Refined Characterizations of Approval-Based Committee Scoring Rules

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Abstract

In approval-based committee (ABC) elections, the goal is to select a fixed-size subset of the candidates, a so-called committee, based on the voters' approval ballots over the candidates. One of the most popular classes of ABC voting rules are ABC scoring rules, for which voters give points to each committee and the committees with maximal total points are chosen. While the set of ABC scoring rules has recently been characterized in a model where the output is a ranking of all committees, no full characterization of these rules exists in the standard model where a set of winning committees is returned. We address this issue by characterizing two important subclasses of ABC scoring rules in the standard ABC election model, thereby both extending the result for ABC ranking rules to the standard setting and refining it to subclasses. In more detail, by relying on a consistency axiom for variable electorates, we characterize *(i)* the prominent class of Thiele rules and *(ii)* a new class of ABC voting rules called ballot size weighted approval voting. Based on these theorems, we also infer characterizations of three well-known ABC voting rules, namely multi-winner approval voting, proportional approval voting, and satisfaction approval voting.

1 Introduction

An important problem for multi-agent systems is collective decision making: given the voters' preferences over a set of alternatives, a common decision has to be made. This problem has traditionally been studied by economists for settings where a single candidate is elected (Arrow, Sen, and Suzumura 2002), but there is also a multitude of applications where a fixed number of the candidates needs to be elected. The archetypal example for this is the election of a city council, but there are also technical applications such as recommender systems (Skowron, Faliszewski, and Lang 2016; Gawron and Faliszewski 2022). In social choice theory, this type of elections is typically called *approval-based committee (ABC) elections* and has recently attracted significant attention (e.g., Aziz et al. 2017; Faliszewski et al. 2017; Lackner and Skowron 2023). In more detail, the research on these elections focuses on *ABC voting rules*, which are functions that choose a set of winning committees (i.e., fixed-size subsets of the candidates) based on the voters' approval ballots (i.e., the sets of candidates that the voters approve).

One of the most important classes of ABC voting rules are ABC scoring rules (see, e.g., Lackner and Skowron 2021b). These rules generalize the idea of single-winner scoring rules to ABC elections: each voter assigns points to each committee according to some scoring function and the winning committees are those with the maximal total score. There are many well-known examples of ABC scoring rules, such as multi-winner approval voting (`AV`), satisfaction approval voting (`SAV`), Chamberlin-Courant approval voting (`CCAV`), and proportional approval voting (`PAV`). Moreover, ABC scoring rules are a superset of the prominent class of Thiele rules.

In a recent breakthrough result, Lackner and Skowron (2021b) have formalized the relation between ABC scoring rules and single-winner scoring rules by characterizing ABC scoring rules with almost the same axioms as Young (1975) uses for his influential characterization of single-winner scoring rules. In more detail, Lackner and Skowron (2021b) show that ABC scoring rules are the only ABC ranking rules that satisfy the axioms of anonymity, neutrality, continuity, weak efficiency, and consistency. However, this result discusses ABC ranking rules, which return transitive rankings of committees, whereas the literature on ABC elections typically focuses on sets of winning committees as output. Hence, this theorem does not allow for characterizations of ABC scoring rules in the standard ABC voting setting.

While Lackner and Skowron (2021a) also present a result for the standard ABC election setting, the proof of this result is incomplete.[1] Moreover, even when the proof could be fixed, this result is not a full characterization of ABC scoring rules as it needs a technical axiom called 2-non-imposition. This axiom is, e.g., violated by `AV` and `SAV`. Hence, characterizations of important ABC voting rules—and more generally tools to easily infer such results—are still missing. Lackner and Skowron (2021a, p. 16) also acknowledge this shortcoming by writing that "a full characterization of ABC scoring rules within the class of ABC choice rules remains as important future work".

[1] Roughly, the proof of Lackner and Skowron (2021a) works by constructing an ABC ranking rule g based on an ABC voting rule f that satisfies the given axioms. Then, Lackner and Skowron (2021a) show that g is an ABC scoring rule, which implies that f is an ABC scoring rule in the choice setting. However, the authors never show that g returns transitive rankings, which is required by definition of ABC ranking rules. Closing this gap seems surprisingly difficult.

Our contribution. We address this problem by presenting full axiomatic characterizations of two important subclasses of ABC scoring rules, namely Thiele rules and ballot size weighted approval voting (BSAV) rules, in the standard ABC election setting. Hence, our results refine the result of Lackner and Skowron (2021b) to subclasses and extend it to the standard ABC voting setting. Thiele rules are ABC scoring rules that do not depend on the ballot size and have attracted significant attention (e.g., Aziz et al. 2017; Skowron, Faliszewski, and Lang 2016; Brill, Laslier, and Skowron 2018). On the other hand, BSAV rules generalize multi-winner approval voting by weighting voters depending on the size of their ballots. So far, the class of BSAV rules has only been studied for single-winner elections (Alcalde-Unzu and Vorsatz 2009) but not for ABC elections. For example, `PAV` and `CCAV` are Thiele rules, `SAV` is a BSAV rule, and `AV` is in both classes. Moreover, every ABC scoring rule that has been studied in the literature is in one of our two classes.

For our results, we mainly rely on the axioms of Lackner and Skowron (2021b): anonymity, neutrality, continuity, weak efficiency, and consistency. The first four of these axioms are mild standard conditions that are satisfied by every reasonable ABC voting rule. By contrast, consistency is central for our proofs. This axiom requires that if some committees are chosen for two disjoint elections, then precisely these committees should win in a joint election, and it features in several prominent results in social choice theory (e.g., Young 1975; Young and Levenglick 1978; Fishburn 1978).

To characterize Thiele rules, we need one more axiom called independence of losers. This condition requires that a winning committee W stays winning if some voters change their ballot by disapproving "losing" candidates outside of W as, intuitively, the quality of W should only depend on its members. Similar conditions are well-known for single-winner elections (e.g., Brandl and Peters 2022) and this axiom has recently been adapted to ABC voting by Dong and Lederer (2023a). We then show that *an ABC voting rule is a Thiele rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and independence of losers (Theorem 1).*

For our characterization of BSAV rules, we introduce a new axiom called choice set convexity. This condition requires that if two committees are chosen, then all committees "in between" those committees are chosen, too: if W and W' are chosen, then all committees W'' with $W \cap W' \subseteq W'' \subseteq W \cup W'$ are also chosen. We believe that this axiom is reasonable for excellence-based elections (where only the individual quality of the candidates matters) as a tie between committees indicates that they are equally good and the candidates in $W \setminus W'$ and $W' \setminus W$ are thus exchangeable. We then prove that *an ABC voting rule is a BSAV rule if and only if it satisfies anonymity, neutrality, consistency, continuity, weak efficiency, and choice set convexity (Theorem 2).*

While our theorems are intuitively related to the results of Lackner and Skowron (2021a,b), they are logically independent. In particular, in contrast to their results, our theorems allow for simple characterizations of *all* Thiele rules and BSAV rules in the *standard ABC voting model*. We also demonstrate this point in Section 3.3 by axiomatizing `AV`, `SAV`, and `PAV`. In more detail, we obtain full characterizations of these rules by analyzing axioms for party-list profiles (where candidates are partitioned into parties and each voter approves all candidates of a single party) that formalize when all candidates of a party are chosen. To the best of our knowledge, the result for `SAV` is the first full characterization of this rule. An overview of our results is given in Figure 1.

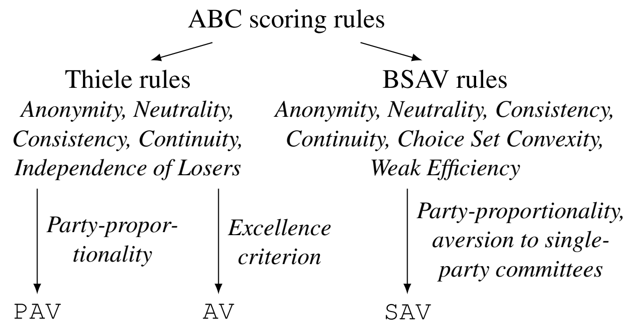

Figure 1: Overview of our results. An arrow from X to Y means that Y is a subset or an element of X. The axioms written on an arrow from X to Y characterize the rule Y within the class X. The axioms written below Thiele rules and BSAV rules characterize these classes of ABC voting rules.

Related work. The lack of axiomatic characterizations is one of the major open problems in the field of ABC voting (see, e.g., Lackner and Skowron 2023, Q1), and there are thus only few closely related papers. Maybe the most important one is due to Lackner and Skowron (2021b) who characterize ABC scoring rules in the context of ABC ranking rules; however, this result does not allow for characterizations of ABC scoring rules in the standard setting. The follow-up paper by Lackner and Skowron (2021a) tries to fix this issue, but its proof is incomplete and the main result requires a technical auxiliary condition that rules out important rules such as `AV` and `SAV`. Moreover, Dong and Lederer (2023a) characterize committee monotone ABC voting rules, which can be seen as greedy approximations of ABC scoring rules. Finally, committee scoring rules have also been analyzed for the case that voters report ranked ballots, but the results for this setting are also restricted to characterizations of committee ranking rules (Skowron, Faliszewski, and Slinko 2019) or partial characterizations within the class of committee scoring rules (Elkind et al. 2017; Faliszewski et al. 2019).

Furthermore, a large amount of papers studies axiomatic properties of ABC scoring rules (e.g., Lackner and Skowron 2018; Aziz et al. 2017; Sánchez-Fernández and Fisteus 2019; Brill, Laslier, and Skowron 2018; Lackner and Skowron 2020). For instance, Aziz et al. (2017) investigate Thiele rules with respect to how fair they represent groups of voters with similar preferences, and Sánchez-Fernández and Fisteus (2019) study monotonicity conditions for several ABC scoring rules. Another important aspect of these rules is their computational complexity. In particular, it is known that all Thiele rules but `AV` are `NP`-hard to compute on the full domain (Aziz et al. 2015; Skowron, Faliszewski, and

Lang 2016). There is thus significant work on how to compute these rules by, e.g., restricting the domain of preference profiles (Elkind and Lackner 2015; Peters 2018), studying approximation algorithms (Dudycz et al. 2020; Barman et al. 2022), or designing FPT algorithms (Bredereck et al. 2020). For a more detailed overview on ABC scoring rules, we refer to the survey by Lackner and Skowron (2023).

Finally, in the broader realm of social choice, there are numerous conceptually related results as consistency features in many prominent theorems: for instance, Young (1975) has characterized scoring rules for single-winner elections based on this axiom (see also Smith 1973; Myerson 1995; Pivato 2013), numerous characterizations of single-winner approval voting rely on consistency (Fishburn 1978; Brandl and Peters 2022), Young and Levenglick (1978) have characterized Kemeny's rule with the help of this axiom, and Brandl, Brandt, and Seedig (2016) characterize a randomized voting rule called maximal lotteries based on this condition.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ denote an infinite set of voters and let $\mathcal{C} = \{c_1, \dots, c_m\}$ denote a set of $m \geq 2$ candidates. Intuitively, we interpret $\mathbb{N}$ as the set of all possible voters and a concrete electorate N is a finite and non-empty subset of $\mathbb{N}$. We thus define $\mathcal{F}(\mathbb{N}) = \{N \subseteq \mathbb{N} \colon N \textit{ is non-empty and finite}\}$ as the set of all possible electorates. Given an electorate $N \in \mathcal{F}(\mathbb{N})$, we assume that each voter $i \in N$ reports her preferences over the candidates as *approval ballot* A_i, i.e., as a non-empty subset of $\mathcal{C}$. $\mathcal{A}$ is the set of all possible approval ballots. An *(approval) profile* A is a mapping from N to $\mathcal{A}$, i.e., it assigns an approval ballot to every voter in the given electorate. Next, we define $\mathcal{A}^* = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \mathcal{A}^N$ as the set of all approval profiles. For every profile $A \in \mathcal{A}^*$, N_A denotes the set of voters that submit a ballot in A. Finally, two approval profiles A, A' are called *disjoint* if $N_A \cap N_{A'} = \emptyset$ and for disjoint profiles A, A', we define the profile $A'' = A + A'$ by $N_{A''} = N_A \cup N_{A'}$, $A''_i = A_i$ for $i \in N_A$, and $A''_i = A'_i$ for $i \in N_{A'}$.

Given an approval profile, our aim is to elect a *committee*, i.e., a subset of the candidates of predefined size. We denote the target committee size by $k \in \{1, \dots, m-1\}$ and the set of all size-k committees by $\mathcal{W}_k = \{W \subseteq \mathcal{C} \colon |W| = k\}$. For determining the winning committees for a given preference profile, we use *approval-based committee (ABC) voting rules* which are mappings from $\mathcal{A}^*$ to $2^{\mathcal{W}_k} \setminus \{\emptyset\}$. Note that we define ABC voting rules for a fixed committee size and may return multiple committees. The first condition is for notational convenience and the second one is necessary to satisfy basic fairness conditions.

2.1 ABC Voting Rules

We focus in this paper on two classes of ABC voting rules, namely Thiele rules and BSAV rules, which are both refinements of the class of ABC scoring rules.

ABC scoring rules. ABC scoring rules rely on a scoring function according to which voters assign points to committees and choose the committees with maximal total score. Formally, a *scoring function* $s(x, y)$ is a mapping from $\{0, \dots, k\} \times \{1, \dots, m\}$ to $\mathbb{R}$ such that $s(x, y) \geq s(x', y)$ for all $x, x' \in \{\max(0, k+y-m), \dots, \min(k, y)\}$ with $x \geq x'$. We define the score of a committee W in a profile A as $\hat{s}(A, W) = \sum_{i \in N_A} s(|A_i \cap W|, |A_i|)$. Then, an ABC voting rule f is an *ABC scoring rule* if there is a scoring function s such that $f(A) = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W}_k \colon \hat{s}(A, W) \geq \hat{s}(A, W')\}$ for all profiles $A \in \mathcal{A}^*$. The set $\{\max(0, k+y-m), \dots, \min(k, y)\}$ contains all "active" intersection sizes: a committee of size k and a ballot of size y intersect at least in $\max(0, k+y-m)$ candidates and at most in $\min(k, y)$ candidates.

Thiele rules. Arguably the most prominent subclass of ABC scoring rules are Thiele rules. These rules, which have first been suggested by their namesake Thiele (1895), are ABC scoring rules that ignore the ballot size. Hence, Thiele rules are defined by a non-decreasing *Thiele scoring function* $s : \{0, \dots, k\} \to \mathbb{R}$ with $s(0) = 0$, and choose the committees that maximize the total score. Formally, an ABC voting rule f is a *Thiele rule* if there is a Thiele scoring function s such that $f(A) = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W}_k \colon \hat{s}(A, W) \geq \hat{s}(A, W')\}$ for all profiles $A \in \mathcal{A}^*$, where $\hat{s}(A, W) = \sum_{i \in N_A} s(|A_i \cap W|)$. There are numerous important Thiele rules such as multi-winner approval voting (AV; defined by $s_{\texttt{AV}}(x) = x$), proportional approval voting (PAV; defined by $s_{\texttt{PAV}}(x) = \sum_{z=1}^{x} \frac{1}{z}$ for $x > 0$), and Chamberlin-Courant approval voting (CCAV; defined by $s_{\texttt{CCAV}}(x) = 1$ for $x > 0$).

BSAV rules. Ballot size weighted approval voting rules form a new subclass of ABC scoring rules which generalize AV by weighting voters based on their ballot size. Formally, a *ballot size weighted approval voting (BSAV) rule* f is defined by a weight vector $\alpha \in \mathbb{R}^m_{\geq 0}$ and chooses for every profile A the committees W that maximize $\hat{s}(A, W) = \sum_{i \in N_A} \alpha_{|A_i|} |A_i \cap W|$. The score of a committee W for a BSAV rule can be represented as the sum of the scores of individual candidates $c \in W$ since $\sum_{i \in N_A} \alpha_{|A_i|} |A_i \cap W| = \sum_{c \in W} \sum_{i \in N_A \colon c \in A_i} \alpha_{|A_i|}$. Clearly, AV is the BSAV rule defined by $\alpha_x = 1$ for all $x \in \{1, \dots, m\}$. Another well-known BSAV rule is satisfaction approval voting (SAV) defined by $\alpha_x = \frac{1}{x}$ for $x \in \{1, \dots, m\}$. This rule has been popularized by Brams and Kilgour (2014) for ABC elections, but it has been studied before by, e.g., Alcalde-Unzu and Vorsatz (2009) and Kilgour and Marshall (2012).

We note that Thiele rules and BSAV rules are diametrically opposing subclasses of ABC scoring rules: Thiele rules do not depend on the ballot size at all, whereas BSAV rules only depend on this aspect. Consequently, if $k < m - 1$, the sets of BSAV rules and Thiele rules only intersect in AV and the trivial rule TRIV (which always chooses all size k committees). So, AV is the only non-trivial ABC voting rule that is in both classes; *non-triviality* means here that there is a profile A such that $f(A) \neq \texttt{TRIV}(A)$. Moreover, both classes are proper subsets of the set of ABC scoring rules if $1 < k < m - 1$. By contrast, the set of BSAV rules is equivalent to the set of ABC scoring rules if $k \in \{1, m-1\}$.

2.2 Basic Axioms

Next, we introduce the axioms used for our characterizations.

Anonymity. Anonymity is one of the most basic fairness properties and requires that all voters should be treated equally. Formally, we say an ABC voting rule f is *anonymous* if $f(A) = f(\pi(A))$ for all profiles $A \in \mathcal{A}^*$ and permutations $\pi : \mathbb{N} \to \mathbb{N}$. Here, we denote by $A' = \pi(A)$ the profile with $N_{A'} = \{\pi(i) \colon i \in N_A\}$ and $A'_{\pi(i)} = A_i$ for all $i \in N_A$.

Neutrality. Similar to anonymity, *neutrality* is a fairness property for the candidates. This axiom requires of an ABC voting rule f that $f(\tau(A)) = \{\tau(W) \colon W \in f(A)\}$ for all profiles $A \in \mathcal{A}^*$ and permutations $\tau : \mathcal{C} \to \mathcal{C}$. This time, $A' = \tau(A)$ denotes the profile with $N_{A'} = N_A$ and $A'_i = \tau(A_i)$ for all $i \in N_A$.

Weak Efficiency. Weak efficiency requires that unanimously unapproved candidates can never be "better" than approved ones. Formally, we say an ABC voting rule f is *weakly efficient* if $W \in f(A)$ for a committee $W \in \mathcal{W}_k$ with $c \in W \setminus (\bigcup_{i \in N_A} A_i)$ implies that $(W \cup \{c'\}) \setminus \{c\} \in f(A)$ for all candidates $c' \in \mathcal{C} \setminus W$.

Continuity. The intuition behind continuity is that a large group of voters should be able to enforce that some of its desired outcomes are chosen. Hence, an ABC voting rule f is *continuous* if for all profiles $A, A' \in \mathcal{A}^*$, there is $\lambda \in \mathbb{N}$ such that $f(\lambda A + A') \subseteq f(A)$. Here, λA denotes the profile consisting of λ copies of A; the names of the voters in $N_{\lambda A}$ will not matter as we will focus on anonymous rules.

Consistency. The central axiom for our results is consistency. This condition states that if some committees are chosen for two disjoint profiles, then precisely those committees are chosen in the joint profile. Formally, an ABC voting rule f is *consistent* if $f(A + A') = f(A) \cap f(A')$ for all disjoint profiles $A, A' \in \mathcal{A}^*$ with $f(A) \cap f(A') \neq \emptyset$. Consistency and the previous four axioms have been introduced by Lackner and Skowron (2021a) for ABC elections. Moreover, except consistency, all these axioms are very mild and satisfied by almost all commonly considered ABC voting rules.

Independence of Losers. Independence of losers has been adapted to ABC elections by Dong and Lederer (2023a) and requires of an ABC voting rule f that a winning committee W should still be a winning committee if voters disapprove candidates outside of W. Or, put differently, whether a committee W wins should not depend on the voters' approvals of "losing" candidates not in W. We hence say an ABC voting rule f is *independent of losers* if $W \in f(A)$ implies that $W \in f(A')$ for all profiles $A, A' \in \mathcal{A}^*$ and committees $W \in \mathcal{W}_k$ such that $N_A = N_{A'}$ and $W \cap A_i = W \cap A'_i$ and $A'_i \subseteq A_i$ for all voters $i \in N_A$. The motivation for this axiom is that the quality of W should only depend on the candidates in W. So, if some voters disapprove candidates $x \notin W$, the quality of this committee is not affected and, when W is chosen initially, it should remain chosen. All commonly studied ABC voting rules that are independent of the ballot size (e.g., Thiele rules, Phragmén's rule, and sequential Thiele rules) satisfy this axiom, whereas all BSAV rules except AV fail it.

Choice Set Convexity. Finally, we introduce a new condition called choice set convexity: an ABC voting rule f is *choice set convex* if $W, W' \in f(A)$ implies that $W'' \in f(A)$ for all committees $W, W', W'' \in \mathcal{W}_k$ and profiles $A \in \mathcal{A}^*$ such that $W \cap W' \subseteq W'' \subseteq W \cup W'$. More informally, this axiom states that if a rule chooses two committees W and W', then all committees "between" W and W' are also chosen. We believe that choice set convexity is reasonable in elections in which only the individual quality of the elected candidates matters. For example, if we want to hire 3 applicants for independent jobs based on the interviewers' preferences, it seems unreasonable that the sets $\{c_1, c_2, c_3\}$ and $\{c_1, c_4, c_5\}$ are good enough to be hired but $\{c_1, c_2, c_4\}$ is not. More generally, we can interpret the membership of a candidate in a chosen committee as certificate for its quality and all candidates $c \in (W \setminus W') \cup (W' \setminus W)$ are then equally good. Many commonly considered voting rules fail this axiom, but one can always compute the "convex hull" of a choice set.

3 Results

We are now ready to state our results. In particular, we discuss the characterizations of Thiele rules and BSAV rules in Sections 3.1 and 3.2, respectively. Moreover, we present characterizations of AV, PAV, and SAV in Section 3.3. Due to space constraints, we defer most proofs to the full version (Dong and Lederer 2023b) and give proof sketches instead.

3.1 Characterization of Thiele Rules

We now turn to our first characterization: Thiele rules are the only ABC voting rules that are anonymous, neutral, consistent, continuous, and independent of losers. We thus turn the result of Lackner and Skowron (2021b) into a characterization of Thiele rules in the standard ABC voting model by replacing weak efficiency with independence of losers.

Theorem 1. *An ABC voting rule is a Thiele rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and independence of losers.*

Proof Sketch. First, suppose that f is a Thiele rule and let $s(x)$ denote its Thiele scoring function. Clearly, f is anonymous, neutral, consistent, and continuous as all ABC scoring rules satisfy these axioms. So, we will only show that f is independent of losers. For this, consider two profiles $A, A' \in \mathcal{A}^*$ and a committee $W \in f(A)$ such that $N_A = N_{A'}$ and $A'_i \subseteq A_i$ and $W \cap A'_i = W \cap A_i$ for all $i \in N_A$. It holds that $\hat{s}(A', W) = \hat{s}(A, W)$ since $W \cap A'_i = W \cap A_i$ for all $i \in N_A$. Moreover, $\hat{s}(A, W) \geq \hat{s}(A, W')$ for all $W' \in \mathcal{W}_k$ because $W \in f(A)$. Finally, $\hat{s}(A, W') \geq \hat{s}(A', W')$ for all $W' \in \mathcal{W}_k$ as $s(x)$ is non-decreasing and $A'_i \subseteq A_i$ for all $i \in N_A$. By chaining the inequalities, we conclude that $\hat{s}(A', W) \geq \hat{s}(A', W')$ for all committees $W' \in \mathcal{W}_k$, so $W \in f(A')$ and f satisfies independence of losers.

For the other direction, we suppose that f is an ABC voting rule that satisfies all axioms of the theorem and aim to show that f is a Thiele rule. For this, we will use the separating hyperplane theorem for convex sets similar to the works of, e.g., Young (1975) and Skowron, Faliszewski, and Slinko (2019). For this, we note first that, if f is trivial, it is the

Thiele rule defined by $s(x) = 0$ for all x. So, we suppose that f is non-trivial and show that for every committee $W \in \mathcal{W}_k$, there is a profile $A \in \mathcal{A}^*$ such that $f(A) = \{W\}$. To apply the separating hyperplane theorem for convex sets, we next extend f to a function $\hat{g}$ of the type $\mathbb{Q}^{|\mathcal{A}|} \rightarrow 2^{\mathcal{W}_k} \setminus \{\emptyset\}$ while keeping all its properties intact. We then define the sets $R_i^f = \{v \in \mathbb{Q}^{|\mathcal{A}|} \colon W^i \in \hat{g}(v)\}$ for all $W^i \in \mathcal{W}_k$ and let $\bar{R}_i^f$ denote the closure of R_i^f with respect to $\mathbb{R}^{|\mathcal{A}|}$. It follows from the properties of $\hat{g}$ that the sets $\bar{R}_i^f$ are convex and have disjoint interiors. The separating hyperplane theorem for convex sets thus shows that there are non-zero vectors $\hat{u}^{i,j} \in \mathbb{R}^{|\mathcal{A}|}$ such that $v\hat{u}^{i,j} \geq 0$ if $v \in \bar{R}_i$ and $v\hat{u}^{i,j} \leq 0$ if $v \in \bar{R}_j$. Moreover, we will show that $\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|} \colon \forall W^j \in \mathcal{W}_k \setminus \{W^i\} \colon v\hat{u}^{i,j} \geq 0\}$, so we study the vectors $\hat{u}^{i,j}$ next.

For this, we first infer from neutrality and independence of losers that there is a function $s^1(x, y)$ such that $\hat{u}_\ell^{i,j} = s^1(|W^i \cap A_\ell|, |W^j \cap A_\ell|)$ for all ballots A_ℓ and committees W^i, W^j with $|W^i \setminus W^j| = 1$. If $k \in \{1, m-1\}$, this insight is already enough for the proof. By contrast, if $k \in \{2, \dots, m-2\}$, we need to analyze the vectors $\hat{u}^{i,j}$ for committees W^i, W^j with $|W^i \setminus W^j| = t > 1$. To this end, we construct a sequence of committees $W^{j_0}, \dots, W^{j_t}$ by replacing the candidates in $W^i \setminus W^j$ one after another with those in $W^j \setminus W^i$. By studying the linear (in)dependence of the vectors $\hat{u}^{i,j}$ and $\hat{u}^{j_{x-1}, j_x}$ for $x \in \{1, \dots, t\}$, we then show that $\hat{u}^{i,j} = \delta \sum_{x=1}^{t} \hat{u}^{j_{x-1}, j_x}$ for some $\delta > 0$. Based on this insight, we can now define the score function s of f: $s(0) = 0$ and $s(x) = s(x-1) + s^1(x, x-1)$ for $x \geq 1$. By our previous observations, it follows that $\hat{u}_\ell^{i,j} = \delta(s(|W^i \cap A_\ell|) - s(|W^j \cap A_\ell|))$, so $\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|} \colon \forall W^j \in \mathcal{W}_k \colon \hat{s}(v, W^i) \geq \hat{s}(v, W^j)\}$. From this, we infer that $\hat{g}(v) \subseteq \{W^i \in \mathcal{W}_k \colon v \in \bar{R}_i^f\} = \{W^i \in \mathcal{W}_k \colon \forall W^j \in \mathcal{W}_k \setminus \{W^i\} \colon \hat{s}(v, W^i) \geq \hat{s}(v, W^j)\}$ for all $v \in \mathbb{Q}^{|\mathcal{A}|}$. Thus, $f(A) \subseteq \{W^i \in \mathcal{W}_k \colon \forall W^j \in \mathcal{W}_k \setminus \{W^i\} \colon \hat{s}(A, W^i) \geq \hat{s}(A, W^j)\}$ and, as the last step, continuity shows that f is the Thiele rule induced by s. □

Remark 1. All axioms are required for Theorem 1. If we omit independence of losers, SAV satisfies all remaining axioms. If we omit continuity, we can refine Thiele rules by applying a second Thiele rule as tie-breaker in case of multiple chosen committees. If we only omit consistency, sequential Thiele rules satisfy all given axioms. These rules compute the winning committees iteratively by always adding the candidate to a winning committee which increases the score the most. If we omit neutrality or anonymity, biased Thiele rules that double the points of every committee that contains a specific candidate or the points assigned by specific voters to the committees satisfy all other axioms.

3.2 Characterization of BSAV Rules

Next, we discuss the characterization of BSAV rules: these are the only ABC voting rules that satisfy anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency. The central axiom for this characterization (aside of consistency) is choice set convexity as it enforces that candidates can be exchanged between chosen committees.

Theorem 2. *An ABC voting rule is a BSAV rule if and only if it satisfies anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency.*

Proof Sketch. First, we assume that f is a BSAV rule and let $\alpha = (\alpha_1, \dots, \alpha_m)$ denote its weight vector. It is simple to verify that f is neutral, anonymous, continuous, and consistent. Moreover, f is weakly efficient as the weights α_i are all non-negative. Finally, we show that f is choice set convex. For this, we consider a profile A and two distinct committees $W, W' \in f(A)$. Next, we choose two candidates $a \in W \setminus W'$ and $b \in W' \setminus W$ and let $W'' = (W \setminus \{a\}) \cup \{b\}$. The central observation is now that BSAV scores are additive, i.e., $\hat{s}(A, W) = \sum_{x \in W} \hat{s}(A, x)$ for $\hat{s}(A, x) = \sum_{i \in N_A \colon x \in A_i} \alpha_{|A_i|}$. Since $W \in f(A)$, $0 \leq \hat{s}(A, W) - \hat{s}(A, W'') = \hat{s}(A, a) - \hat{s}(A, b)$. By applying this argument also to W' and $W''' = (W' \setminus \{b\}) \cup \{a\}$, we obtain $0 \leq \hat{s}(A, b) - \hat{s}(A, a)$, so $\hat{s}(A, a) = \hat{s}(A, b)$ and $\hat{s}(A, W) = \hat{s}(A, W'')$. This proves that $W'' \in f(A)$ and by repeating the argument, we infer that $\bar{W} \in f(A)$ for all $\bar{W}$ with $W \cap W' \subseteq \bar{W} \subseteq W \cup W'$.

For the converse direction, we give again only a rough proof sketch and note that the outline of this proof is very similar to the one of Theorem 1 as mainly the technical details differ. In more detail, we first extend f to a function $\hat{g}$ on $\mathbb{Q}^{|\mathcal{A}|}$ and then use the same hyperplane argument as for Theorem 1. Hence, we will again analyze the sets $R_i^f = \{v \in \mathbb{Q}^{|\mathcal{A}|} \colon W^i \in \hat{g}(v)\}$ and the vectors $\hat{u}^{i,j}$ with $v\hat{u}^{i,j} \geq 0$ if $v \in \bar{R}_i^f$ and $v\hat{u}^{i,j} \leq 0$ if $v \in \bar{R}_j^f$. In particular, based on choice set convexity, we show for every ballot size $r \in \{1, \dots, m\}$ that there is a constant $\alpha_r \geq 0$ such that $\hat{u}_\ell^{i,j} = \alpha_r$ for all ballots $A_\ell \in \mathcal{A}$ with $|A_\ell| = r$ and committees $W^i, W^j \in \mathcal{W}_k$ with $|W_i \cap A_\ell| = |W_j \cap A_\ell| + 1$. Based on this insight, it is simple to complete the proof if $k \in \{1, m-1\}$. On the other hand, if $k \in \{2, \dots, m-2\}$, we again consider committees W^i, W^j such that $|W^i \setminus W^j| = t > 1$. Just as for Theorem 1, we consider a sequence of committees $W^{j_0}, \dots, W^{j_t}$ such that $W^{j_0} = W^i$, $W^{j_t} = W^j$, and $|W^{j_{x-1}} \setminus W^{j_x}| = 1$ for $x \in \{1, \dots, t\}$, and show that $\hat{u}^{i,j} = \delta \sum_{x=1}^{t} \hat{u}^{j_{x-1}, j_x}$ for some $\delta > 0$. This implies that $\hat{u}_\ell^{i,j} = \alpha_r(|W^i \cap A_\ell| - |W^j \cap A_\ell|)$ for all committees $W^i, W^j \in \mathcal{W}_k$ and ballots $A_\ell \in \mathcal{A}$ with $|A_\ell| = r$. Finally, we can now prove that f is the BSAV rule defined by the score function $s(|W \cap A_\ell|, |A_\ell|) = \alpha_{|A_\ell|}|W \cap A_\ell|$. □

Remark 2. All axioms are required for Theorem 2. For anonymity, neutrality, and continuity, we can define examples similar to the ones given for Thiele rules. When omitting consistency, the "convex hull" of Phragmén's rule satisfies all remaining axioms and is no BSAV rule. The rule that elects the k candidates with minimal approval scores satisfies all given axioms but weak efficiency. Finally, every Thiele rule other than AV only fails choice set convexity.

Remark 3. AV is the only non-trivial ABC voting rule that is both a BSAV rule and a Thiele rule if $k \leq m-2$. Theorems 1 and 2 thus characterize AV as the only non-trivial ABC voting rule that is anonymous, neutral, continuous, consistent, independent of losers, and choice set convex if $k \leq m-2$.

Remark 4. We define ABC voting rules for a fixed committee size k, but in the literature k is often part of the input. For such rules, Theorems 1 and 2 imply that for every $k \in \{1, \dots, m-1\}$, $f(A, k)$ is a Thiele rule or a BSAV rule, respectively, if it satisfies the required axioms. However, our conditions do not enforce consistency with respect to the committee size, so we can, e.g., use AV for $k = 2$ and PAV for $k = 3$. It is not difficult to exclude such rules. For instance, the well-known axiom of committee monotonicity (Elkind et al. 2017) entails for every BSAV rule that it must use the same weight vector for every committee size k. Similar, committee separability, an axiom introduced by Dong and Lederer (2023a), can be used to enforce that non-imposing Thiele rules use the same Thiele scoring function for every committee size. Thus, our results can be easily extended to the setting where the committee size is part of the input.

3.3 Characterizations of AV, PAV, and SAV

Finally, we demonstrate in this section how Theorems 1 and 2 can be used to characterize specific ABC voting rules. To this end, we first note that there are numerous characterizations of ABC voting rules within the class of Thiele rules in the literature, and Theorem 1 can typically be used to extend these results to full characterizations. For instance, Lackner and Skowron (2018) characterize AV among the class of Thiele rules based on a strategyproofness notion and it is easy to extend this result to a full characterization of AV based on Theorem 1. Similar claims are true for, e.g., the characterization of AV based on committee monotonicity (Janson 2016), the characterization of PAV based on D'Hondt proportionality (e.g., Brill, Laslier, and Skowron 2018), or characterizations of CCAV (e.g., Delemazure et al. 2023). In this paper, we will, however, give characterizations of three ABC scoring rules (namely AV, PAV, and SAV) that are largely independent of the literature. The reason for this is that our technique seems rather universal and may thus also be used to characterize further Thiele rules or BSAV rules. Finally, we will state our results only within the class of Thiele rules and BSAV rules, respectively; Theorems 1 and 2 then generalize these results to full characterizations.

In more detail, for all three results in this subsection, we study axioms defined for special profiles. To this end, we say a profile $A \in \mathcal{A}^*$ is a *party-list profile* if there is a partition $\mathcal{P}_A = \{P_1, \dots, P_\ell\}$ of the candidates such that every voter approves all candidates in one set P_j, i.e., for every voter $i \in N_A$, there is a set $P_j \in \mathcal{P}_A$ such that $A_i = P_j$. Less formally, in a party-list profile, the candidates are grouped into disjoint parties and every voter supports a single party by approving all of its members. We denote by n_j the number of voters who support party P_j in a party-list profile A. For these profiles, we will investigate the question when a voting rule elects all members of a party. The reason for this design choice is twofold: firstly, this will lead to rather mild axioms which makes our characterizations only stronger. Secondly, on party-list profiles, BSAV rules typically elect one party after another by first electing all members of the first party, then electing all members of the second party, and so on. Hence, axioms describing when all candidates of a party are elected are well-suited for characterizing these rules.

Clearly, any justification for when all members of a party should be chosen needs to consider the purpose of the election. For instance, if the goal of an election is to find the best k candidates only based on their individual quality (a setting known as excellence-based elections), the main criterion for deciding whether to choose a candidate is the number of voters supporting it. Hence, if a party P_i is approved by more voters than another party P_j, then every candidate in P_i seems better than every candidate in P_j. Thus, if all candidates of party P_j are chosen, all candidates of party P_i should also be chosen. We formalize this idea as follows: An ABC voting rule f satisfies the *excellence criterion* if for all party-list profiles A, committees $W \in f(A)$, and parties $P_i, P_j \in \mathcal{P}_A$ with $n_i < n_j$, it holds that $P_i \subseteq W$ implies that $P_j \subseteq W$. As we show next, this condition characterizes AV among Thiele rules.

Proposition 1. *AV is the only Thiele rule that satisfies the excellence criterion.*

Proof. Clearly, AV satisfies the excellence criterion and we thus focus on the converse direction. For this, let f denote a Thiele rule that satisfies the excellence criterion and let s denote its Thiele scoring function. Our first goal is to show that $s(1) > 0$ and we consider for this the party-list profile A in which 2 voters approve $P_1 = \{c_1\}$ and 1 voter approves $P_2 = \{c_2, \dots, c_{k+1}\}$. Now, if $s(1) = 0$, then $P_2 \in f(A)$ as s is non-decreasing. This, however, violates the excellence criterion as there is a winning committee that contains all members of P_2 but none of P_1, even though $n_1 > n_2$. Hence, $s(1) > 0$ and we subsequently suppose that $s(1) = 1$ as Thiele rules are invariant under scaling the scoring function.

Next, we assume for contradiction that there is an index $\ell \in \{2, \dots, k\}$ such that $s(\ell) \neq \ell$ and $s(x) = x$ for all $x < \ell$. Moreover, we define $\Delta = |s(\ell) - \ell| \neq 0$ and let $t \in \mathbb{N}$ such that $t \geq 2$ and $t\Delta > k$. We now use a case distinction with respect to $s(\ell)$ and first suppose that $s(\ell) = \ell + \Delta$. In this case, consider the party-list profile A where t voters approve $P_1 = \{c_1, \dots, c_\ell\}$ and each other candidate $c \in \mathcal{C} \setminus P_1$ is uniquely approved by $t + 1$ voters. It is easy to verify that every committee W with $P_1 \subseteq W$ has a score of $\hat{s}(A, W) = ts(\ell) + (k - \ell)(t + 1) = t\Delta + t\ell + (k - \ell)(t + 1) > k + tk$. By contrast, every committee W' with $\ell' = |P_1 \cap W'| < \ell$ has a score of $\hat{s}(A, W') = ts(\ell') + (k - \ell')(t + 1) = t\ell' + (k - \ell')(t + 1) \leq tk + k$. Thus, $f(A) = \{W \in \mathcal{W}_k : P_1 \subseteq W\}$. However, this contradicts the excellence criterion since $P_1 \subseteq W$ for every $W \in f(A)$ and there is a party $P_j = \{c\}$ with $c \notin W$ and $n_j > n_1$.

For the second case, we suppose that $s(\ell) = \ell - \Delta$ and consider the profile A in which t voters approve the party $P_1 = \{c_1, \dots, c_\ell\}$ and each candidate $c \in \mathcal{C} \setminus P_1$ is uniquely approved by $t - 1$ voters. We compute again the scores of committees $W \in \mathcal{W}_k$: if $P_1 \subseteq W$, then $\hat{s}(A, W) = ts(\ell) + (k - \ell)(t - 1) = t\ell - t\Delta + (k - \ell)(t - 1) < tk - k$, and if $|W' \cap P_1| = \ell - 1$, then $\hat{s}(A, W') = ts(\ell - 1) + (k - \ell + 1)(t - 1) = t(\ell - 1) + (k - \ell + 1)(t - 1) \geq kt - k$. Hence, $P_1 \not\subseteq W$ for all $W \in f(A)$. However, this violates the excellence criterion since for every $W \in f(A)$, there is a party $P_j = \{c\}$ with $P_j \subseteq W$ and $n_j < n_1$. We thus have a contradiction in both cases, so $s(\ell) = \ell$ and f is AV. □

Another frequent goal in committee elections is proportional representation: the chosen committee should proportionally represent the voters' preferences. To this end, we note that if a party P_i with n_i votes gets x_i seats in the chosen committee, then each of the elected candidates in P_i represents on average n_i/x_i voters. Hence, if $n_i/x_i < n_j/x_j{+}1$, then reassigning one seat from party P_i to party P_j intuitively results in a more representative outcome. We will formalize this intuition with a new proportionality notion since we aim to show that SAV is more proportional than AV, but SAV violates all commonly considered proportionality axioms. In more detail, we say that an ABC voting rule f is *party-proportional* if for all party-list profiles A, committees $W \in f(A)$, and parties $P_i, P_j \in \mathcal{P}_A$ with $n_i/|P_i| < n_j/|P_j|$, it holds that $P_i \subseteq W$ implies $P_j \subseteq W$. Intuitively, this axiom states that we can only choose all members of a party if there is no party that represents on average more voters and is not fully chosen yet. Hence, this axiom combines the idea of proportionality with the native behavior of BSAV rules. Even though party-proportionality is a rather weak axiom as it is, e.g., implied by D'Hondt proportionality (Lackner and Skowron 2021b), we show next that this condition characterizes PAV within the class of Thiele rules. This demonstrates that our new axiom is indeed a reasonable and non-trivial proportionality notion.

Proposition 2. *PAV is the only Thiele rule that satisfies party-proportionality.*

Proof Sketch. First, we show that PAV is party-proportional. To this end, let A denote a party-list profile, consider two parties P_i, $P_j \in \mathcal{P}_A$ with $\frac{n_i}{|P_i|} < \frac{n_j}{|P_j|}$, and suppose for contradiction that there is a committee $W \in \text{PAV}(A)$ such that $P_i \subseteq W$, $P_j \not\subseteq W$. In this case, exchanging a candidate $x \in W \cap P_i$ with a candidate $y \in P_j \setminus W$ leads to a committee W' with higher PAV-score than W, which contradicts that $W \in \text{PAV}(A)$. Thus, PAV is party-proportional. For the other direction, we proceed similarly to the proof of Proposition 1 and let f denote a Thiele rule that is party-proportional and s its Thiele scoring function. First, we show that $s(1) > 0$ by the same construction as in the proof of Proposition 1 and rescale s such that $s(1) = 1$. Then, we construct two profiles showing that f fails party-proportionality if $s(\ell) \neq \sum_{x=1}^{\ell} 1/x$ for some $\ell \in \{2, \dots, k\}$. So, f is indeed PAV. □

It is easy to see that SAV satisfies—in contrast to AV—party-proportionality, so SAV is more proportional than AV. Even more, party-proportionality characterizes SAV within the class of BSAV rules when only allowing voters to approve at most k candidates. However, if there is a party P_i with $|P_i| > k$, this is no longer true as not all member of such parties can be elected. We thus introduce another axiom to characterize SAV: an ABC voting rule f satisfies *aversion to single-party committees* if for all party-list profiles A and parties $P_i \in \mathcal{P}_A$, it holds that $W \subseteq P_i$ for all $W \in f(A)$ implies that $\frac{n_i}{|P_i|} > n_j$ for all other parties $P_j \in \mathcal{P}_A$ with $|P_j| = 1$. Intuitively, this axiom is a mild diversity criterion which requires that a single party can only get all seats in the chosen committee if it is approved by a sufficient number of voters when compared to singleton parties. We next characterize SAV based on this this axiom and party-proportionality.

Proposition 3. *SAV is the only BSAV rule that satisfies party-proportionality and aversion to single-party committees.*

Proof Sketch. First, it follows immediately from the definition of SAV that it satisfies party-proportionality and aversion to single-party committees. For the other direction, we consider a BSAV rule f that satisfies the given axioms and let $\alpha \in \mathbb{R}^m_{\geq 0}$ denote its weight vector. From here on, the proof proceeds again just as the one of Proposition 1: we first show that $\alpha_1 > 0$, rescale such that $\alpha_1 = 1$, and then use a similar construction to infer that $\alpha_\ell = \frac{1}{\ell}$ for all $\ell \in \{1, \dots, m\}$. □

Remark 5. We note that PAV fails aversion to single-party committees as the ratio is chosen too restrictive: there are party-list profiles A with a party P_i such that $W \subseteq P_i$ for all $W \in \text{PAV}(A)$ even though $n_j > n_i/|P_i|$ for a singleton party P_j. However, for all such profiles, it holds that $n_i/k > n_j$. In the context of proportional representation, this bound seems more reasonable as it states that each elected member of P_i represents more voters than the single member of P_j. Interestingly, party-proportionality together with a variant of this condition (for all party-list profiles A and parties P_i, it holds that $W \subseteq P_i$ for all $W \in f(A)$ if and only if $n_j < n_i/k$ for all singleton parties P_j) characterize the BSAV rule defined by the weight vector $\alpha_\ell = \max(1/\ell, 1/k)$ for all ℓ. This rule is known as modified satisfaction approval voting (Kilgour and Marshall 2012) and this observation shows that it might be more desirable than SAV.

4 Conclusion

In this paper, we axiomatically characterize two important classes of approval-based committee (ABC) voting rules, namely Thiele rules and BSAV rules. Thiele rules choose the committees that maximize the total score according to a score function that only depends on the intersection size of the considered committee and the ballots of the voters. On the other hand, BSAV rules are a new generalization of multi-winner approval voting which weight voters depending on the size of their ballot. For both of our characterizations, the central axiom is consistency which has famously been used by Young (1975) for a characterization of single-winner scoring rules or by Lackner and Skowron (2021b) for a characterization of ABC scoring rules in the context of committee ranking rules. In particular, our results allow for simple characterizations of all important ABC scoring rules as all such rules belong to one of our classes. We also demonstrate this point by characterizing the well-known ABC voting rules AV, SAV, and PAV. In particular, the result for SAV is, to the best of our knowledge, the first full characterization of this rule. Figure 1 shows a more detailed overview of our results.

Our paper offers several directions for future work. Firstly, our main results allow, of course, to characterize further ABC scoring rules. Secondly, characterizations of many important ABC voting rules (e.g., Phragmén's rule and the method of equal shares) are still missing and some of our ideas might be helpful to derive such results. Finally, even though all relevant ABC scoring rules belong to one of our classes, we would still find a full characterization of the set of ABC scoring rules interesting.

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References

Alcalde-Unzu, J.; and Vorsatz, M. 2009. Size approval voting. *Journal of Economic Theory*, 144(3): 1187–1210.

Arrow, K. J.; Sen, A. K.; and Suzumura, K., eds. 2002. *Handbook of Social Choice and Welfare*, volume 1. North-Holland.

Aziz, H.; Brill, M.; Conitzer, V.; Elkind, E.; Freeman, R.; and Walsh, T. 2017. Justified Representation in Approval-Based Committee Voting. *Social Choice and Welfare*, 48(2): 461–485.

Aziz, H.; Gaspers, S.; Gudmundsson, J.; Mackenzie, S.; Mattei, N.; and Walsh, T. 2015. Computational Aspects of Multi-Winner Approval Voting. In *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 107–115.

Barman, S.; Fawzi, O.; Ghoshal, S.; and Gürpınar, E. 2022. Tight Approximation Bounds for Maximum Multi-Coverage. *Mathematical Programming*, 192(1–2): 443–476.

Brams, S. J.; and Kilgour, D. M. 2014. Satisfaction Approval Voting. In *Voting Power and Procedures*, Studies in Choice and Welfare, 323–346. Springer.

Brandl, F.; Brandt, F.; and Seedig, H. G. 2016. Consistent Probabilistic Social Choice. *Econometrica*, 84(5): 1839–1880.

Brandl, F.; and Peters, D. 2022. Approval Voting under Dichotomous Preferences: A Catalogue of Characterizations. *Journal of Economic Theory*, 205.

Bredereck, R.; Faliszewski, P.; Kaczmarczyk, A.; Knop, D.; and Niedermeier, R. 2020. Parameterized Algorithms for Finding a Collective Sets of Items. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI)*, 1838–1845.

Brill, M.; Laslier, J.-F.; and Skowron, P. 2018. Multiwinner Approval Rules as Apportionment Methods. *Journal of Theoretical Politics*, 30(3): 358–382.

Delemazure, T.; Demeulemeester, T.; Eberl, M.; Israel, J.; and Lederer, P. 2023. Strategyproofness and Proportionality in Party-approval Multiwinner Elections. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI)*, 5591–5599.

Dong, C.; and Lederer, P. 2023a. Characterizations of Sequential Valuation Rules. In *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 1697–1705.

Dong, C.; and Lederer, P. 2023b. Refined Characterizations of Approval-Based Committee Scoring Rules. Technical report, https://arxiv.org/abs/2312.08799.

Dudycz, S.; Manurangsi, P.; Marcinkowski, J.; and Sornat, K. 2020. Tight Approximation for Proportional Approval Voting. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI)*, 276–282.

Elkind, E.; Faliszewski, P.; Skowron, P.; and Slinko, A. 2017. Properties of Multiwinner Voting Rules. *Social Choice and Welfare*, 48: 599–632.

Elkind, E.; and Lackner, M. 2015. Structure in Dichotomous Preferences. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI)*, 2019–2025.

Faliszewski, P.; Skowron, P.; Slinko, A.; and Talmon, N. 2017. Multiwinner Voting: A New Challenge for Social Choice Theory. In Endriss, U., ed., *Trends in Computational Social Choice*, chapter 2.

Faliszewski, P.; Skowron, P.; Slinko, A.; and Talmon, N. 2019. Committee Scoring Rules: Axiomatic Characterization and Hierachy. *ACM Transactions on Economics and Computation*, 7(1): Article 3.

Fishburn, P. C. 1978. Axioms for approval voting: Direct proof. *Journal of Economic Theory*, 19(1): 180–185.

Gawron, G.; and Faliszewski, P. 2022. Using Multiwinner Voting to Search for Movies. In *Proceedings of the 19th European Conference on Multi-Agent Systems (EUMAS)*, Lecture Notes in Computer Science (LNCS), 134–151. Springer-Verlag.

Janson, S. 2016. Phragmén's and Thiele's election methods. Technical Report arXiv:1611.08826 [math.HO], arXiv.org.

Kilgour, D. M.; and Marshall, E. 2012. Approval balloting for fixed-size committees. In *Electoral Systems*, Studies in Choice and Welfare, chapter 12, 305–326. Springer.

Lackner, M.; and Skowron, P. 2018. Approval-Based Multi-Winner Rules and Strategic Voting. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI)*, 340–346.

Lackner, M.; and Skowron, P. 2020. Utilitarian Welfare and Representation Guarantees of Approval-based Multiwinner Rules. *Artificial Intelligence*, 288: 103366.

Lackner, M.; and Skowron, P. 2021a. Axiomatic characterizations of consistent approval-based committee choice rules. Technical report, https://arxiv.org/abs/2112.10407v1.

Lackner, M.; and Skowron, P. 2021b. Consistent Approval-Based Multi-Winner Rules. *Journal of Economic Theory*, 192: 105173.

Lackner, M.; and Skowron, P. 2023. *Multi-Winner Voting with Approval Preferences*. Springer-Verlag.

Myerson, R. B. 1995. Axiomatic derivation of scoring rules without the ordering assumption. *Social Choice and Welfare*, 12(1): 59–74.

Peters, D. 2018. Single-Peakedness and Total Unimodularity: New Polynomial-Time Algorithms for Multi-Winner Elections. In *Proceedings of the 32th AAAI Conference on Artificial Intelligence (AAAI)*, 1169–1176.

Pivato, M. 2013. Variable-population voting rules. *Journal of Mathematical Economics*, 49(3): 210–221.

Sánchez-Fernández, L.; and Fisteus, J. A. 2019. Monotonicity axioms in approval-based multi-winner voting rules. In *Proceedings of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 485–493.

Skowron, P.; Faliszewski, P.; and Lang, J. 2016. Finding a Collective Set of Items: From Proportional Multirepresentation to Group Recommendation. *Artificial Intelligence*, 241: 191–216.

Skowron, P.; Faliszewski, P.; and Slinko, A. 2019. Axiomatic characterization of committee scoring rules. *Journal of Economic Theory*, 180: 244–273.

Smith, J. H. 1973. Aggregation of Preferences with Variable Electorate. *Econometrica*, 41(6): 1027–1041.

Thiele, T. N. 1895. Om Flerfoldsvalg. *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger*, 415–441.

Young, H. P. 1975. Social Choice Scoring Functions. *SIAM Journal on Applied Mathematics*, 28(4): 824–838.

Young, H. P.; and Levenglick, A. 1978. A Consistent Extension of Condorcet's Election Principle. *SIAM Journal on Applied Mathematics*, 35(2): 285–300.