex}}}(\delta_5)\colon \eta_5 > \min\{0\} - \min\{0\} \qquad = 0.$$

For instance, $C^{\psi}_{\Delta_{\mathsf{ex}}}(\delta_1)$ is obtained by applying the rewriting rule R1 to $V_{\Delta_{\mathsf{ex}}}(\delta_1)$ and rule R2 to $F_{\Delta_{\mathsf{ex}}}(\delta_1)$. The constraint $C^{\psi}_{\Delta_{\mathsf{ex}}}(\delta_5)$ can be obtained by applying either rule R3 or R4.

In (Beierle, Haldimann, and Kern-Isberner 2021) spcrs where studied in the context of so-called *constraint splittings* where $\mathsf{CSP}^{\phi}(\Delta)$ decomposes into a set of smaller independent constraint satisfaction problems. Constraint splittings subsume many splitting techniques such as *syntax splitting* (Parikh 1999; Kern-Isberner and Brewka 2017; Kern-Isberner, Beierle, and Brewka 2020; Beierle and Kern-Isberner 2021), *safe conditional syntax splitting* (Heyninck, Kern-Isberner, and Meyer 2022), and *case splitting* (Sezgin, Kern-Isberner, and Beierle 2021). Here, we generalize this line of research and consider *constraint networks* based on *safe covers* of Δ where $\mathsf{CSP}^{\phi}(\Delta)$ and Δ do not need to split into independent subsets but the subsets may overlap. Instead, we hierarchically organize the constraints in $\mathsf{CSP}^{\phi}(\Delta)$ and solve them iteratively.

Because the *trivial constraint reduction* ϕ_0 which is given by $V^{\phi_0}_{\Delta}(\delta) = V_{\Delta}(\delta)$ and $F^{\phi_0}_{\Delta}(\delta) = F_{\Delta}(\delta)$ is obviously solution preserving, the following results also hold when no spcr is applied.

Safe Covers

Safe covers are specific decompositions of belief bases for which the constraint satisfaction problems of the sub-bases are self-contained. For a formal definition of safe covers, we introduce a notion of *adjacency* between conditionals first.

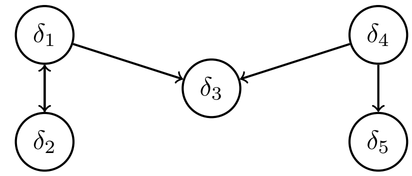

Figure 1: Adjacency graph of Δ_{ex} wrt. ψ.

Definition 1 (Adjacent Conditionals)**.** *Let Δ be a consistent belief base, $\delta \in \Delta$, and ϕ an* spcr*. Then, we call*

$$\mathcal{D}^{\phi}_{\Delta}(\delta) = \{\delta\} \cup \bigcup_{S \in V^{\phi}_{\Delta}(\delta) \cup F^{\phi}_{\Delta}(\delta)} S$$

the adjacent conditionals *of δ wrt. Δ and ϕ. We generalize this notion of adjacency to sub-bases $\Delta' \subseteq \Delta$ by*

$$\mathcal{D}^{\phi}_{\Delta}(\Delta') = \bigcup_{\delta \in \Delta'} \mathcal{D}^{\phi}_{\Delta}(\delta).$$

$\mathcal{D}^{\phi}_{\Delta}(\delta)$ is the set of conditionals the penalty points of which occur in the constraint $C^{\phi}_{\Delta}(\delta)$ within the constraint satisfaction problem $\mathsf{CSP}^{\phi}(\Delta)$. In particular, $\delta \in \mathcal{D}^{\phi}_{\Delta}(\delta)$. In this sense, $\mathcal{D}^{\phi}_{\Delta}(\delta)$ is the set of conditionals that (maybe) have a *direct influence* on the penalty point $\eta_{\Delta}(\delta)$ which justifies the term *adjacent*.

Example 5. *The adjacent conditionals of the conditionals in Δ_{ex} wrt. the natural* spcr *ψ are shown in Table 3.*

The adjacency between conditionals can be represented as a directed graph.

Definition 2 (Adjacency Graph)**.** *Let Δ be a consistent belief base and ϕ an* spcr*. Further, let $\mathcal{V} = \Delta$ and*

$$\mathcal{E} = \{(\delta, \delta') \mid \delta \in \Delta, \delta' \in \mathcal{D}^{\phi}_{\Delta}(\delta) \setminus \{\delta\}\}.$$

Then, we call the directed graph $\mathcal{G}^{\phi}(\Delta) = \langle \mathcal{V}, \mathcal{E} \rangle$ adjacency graph *of Δ wrt. ϕ.*

We exclude tuples of the form (δ, δ), $\delta \in \Delta$, from $\mathcal{E}$ and, therewith, omit loops in adjacency graphs for convenience. Otherwise, every vertex in $\mathcal{G}^{\phi}(\Delta)$ would have a loop.

Example 6. *The adjacency graph of Δ_{ex} wrt. the natural* spcr *ψ is shown in Figure 1.*

Because $\Delta' \subseteq \mathcal{D}^{\phi}_{\Delta}(\Delta') \subseteq \Delta$ and Δ is finite, the repeated application of $\mathcal{D}^{\phi}_{\Delta}$ on any sub-base $\Delta' \subseteq \Delta$ becomes stationary after finitely many iteration steps and we can establish the following notions of milieus and safe sub-bases.

Definition 3 (Milieu, Safe Sub-base)**.** *Let Δ be a consistent belief base, $\Delta' \subseteq \Delta$, and ϕ an* spcr*. Further, let $l \in \mathbb{N}_0$ be such that $(\mathcal{D}^{\phi}_{\Delta})^k(\Delta') = (\mathcal{D}^{\phi}_{\Delta})^l(\Delta')$ for all $k \geq l$. Then,*

$$\mathcal{M}^{\phi}_{\Delta}(\Delta') = (\mathcal{D}^{\phi}_{\Delta})^l(\Delta')$$

is called the milieu *of Δ' wrt. Δ and ϕ. If $\mathcal{M}^{\phi}_{\Delta}(\Delta') = \Delta'$, then Δ' is called a* safe sub-base *of Δ wrt. ϕ.*

The milieu of a sub-base $\Delta' \subseteq \Delta$ wrt. ϕ coincides with the set of conditionals that are reachable from Δ' in $\mathcal{G}^\phi(\Delta)$. Hereby, a conditional $\delta \in \Delta$ is *reachable* from Δ' if there is a directed path from any $\delta' \in \Delta'$ to δ in $\mathcal{G}^\phi(\Delta)$. The constraints wrt. the conditionals in Δ' mention exactly the penalty points $\eta_\Delta(\delta')$ for $\delta' \in \Delta'$. Thus, $C^\phi_{\Delta'}(\delta) = C^\phi_\Delta(\delta)$ for $\delta \in \Delta'$ holds, and, we have

$$\mathsf{CSP}^\phi(\Delta') = \{C^\phi_\Delta(\delta) \mid \delta \in \Delta'\} \subseteq \mathsf{CSP}^\phi(\Delta). \qquad (7)$$

Note that this does not imply that $\mathsf{CSP}^\phi(\Delta)$ automatically splits into two independent sub-problems $\mathsf{CSP}^\phi(\Delta')$ and $\mathsf{CSP}^\phi(\Delta \setminus \Delta')$ unless $\mathsf{CSP}^\phi(\Delta \setminus \Delta')$ is safe, too, because, in general, the constraints $C^\phi_\Delta(\delta)$ with $\delta \in \Delta \setminus \Delta'$ may still mention penalty points $\eta_\Delta(\delta')$ wrt. $\delta' \in \Delta'$.

Example 7. *It is $\mathcal{D}^\psi_{\Delta_{\mathsf{ex}}}(\delta_2) = \{\delta_1, \delta_2\}$, $(\mathcal{D}^\psi_{\Delta_{\mathsf{ex}}})^2(\delta_2) = \Delta'$ with $\Delta' = \{\delta_1, \delta_2, \delta_3\}$, and $(\mathcal{D}^\psi_{\Delta_{\mathsf{ex}}})^k(\delta_2) = (\mathcal{D}^\psi_{\Delta_{\mathsf{ex}}})^2(\delta_2)$ for $k \geq 3$ so that $\mathcal{M}^\psi_{\Delta_{\mathsf{ex}}}(\{\delta_2\}) = \Delta' = \mathcal{M}^\psi_{\Delta_{\mathsf{ex}}}(\Delta')$ holds (cf. also Figure 1). Consequently, Δ' is a safe sub-base of Δ_{ex} wrt. ψ while $\{\delta_2\}$ and, likewise, $\{\delta_1, \delta_2\}$ are not. Note that $C^\psi_{\Delta_{\mathsf{ex}}}(\delta_4)$ mentions η_3 nevertheless.*

We put some basic properties of milieus on record.

Proposition 1. *Let Δ be a consistent belief base, $\delta \in \Delta$, $\Delta', \Delta'' \subseteq \Delta$, and ϕ an* spcr*. Then,*

1. *$\Delta' \subseteq \mathcal{M}^\phi_\Delta(\Delta') \subseteq \Delta$ and, thus, $\mathcal{M}^\phi_\Delta(\Delta) = \Delta$,*
2. *$\mathcal{M}^\phi_\Delta(\mathcal{M}^\phi_\Delta(\Delta')) = \mathcal{M}^\phi_\Delta(\Delta')$,*
3. *$\mathcal{M}^\phi_\Delta(\Delta') = \bigcup_{\delta' \in \Delta'} \mathcal{M}^\phi_\Delta(\{\delta'\})$ and, thus, $\mathcal{M}^\phi_\Delta(\Delta' \cup \Delta'') = \mathcal{M}^\phi_\Delta(\Delta') \cup \mathcal{M}^\phi_\Delta(\Delta'')$,*
4. *$\mathcal{M}^\phi_\Delta(\delta)$ is minimal in the sense that there is no safe sub-base Δ'' of Δ wrt. ϕ with $\delta \in \Delta''$ and $\Delta'' \subsetneq \mathcal{M}^\phi_\Delta(\{\delta\})$,*
5. *$\mathcal{M}^\phi_\Delta$ is monotone: $\Delta' \subseteq \Delta'' \Rightarrow \mathcal{M}^\phi_\Delta(\Delta') \subseteq \mathcal{M}^\phi_\Delta(\Delta'')$.*

According to Proposition 1.2, milieus are always safe sub-bases and vice versa. The next proposition shows that intersections of safe sub-bases are safe sub-bases, too.

Proposition 2. *Let Δ be a consistent belief base, ϕ an* spcr*, and, for $i \in [m]$, let Δ_i be a safe sub-base of Δ wrt. ϕ. Then, for $I \subseteq [m]$, $I \neq \emptyset$, $\bigcap_{i \in I} \Delta_i$ is a safe sub-base of Δ wrt. ϕ.*

Now we combine safe sub-bases to *safe covers* of Δ.

Definition 4 (Safe Cover)**.** *Let Δ be a consistent belief base. A finite set of sub-bases of Δ, $\mathcal{O} = \{\Delta_i \mid i \in [m]\}$, is a* cover *of Δ if $\bigcup_{i \in [m]} \Delta_i = \Delta$. We call such a cover $\mathcal{O}$* safe *if there is an* spcr *ϕ such that, for $i \in [m]$, Δ_i is safe wrt. ϕ.*

Safe covers $\mathcal{O}$ of Δ can be represented as the hypergraph $\mathcal{H}_\Delta(\mathcal{O}) = \langle \Delta, \mathcal{O} \rangle$.

Example 8. *For the cover $\mathcal{O}_{\mathsf{ex}} = \{\Delta_1, \Delta_2\}$ of Δ_{ex} with $\Delta_1 = \{\delta_1, \delta_2, \delta_3\}$ and $\Delta_2 = \{\delta_3, \delta_4, \delta_5\}$ we already showed that $\mathcal{M}^\psi_{\Delta_{\mathsf{ex}}}(\Delta_1) = \Delta_1$ holds (cf. Example 7). It is easy to see that $\mathcal{M}^\psi_{\Delta_{\mathsf{ex}}}(\Delta_2) = \Delta_2$ holds as well so that $\mathcal{O}_{\mathsf{ex}}$ is safe. As $\mathcal{M}^\psi_{\Delta_{\mathsf{ex}}}(\{\delta_3\}) = \{\delta_3\}$ and $\mathcal{M}^\psi_{\Delta_{\mathsf{ex}}}(\{\delta_5\}) = \{\delta_5\}$ hold, too, $\mathcal{O}'_{\mathsf{ex}} = \mathcal{O}_{\mathsf{ex}} \cup \{\{\delta_3\}, \{\delta_5\}\}$ is also a safe cover of Δ_{ex}. The hypergraph $\mathcal{H}_{\Delta_{\mathsf{ex}}}(\mathcal{O}'_{\mathsf{ex}})$ is shown in Figure 2.*

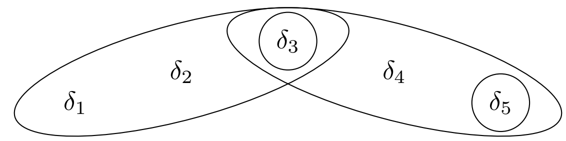

Figure 2: Hypergraph $\mathcal{H}_{\Delta_{\mathsf{ex}}}(\mathcal{O}'_{\mathsf{ex}})$ (cf. Example 8).

Note that following the OCF-LEG-network approach (Eichhorn and Kern-Isberner 2014), the belief base Δ_{ex} can not be decomposed at all because of the strong syntactical linkage between the conditionals in Δ_{ex}.

Constraint splittings (Beierle, Haldimann, and Kern-Isberner 2021) constitute a specific sub-class of safe covers for which the sub-bases and, thus, the edges of the corresponding hypergraph are pairwise disjoint.

Definition 5 (Constraint Splitting (adapted from (Beierle, Haldimann, and Kern-Isberner 2021)))**.** *Let Δ be a consistent belief base. If $\mathcal{O} = \{\Delta_i \mid i \in [m]\}$ is a cover of Δ such that, for some* spcr *ϕ, $\mathcal{M}^\phi_\Delta(\Delta_i) \cap \mathcal{M}^\phi_\Delta(\Delta_j) = \emptyset$ for $i, j \in [m]$, $i \neq j$, then $\mathcal{O}$ is called a* constraint splitting *of Δ.*

Proposition 3. *Let Δ be a consistent belief base and $\mathcal{O}$ a constraint splitting of Δ. Then, $\mathcal{O}$ is a partition of Δ and also a safe cover of Δ.*

Example 8 shows that belief bases can have several safe covers. However, there is a canonical safe cover that we want to focus on in the following.

Definition 6 (Canonical Safe Cover)**.** *Let Δ be a consistent belief base and ϕ an* spcr*. Then, we call*

$$\mathcal{O}^\phi(\Delta) = \{\mathcal{M}^\phi_\Delta(\{\delta\}) \mid \delta \in \Delta\}$$

the canonical safe cover *of Δ wrt. ϕ.*

Canonical safe covers are the finest-grained safe covers.

Proposition 4. *Let Δ be a consistent belief base, ϕ an* spcr*, $\mathcal{O}^\phi(\Delta)$ the canonical safe cover of Δ wrt. ϕ, and $\mathcal{O}(\Delta)$ any further safe cover of Δ wrt. ϕ. Then, for every $\Delta' \in \mathcal{O}(\Delta)$, there is some $\Delta'' \in \mathcal{O}^\phi(\Delta)$ with $\Delta'' \subseteq \Delta'$.*

As a consequence, canonical safe covers are closed under subset formation and, therewith, satisfy a strong version of the *running intersection property (RIP)* (Pearl 1988).

Corollary 1. *Let Δ be a consistent belief base, ϕ an* spcr*, and $\mathcal{O}^\phi(\Delta)$ the canonical safe cover of Δ wrt. ϕ. For all sets $\mathcal{O}' \subseteq \mathcal{O}^\phi(\Delta)$ of safe sub-bases of Δ from $\mathcal{O}^\phi(\Delta)$, if $\Delta'' = \bigcap_{\Delta' \in \mathcal{O}'} \Delta'$ with $\Delta'' \neq \emptyset$, then $\Delta'' \in \mathcal{O}^\phi(\Delta)$.*

The next theorem shows that safe covers allow for local computations of c-representations provided that a so-called *c-consistency condition* is satisfied.

Theorem 1. *Let $\mathcal{O} = \{\Delta_i \mid i \in [m]\}$ be a safe cover of a consistent belief base Δ.*

1. *If there are c-representations $\kappa_{\eta_{\Delta_i}}$ of Δ_i, $i \in [m]$, which satisfy the* c-consistency condition
$$\forall i, j \in [m]\colon \delta \in \Delta_i \cap \Delta_j \Rightarrow \eta_{\Delta_i}(\delta) = \eta_{\Delta_j}(\delta), \qquad (8)$$
then they satisfy the consistency condition (4), and κ_{η_Δ} defined by $\eta_\Delta(\delta) = \eta_{\Delta_i}(\delta)$ for any $i \in [m]$ with $\delta \in \Delta_i$ is a c-representation of Δ.

2. *For every c-representation κ_{η_Δ} of Δ there are c-representations $\kappa_{\eta_{\Delta_i}}$ of Δ_i, $i \in [m]$, such that the c-consistency condition (8) is satisfied.*

Essential for the proof of Theorem 1 is the fact that, because of (7), $\bigcup_{i\in[m]} \mathsf{CSP}^\phi(\Delta_i) = \mathsf{CSP}^\phi(\Delta)$ holds. The need of the c-consistency condition (8) causes unwanted dependencies among the local c-representations, though, which we will address in the next section by introducing *constraint networks*. Before that, we prove that based on safe covers it is possible to answer skeptical c-inference queries locally. For the proof, we need some preparatory work.

Lemma 1. *Let Δ be a consistent belief base and ϕ an* spcr. *If κ_{η_Δ} is a c-representation of Δ, so is $\kappa_{\eta'_\Delta}$ which is defined by $\eta'_\Delta(\delta) = s \cdot \eta_\Delta(\delta)$ for $\delta \in \Delta$ and an arbitrary $s \in \mathbb{N}_{>0}$. Moreover, η'_Δ satisfies*

$$\eta'_\Delta(\delta) > \min_{S\in V^\phi_\Delta(\delta)}\{\textstyle\sum_{\delta'\in S}\eta'_\Delta(\delta')\} \\ - \min_{S\in F^\phi_\Delta(\delta)}\{\textstyle\sum_{\delta'\in S}\eta'_\Delta(\delta')\} + (s-1)$$

for $\delta \in \Delta$. That is, $\kappa_{\eta'_\Delta}$ accepts the conditionals in Δ with firmness $s-1$ *(cf. (Spohn 2014; Kern-Isberner 2001)).*

Based on Lemma 1, one can show that every c-representation of a safe sub-base of a consistent belief base Δ can be extended to a c-representation of Δ.

Proposition 5. *Let Δ be a consistent belief base, ϕ an* spcr, *Δ' a safe sub-base of Δ wrt. ϕ, and $\kappa_{\eta_{\Delta'}}$ a c-representation of Δ'. Then, there is a c-representation κ_{η_Δ} of Δ with $\eta_\Delta(\delta) = \eta_{\Delta'}(\delta)$ for $\delta \in \Delta'$.*

Eventually, we come up with our result on local skeptical c-inference.

Theorem 2. *Let Δ be a consistent belief base, q a conditional, ϕ an* spcr, *and $\Delta^\phi_q = V^\phi_\Delta(q) \cup F^\phi_\Delta(q) \subseteq \Delta$. Then,*

$$\Delta \mathrel{|\!\!\sim}_c q \quad \textit{iff} \quad \mathcal{M}^\phi_\Delta(\Delta^\phi_q) \mathrel{|\!\!\sim}_c q.$$

Theorem 2 is especially meaningful from the viewpoint that $\mathrel{|\!\!\sim}_c$ is nonmonotonic in general. In particular, it reduces the number of possible worlds that has to be considered for drawing the inference from $2^{|\Sigma|}$ to $2^{|\Sigma(\Delta^\phi_q)|}$ where $\Sigma(\Delta^\phi_q)$ is the set of atoms that occur in Δ^ϕ_q which means a natural reduction of complexity by lowering the problem size.

Example 9. *For $q_{\mathsf{ex}} = (a|\top)$ and the natural* spcr *ψ, we have $V^\psi_{\Delta_{\mathsf{ex}}}(q_{\mathsf{ex}}) = \{\emptyset\}$ and $F^\psi_{\Delta_{\mathsf{ex}}}(q_{\mathsf{ex}}) = \{\{\delta_1,\delta_2\},\{\delta_3\}\}$ (cf. Table 1). Thus, for every c-representation $\kappa_{\eta_{\Delta_{\mathsf{ex}}}}$ of Δ_{ex}, $\kappa_{\eta_{\Delta_{\mathsf{ex}}}}(a) = 0$ and $\kappa_{\eta_{\Delta_{\mathsf{ex}}}}(\bar{a}) = \min\{\eta_1+\eta_2,\eta_3\}$ hold. Because $(1,1,1,3,1)$, $(2,0,1,3,1)$, and $(0,2,1,3,1)$ are the pareto-minimal solutions of $\mathsf{CSP}(\Delta_{\mathsf{ex}})$, $\kappa_{\eta_{\Delta_{\mathsf{ex}}}}(\bar{a}) \geq 1$ follows, such that $\Delta_{\mathsf{ex}} \mathrel{|\!\!\sim}_c q_{\mathsf{ex}}$ holds. Theorem 2 now states that this inference can be validated based on the sub-base $(\Delta_{\mathsf{ex}})^\psi_{q_{\mathsf{ex}}} = \{\delta_1,\delta_2,\delta_3\}$ already.*

In the next section, we develop *constraint networks* which allow us to iteratively solve the constraint satisfaction problems of the sub-bases in canonical safe covers. In this way, we can ensure the c-consistency condition (8).

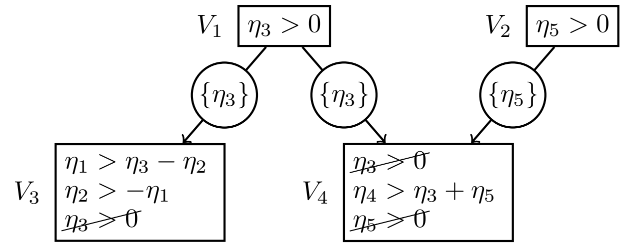

Figure 3: Constraint network $\mathcal{N}^\psi(\Delta_{\mathsf{ex}})$.

Constraint Networks

Constraint networks $\mathcal{N}^\phi(\Delta)$ are directed graphs that can be deduced from canonical safe covers $\mathcal{O}^\phi(\Delta)$ in which the constraints of the constraint satisfaction problem $\mathsf{CSP}^\phi(\Delta)$ are hierarchically organized. With the help of constraint networks, the penalty points η_Δ of c-representations κ_{η_Δ} of Δ can be calculated locally based on the constraints wrt. conditionals of sub-bases in $\mathcal{O}^\phi(\Delta)$ and propagated along the edges of the network to the other sub-bases. Every penalty point $\eta_\Delta(\delta)$ for $\delta \in \Delta$ has to be calculated only once so that the c-consistency condition (8) is automatically satisfied.

Definition 7 (Constraint Network)**.** *Let Δ be a consistent belief base and ϕ an* spcr. *Then, we call the directed acyclic graph $\mathcal{N}^\phi(\Delta) = \langle \mathcal{V}, \mathcal{E}\rangle$ with*

$$\mathcal{V} = \{\mathsf{CSP}^\phi(\Delta') \mid \Delta' \in \mathcal{O}^\phi(\Delta)\},$$
$$\mathcal{E} = \{(V, V') \mid V, V' \in \mathcal{V} \colon V \subsetneq V'\},$$

where $\mathcal{O}^\phi(\Delta)$ is the canonical safe cover of Δ wrt. ϕ the constraint network *of Δ wrt. ϕ. We label the edges in $\mathcal{N}^\phi(\Delta)$ with the* separating penalty points, *i.e., for $E \in \mathcal{E}$ where $E = (\mathsf{CSP}^\phi(\Delta'), \mathsf{CSP}^\phi(\Delta''))$, we label E with*

$$\mathsf{lab}(E) = \{\eta_\Delta(\delta) \mid \delta \in \Delta' \cap \Delta''\}.$$

The constraint network $\mathcal{N}^\phi(\Delta)$ organizes the sub-bases of the canonical safe cover $\mathcal{O}^\phi(\Delta)$ wrt. strict set inclusion. Recall that, for every two sub-bases $\Delta', \Delta'' \in \mathcal{O}^\phi(\Delta)$, the intersection $\Delta' \cap \Delta''$ is also in $\mathcal{O}^\phi(\Delta)$ (unless $\Delta' \cap \Delta'' = \emptyset$; cf. Corollary 1). Hence, if Δ' and Δ'' with $\Delta' \neq \Delta''$ overlap, they have a common predecessor in $\mathcal{N}^\phi(\Delta)$. Constraint networks are directed and acyclic because the strict set inclusion is directed and acyclic.

In order to keep the graphical representation of constraint networks compact, we usually denote the "new" constraints in vertices only. That is, the vertex $\mathcal{V} = \mathsf{CSP}^\phi(\Delta')$ is annotated by $\{C^\phi(\delta) \mid \delta \in \mathcal{R}^\phi_\Delta(\Delta')\}$ where

$$\mathcal{R}^\phi_\Delta(\Delta') = \Delta' \setminus (\textstyle\bigcup_{\Delta''\in\mathcal{O}^\phi(\Delta)\colon \Delta''\subsetneq\Delta'} \Delta'')$$

is the *residuum* of Δ' wrt. Δ and ϕ. The motivation for doing so is that the satisfaction of the constraints wrt. the conditionals in $\Delta' \setminus \mathcal{R}^\phi_\Delta(\Delta')$ (cf. the canceled constraints in Figure 3) is already guaranteed by the penalty points which are propagated to $\mathsf{CSP}^\phi(\Delta')$.

Algorithm 1: Calculation of c-representations on the basis of constraint networks

Input: Constraint network $\mathcal{N}^{\phi}(\Delta) = \langle \mathcal{V}, \mathcal{E} \rangle$
Output: Penalty points η_Δ of a c-representation κ_{η_Δ} of Δ

1 $\mathcal{V}' = \mathcal{V}$ and $\mathcal{E}' = \mathcal{E}$
2 while $\mathcal{V}' \neq \emptyset$:
3   select $V \in \mathcal{V}'$ without $(V', V) \in \mathcal{E}'$ for all $V' \in \mathcal{V}'$
4   calculate solution η_V of V
5   for $V' \in \mathcal{V}'$ with $(V, V') \in \mathcal{E}'$:
6   propagate η_V to V'
7   remove V from $\mathcal{V}'$ and (V, V') from $\mathcal{E}'$
8 for $\delta \in \Delta$:
9   $\eta_\Delta(\delta) = \eta_V(\delta)$ for any $V \in \mathcal{V}$ with $C^{\phi}_{\Delta}(\delta) \in V$
10 return η_Δ

Example 10. *The canonical safe cover of Δ_{ex} wrt. the natural* spcr *ψ is $\mathcal{O}'_{\mathsf{ex}}$ from Example 8. The corresponding constraint network is shown in Figure 3. For instance, the vertex V_3 represents the safe sub-base $\{\delta_1, \delta_2, \delta_3\}$. We may remove the constraint $\eta_3 > 0$ from V_3 because this constraint is mentioned in V_1 already and a solution of this constraint will be propagated from V_1 to V_3 when solving the local constraint satisfaction problems iteratively.*

A systematic solving of $\mathsf{CSP}^{\phi}(\Delta)$ based on the constraint network $\mathcal{N}^{\phi}(\Delta)$ proceeds as follows (cf. Algorithm 1): First, one solves the constraint satisfaction problems of the sub-bases in $\mathcal{O}^{\phi}(\Delta)$ which correspond to vertices in $\mathcal{N}^{\phi}(\Delta)$ without incoming edges (which always exist, cf. Theorem 3). Then, the calculated penalty points are propagated along the edges to the successors of these vertices and so on. The propagation of a penalty point means that each of its occurrences in the constraint satisfaction problem of the target vertex is replaced by its assignment. Then, the constraint satisfaction problems of vertices with incoming edges can be solved once all predecessors are processed.

The order in which vertices are processed is irrelevant to the outcome of the algorithm as long as all predecessors where processed first. In contrast, the selection of particular solutions of the constraint satisfaction problems in the vertices affect the outcome of Algorithm 1 (cf. Algorithm 1, line 4). However, for every vertex, any solution of the respective constraint satisfaction problem can be selected thanks to Proposition 5 so that this freedom of choice is a feature and not a shortage of the algorithm. Consequently, because of Theorem 1, every c-representation of Δ can be calculated with Algorithm 1. Before we prove the soundness of Algorithm 1, we give an example.

Example 11. *We apply Algorithm 1 to the constraint network $\mathcal{N}^{\psi}(\Delta_{\mathsf{ex}})$ in Figure 3. The vertices $V_1 = \{\eta_3 > 0\}$ and $V_2 = \{\eta_5 > 0\}$ have no incoming edges and can be selected as the starting point of the loop in line 3 of Algorithm 1. We assume that V_1 is selected. Then, a solution of V_1 is calculated, for instance $\eta_3 = 1$. Afterwards η_3 is propagated to $V_3 = \{\eta_1 > \eta_3 - \eta_2, \eta_2 > -\eta_1\}$ and $V_4 = \{\eta_4 > \eta_3 + \eta_5\}$ and the vertex V_1 as well as the edges (V_1, V_3) and (V_1, V_4) are removed (line 7). Now, V_2 and V_3 can be processed. For instance, V_2 is selected and $\eta_5 = 1$ is calculated which is propagated to V_4. After that, V_4 mentions $\eta_4 > 2$ which leads to $\eta_4 = 3$ if chosen minimally. Finally V_3 is processed which has the pareto-minimal solutions $\vec{\eta}_{V_3} = (2, 0, 1)$, $(1, 1, 1)$, and $(0, 2, 1)$, for instance.*

Theorem 3. *Algorithm 1 terminates and is sound.*

The runtime of Algorithm 1 is dominated by the solving of the constraint satisfaction problems in the vertices of $\mathcal{N}^{\phi}(\Delta)$ (cf. Algorithm 1, line 4). Actually, only the residues have to be solved so that the runtime of Algorithm 1 essentially depends on the size of the largest residuum. Typically, we have $\max_{\Delta' \in \mathcal{O}^{\phi}(\Delta)} |\mathcal{R}^{\phi}_{\Delta}(\Delta')| \ll |\Delta|$ which means a natural reduction of complexity compared to a direct solving of $\mathsf{CSP}(\Delta)$ by a decrease of the problem size. For instance, we have $\max_{\Delta' \in \mathcal{O}^{\psi}(\Delta_{\mathsf{ex}})} |\mathcal{R}^{\psi}_{\Delta_{\mathsf{ex}}}(\Delta')| = 2$ and $|\Delta_{\mathsf{ex}}| = 5$ (cf., particularly, V_3 in Figure 3). Further note that constraint satisfaction problems of the form $\mathsf{CSP}^{\phi}(\Delta)$ can be transformed into integer linear programs and integer programming is NP-complete (Papadimitriou 1981).

Finally, we have the following sum representation of c-representations when calculated with our approach.

Corollary 2. *Let Δ be a consistent belief base, ϕ an* spcr, *$\mathcal{O}^{\phi}(\Delta) = \{\Delta_i \mid i \in [m]\}$ the canonical safe cover of Δ wrt. ϕ, and $\kappa_{\eta_{\Delta_i}}$, $i \in [m]$, c-representations of Δ_i satisfying the c-consistency condition (8) (for instance, because they were calculated with Algorithm 1). Then,*

$$\kappa(\omega) = \sum\nolimits_{i \in [m]} \sum\nolimits_{\delta \in \mathsf{fal}_{\Delta_i}(\omega) \cap \mathcal{R}^{\phi}_{\Delta}(\Delta_i)} \eta_{\Delta_i}(\delta), \quad (9)$$

$\omega \in \Omega(\Sigma)$, is a c-representation of Δ.

Corollary 2 makes clear that the calculation of c-representations is entirely localized by our approach. The restriction to conditionals from the residuum in the second sum in (9) guarantees that penalty points are not considered twice if a conditional occurs in several sub-bases of $\mathcal{O}^{\phi}(\Delta)$.

Conclusions and Future Work

We investigated the task of localizing default reasoning based on Spohn's ranking functions for conditional belief bases. Hereby, we focused on c-representations as a prominent class of ranking models which are characterized by penalizing possible worlds for falsifying conditionals. Based on *safe covers* which decompose a belief base into sub-bases their corresponding penalty points can be calculated independently of the remaining conditionals in the belief base, we developed with *constraint networks* a graph structure which can be used to iteratively calculate these penalty points. The iterative calculations ensure that conditionals in the intersections of several sub-bases are assigned a distinct penalty point (*c-consistency condition*) so that the local c-representations can be assembled to a global c-representation of the whole belief base. Therewith, we successfully localized not only the computation of c-representations but we also localized drawing skeptical c-inference.

In future work, we want to test our approach on belief bases from real world applications, investigate whether our approach can benefit from the syntax-based concepts of OCF-LEG-networks, and lift it to first-order conditionals.

Acknowledgments

This work was supported by DFG Grant KE 1413/14-1 of the German Research Foundation (DFG) awarded to Gabriele Kern-Isberner.

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