How to Cut a Discrete Cake Fairly

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Abstract

Cake-cutting is a fundamental model of dividing a heterogeneous resource, such as land, broadcast time, and advertisement space. In this study, we consider the problem of dividing a discrete cake fairly in which the indivisible goods are aligned on a path and agents are interested in receiving a connected subset of items. We prove that a connected division of indivisible items satisfying a discrete counterpart of envy-freeness, called *envy-freeness up to one good* (EF1), always exists for any number of agents n with monotone valuations. Our result settles an open question raised by Bilò et al. (2019), who proved that an EF1 connected division always exists for the number of agents $n \leqslant 4$. Moreover, the proof can be extended to show the following (1) "secretive" and (2) "extra" versions: (1) for n agents with monotone valuations, the path can be divided into n connected bundles such that an EF1 assignment of the remaining bundles can be made to the other agents for any selection made by the "secretive agent"; (2) for $n + 1$ agents with monotone valuations, the path can be divided into n connected bundles such that when any "extra agent" leaves, an EF1 assignment of the bundles can be made to the remaining agents.

1 Introduction

Imagine a group of researchers scheduling time slots for meetings. Their preferences may be heterogeneous: for example, one researcher may prefer a morning meeting, whereas another may prefer an afternoon meeting. This situation raises the question: how can we allocate time slots fairly? This problem falls within the field of the well-known cake-cutting problem, in which the *cake*, often represented by the unit interval $[0, 1]$, has to be divided between n agents with different preferences. Here, the term "cake" is a metaphor for a heterogeneous divisible resources, such as land or time.

A central notion of fairness in the literature is *envy-freeness* (Foley 1967), which requires that each agent receives their personal best piece out of the allocated pieces. In this scenario, no agent wishes to replace their allocated portion with that of any other agent. The classical result shows that under mild assumptions on the agents' preferences, there is an envy-free division offering each agent a connected piece (Stromquist 1980; Su 1999; Woodall 1980). Note that the connectivity constraint is crucial in various contexts, particularly when the resource has temporal or spacial structure.

In many application domains, the resource may be indivisible. For example, time is often divided into discrete time units, such as scheduled shifts and research seminars. As another example, land may be divided into discrete land plots, due to geographical or historical constraints. A discrete version of the cake-cutting problem has been considered in several papers recently (Bilò et al. 2019; Bouveret et al. 2017; Marenco and Tetzlaff 2014; Suksompong 2019). In this framework, the indivisible items are aligned on a path and each agent is allocated to a connected bundle of items.

In allocation of indivisible resources, envy-freeness is not guaranteed. Indeed, in the case of one item and two agents, one agent necessarily receives nothing and therefore envies the other. Nevertheless, the objective can be naturally relaxed via approximations. An approximate notion of envy-freeness, called *envy-freeness up to one good* (EF1), has been intensively investigated in recent years (Budish 2011). EF1 allows agents to envy other agents but the envy can be eliminated after removing one item from others' bundles. Several algorithms achieve EF1 within the standard setting of fair division of indivisible items (Lipton et al. 2004; Caragiannis et al. 2016). For agents with monotone valuations, the envy-cycle algorithm in Lipton et al. (2004) returns an EF1 division. For agents with monotone additive valuations, an allocation maximizing the Nash product of agents' valuations satisfies EF1 (Caragiannis et al. 2016). Can we achieve EF1 under the connectivity constraint of a path?

This question was partially answered by Bilò et al. (2019). They showed that an EF1 connected division exists for any monotone valuations when there are at most four agents. They followed Su's approach (Su 1999) using Sperner's lemma: The possible divisions of the path can be encoded by the vertices of a triangulated $(n - 1)$-dimensional simplex; see Figure 1 for an illustration with $n = 3$. Each agent colors each vertex with the index of the most preferred bundle of the partition represented by each vertex. Sperner's lemma then implies the existence of a small simplex labeled with distinct agents and colored with different indices of bundles. Loosely speaking, this simplex corresponds to a sequence of "similar" divisions, each of which satisfies different agents

with different bundles. Bilò et al. (2019) developed a rounding technique of such a simplex that works for four or fewer agents. However, the existence of EF1 is unclarified when the number of agents n is more than four, while a connected division satisfying the weaker fairness notion of EF2, requiring the envy to be bounded up to two items, exists for any number of agents (Bilò et al. 2019). A key issue in this proof lies in the left-right *symmetry* in the agents' evaluation. Since agents treat left-most and right-most items of a bundle symmetrically, they may face a conflict with another agent who want neighboring bundles. See Section 3.1 for details.

To settle this open question, we show that an EF1 connected division exists for any number of agents with monotone valuations. Our proof adopts the rounding technique of a simplex, similar to that developed in Bilò et al. (2019), but our final path division has left-right *asymmetry*. Namely, our rounding algorithm prioritizes an agent who wants a left bundle over an agent who wants a right bundle on the simplex. In our proof, each agent is pessimistic about obtaining the left-most item and optimistic about obtaining the right-most item, which ensures that the estimate of each agent i on the j-th bundle of the final output is neither overly optimistic nor overly pessimistic. In this way, we can successfully circumvent the difficulty arising when $n \geq 5$. In fact, we show the existence of a connected division satisfying a slightly stronger notion of EF1$_{outer}$, which additionally requires that the envied bundles remain connected after removing an item.[1]

Exploiting the proof of this theorem, we further obtain the discrete analogs of the existential result concerning a secretive envy-free divisions of the cake (Woodall 1980; Asada et al. 2018; Meunier and Su 2019): for any number n of agents with monotone valuations over a path, one can divide the path into n parts such that whichever part a secretive agent chooses, an EF1$_{outer}$ assignment of the remaining bundles can be made to the other agents.

For two agents, such existence directly follows from a discrete version of the cut-and-choose protocol over the path: the first agent computes an EF1$_{outer}$ connected division among two agents as if the other agent has the same valuation, and then the second agent (called the *secretive agent*) selects a preferred bundle, leaving the remainder for the first agent. Here, the valuation of one agent is sufficient to find a partition such that whichever part another agent chooses, the resulting assignment is EF1$_{outer}$. More generally, our result shows that the valuations of $n - 1$ agents suffice to find an EF1$_{outer}$ connected division among n agents. Note that without connectivity constraints, a secretive EF1 division is known to exist and can be computed in polynomial time when the agents have monotone valuations (Arunachaleswaran, Barman, and Rathi 2019). However, our result is the first to show that the existential result holds in conjunction with connectivity requirements.

Finally, we show the dual statement that for $n + 1$ agents with monotone valuations, the path can be partitioned into n connected subsets such that when any extra agent leaves, an

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[1]The result of Bilò et al. (2019) also holds with EF2$_{outer}$ for any number of agents with monotone valuations.
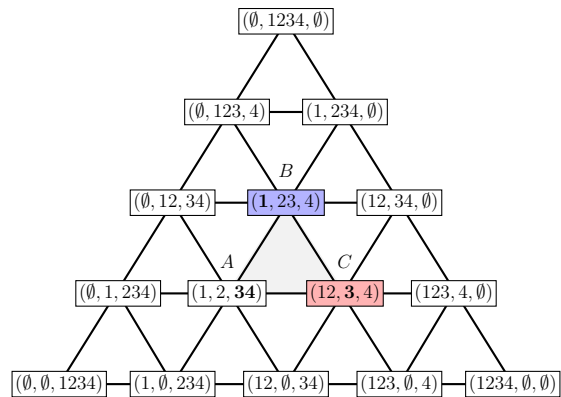

Figure 1: Illustration of a simplex whose vertices represent divisions of a path with four vertices into three bundles. In the small triangle colored in light gray, agent A most-prefers the third bundle at the left corner vertex, B most-prefers the first bundle at the middle corner vertex, and C most-prefers the second bundle at the right corner vertex.

EF1$_{outer}$ assignment of the bundles can be made to the remaining agents, thereby establishing the discrete counterpart of the existence of an extra envy-free cake division, recently shown by Meunier and Su (2019).

An important application of our results is that for graph fair division proposed by Bouveret et al. (2017). This setting captures, e.g., the division of road networks, where the items can be aligned on a graph and the agents value connected bundles of the items. Our results on a path apply to a wider class of *traceable graphs* that admit a Hamiltonian path. Indeed, for such graphs, one can take a Hamiltonian path of the original graph and apply our result to obtain an EF1$_{outer}$ connected division of the Hamiltonian path; the resulting division is trivially both EF1$_{outer}$ and connected in the original graph.[2] Thus, for any number of agents with monotone valuations, an EF1$_{outer}$ connected division of a graph exists whenever the graph is traceable.

Related work To the best of our knowledge, the problem of dividing a discrete cake, i.e., a path, has been first considered in (Marenco and Tetzlaff 2014; Suksompong 2019; Bárány and Grinberg 2015). Marenco and Tetzlaff (2014) studied a special valuation in which each item is liked by exactly one agent and showed that an envy-free division exists for such valuations. Suksompong (2019) considered approximation of envy-freeness, showing that a simple rounding of an envy-free division gives us a division such that the envy is bounded by at most $2v_{max}$ for agents with additive valuations, where v_{max} is the maximum value of the agents for the single items. A result similar to this has been obtained in Bárány and Grinberg (2015). Bouveret et al. (2017) proposed a model for allocating indivisible goods under connectivity constraints of a graph. The graph fair division has attracted a great deal of attention since then (Bouveret, Cechlárová, and Lesca 2019; Igarashi and Peters 2019; Bei et al. 2021; Truszczynski and Lonc 2020; Bilò et al. 2019;

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[2]This observation has already been made in Bilò et al. (2019).

Bouveret et al. 2017; Greco and Scarcello 2020; Deligkas et al. 2021). Bilò et al. (2019) developed several methods to obtain an EF1 division under connectivity constraints of a path when the number of agents is two, three, or four, or when the agents have identical monotone valuations.

2 Preliminaries

For each natural number $s \in \mathbb{N}$, we write $[s] = \{1, 2, \ldots, s\}$. For each pair of natural numbers $s, t \in \mathbb{N}$ with $s \leqslant t$, we write $[s, t] = \{s, s+1, \ldots, t\}$. We are given n *agents* and m *items* (or *goods*). We may refer to subsets of items as *bundles*. The items are aligned along a path $(1, 2, \ldots, m)$. Each agent i has a *valuation function* v_i that assigns a real value to every connected subset of the path. For two connected subsets S and T, an agent i *weakly prefers* (resp. *strictly prefers*) S to T if $v_i(S) \geq v_i(T)$ (resp. $v_i(S) > v_i(T)$). We assume that all agents have *monotone valuations*, i.e., each agent i weakly prefers T to S whenever $S \subseteq T$ and that $v_i(\emptyset) = 0$ for each $i \in N$.

A *division* is a partition $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ of the path into n connected bundles, where I_i is the ith bundle from the left. A division $\mathcal{I}$ is *envy-free* if there exists a permutation $\pi \colon [n] \to [n]$ such that $v_i(I_{\pi(i)}) \geqslant v_i(I_{\pi(j)})$ for any pair i, j of agents.

An envy-free division is not guaranteed when the items are indivisible. For instance, when two agents desire one item, one agent gets the item, whereas the other gets nothing. Thus, Budish (2011) relaxed the envy-freeness condition to *envy-freeness up to one good* (EF1). In an EF1 division, an agent can envy another agent, but envy will disappear after one item is removed from the other's bundle. We adopt a slightly more robust version of EF1, introduced in Bilò et al. (2019), requiring that removing the items leave the envied bundles connected.

Definition 2.1 (EF1$_{outer}$: envy-freeness up to one *outer* good). A division $\mathcal{I}$ satisfies *EF1$_{outer}$* if there exists a permutation $\pi \colon [n] \to [n]$ such that for any pair i, j of agents, $v_i(I_{\pi(i)}) \geqslant v_i(I_{\pi(j)})$, or there exists a good $g \in I_{\pi(j)}$ such that $I_{\pi(j)} \setminus \{g\}$ is connected[3] and $v_i(I_{\pi(i)}) \geqslant v_i(I_{\pi(j)} \setminus \{g\})$.

In our context, EF1$_{outer}$ is fairer than EF1. In particular, EF1 may not be binding at all when nonconnected subsets are less preferred compared with connected subsets or even undesirable; for instance, when allocating time slots for certain tasks among multiple employees, people often value being allocated a contiguous chuck of time, instead of being allocated to a disconnected one.

We introduce the following notation in Bilò et al. (2019). For every connected subset I, we define the *up-to-one valuation* v_i^- of agent i as

$$v_i^-(I) := \begin{cases} 0 & \text{if } I = \emptyset, \\ \min \{v_i(I \setminus \{g\}) : \\ g \in I \text{ such that } I \setminus \{g\} \text{ is connected}\} & \text{if } I \neq \emptyset. \end{cases}$$

Clearly, a division $\mathcal{I}$ satisfies EF1$_{outer}$ if and only if there exists a permutation $\pi \colon [n] \to [n]$ such that $v_i(I_{\pi(i)}) \geqslant v_i^-(I_{\pi(j)})$ for any pair i, j of agents.

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[3] We consider the empty set to be connected.

Throughout this paper, we assume that the number of items exceeds the number of agents, i.e., $m \geqslant n$. Note that if $m < n$, a trivial EF1$_{outer}$ division exists.

2.1 Sperner's Lemma and Envy-free Divisions

We review basic notions of combinatorial topology and explain the link between Sperner's lemma and the classical cake cutting problem. An $(n-1)$-simplex S is the convex hull of n *main vertices* $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$; we write $S = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$. For $j \in [n]$, we write $\boldsymbol{e}^j \in \{0, 1\}^n$ as the j-th unit vector, where $e_h^j = 1$ if $h = j$ and $e_h^j = 0$ otherwise. The $(n-1)$ standard simplex Δ^{n-1} is the $(n-1)$-simplex whose main vertices are given by $\boldsymbol{e}^1, \boldsymbol{e}^2, \ldots, \boldsymbol{e}^n$. A *triangulation* T of an $(n-1)$-simplex S is a collection of smaller $n-1$ simplices $S_1, S_2, \ldots, S_k$ where $S = \bigcup_{j=1}^{k} S_j$, and for each pair of distinct indices $i, j \in [k]$, the intersection $S_i \cap S_j$ is either empty or a face common to them. We refer to $S_1, S_2, \ldots, S_k$ as *elementary* simplices. We denote by $V(\mathsf{T})$ the set of vertices of a triangulation T.

Given a triangulation T of an $(n-1)$-simplex S, a *coloring* is a function $\lambda \colon V(\mathsf{T}) \to 2^{[n]}$ that assigns to each vertex $\boldsymbol{x} \in V(\mathsf{T})$ a subset $\lambda(\boldsymbol{x}) \subseteq [n]$, where each element of $[n]$ is (called) a *color*. A coloring $\lambda \colon V(\mathsf{T}) \to 2^{[n]}$ is (called) *proper* if we can write $S = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$ such that if a vertex $\boldsymbol{x} \in V(\mathsf{T})$ is colored by index j, i.e., $j \in \lambda(\boldsymbol{x})$, then $\boldsymbol{x}_j$ is a vertex of a minimal face containing $\boldsymbol{x}$.

A *fully-colored* elementary simplex $S^* = \langle \boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_n^* \rangle$ has a complete set of colors, i.e., there exists a permutation $\pi \colon [n] \to [n]$ such that $\pi(i) \in \lambda(\boldsymbol{x}_i^*)$ for each $i \in [n]$. Sperner's lemma states that a triangulated simplex with a proper coloring admits a fully-colored elementary simplex.

Theorem 2.2 (Sperner's lemma). *Any triangulation T of an $(n-1)$-simplex with a proper coloring $\lambda \colon V(\mathsf{T}) \to 2^{[n]}$ admits a fully-colored elementary simplex.*

Su (1999) was the first to demonstrate the usefulness of Sperner's lemma in the context of cake-cutting, citing Simmons as the one who conceptualized the proof. The Simmons–Su method encoded possible divisions of the cake $[0, 1]$ by the points of an $(n-1)$ standard simplex where the i-th coordinate can be interpreted as the i-th knife position, obtaining an envy-free division based on a *labeling* and *coloring* of the $(n-1)$ standard simplex as follows.

First, we assign an *agent label* to each vertex so that the vertices of each elementary simplex have n distinct agent labels. Formally, an *owner labeling* of the triangulation T of an $(n-1)$-simplex is a function $a \colon V(\mathsf{T}) \to [n]$ such that for each pair of distinct vertices $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ of S with $i \neq j$, $a(\boldsymbol{x}_i) \neq a(\boldsymbol{x}_j)$, where each $a(\boldsymbol{x})$ is called the *owner* of a vertex. There is a triangulation of the simplex that does not admit an owner labeling; see, e.g., Figure 1 in Deng, Qi, and Saberi (2012). Nevertheless, some triangulations, e.g., Kuhn's triangulation and the barycentric triangulation, do admit owner labelings, while allowing small mesh size (Su 1999; Deng, Qi, and Saberi 2012).

Second, since the vertices of the triangulation correspond to divisions of the cake, we go to each vertex of the triangu-
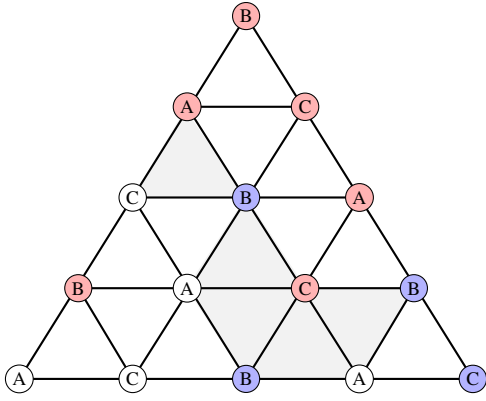
Figure 2: Illustration of the labeling-coloring proof when there are three agents A, B, C. The blue, red, white vertices correspond to the divisions where the first, second, and third pieces are the most favorite pieces of an owner agent, respectively. The three elementary simplices colored in light gray correspond to fully-colored simplices.

lation and ask the owner to *color* that vertex with the indices of the most preferred piece of the corresponding division. An illustration of a triangulation and the labeling-coloring approach is given in Figure 2. Such a coloring is proper if each agent never chooses the piece of zero-length. For example, when $n = 3$, the corner vertices have distinct colors as they correspond to divisions in which one piece includes the entire cake; further, the vertices on each side correspond to divisions in which one piece is empty, thus missing one color that corresponds to the empty piece. By Sperner's lemma, this construction yields a fully-colored elementary simplex S^*. Now, observe that in S^*, each agent points out a different piece of her owned division as a most preferred piece. If the simplex vertices are close enough, it corresponds to an approximate solution that converges to an envy-free contiguous division for a sequence of finer triangulations. In the next section, we will adopt the Simmons–Su method to the setting of discrete cake-cutting.

3   Existence of Connected EF1 Divisions

In this section, we prove the the main result of this paper.

Theorem 3.1. *For any number of agents with monotone valuation functions on a path, a connected EF1$_{outer}$ division exists.*

Let us first illustrate potential approaches to prove the existence of a connected EF1$_{outer}$ division. We will then proceed to identifying some of the problems with these approaches, and explain how we overcome them.

3.1   Potential Approach

One possible approach to prove Theorem 3.1 is to treat a path $(1, 2, \ldots, m)$ as an interval $[0, m]$, extend valuation functions to continuous ones, and apply some rounding of an envy-free connected division of the cake, which is guaranteed to exist (Su 1999; Woodall 1980; Stromquist 1980). This approach was also investigated by Suksompong (2019);

however, a division obtained via this approach satisfies a weaker fairness property. In particular, it can lead agents to steal an item instead of removing it from another agent's bundle to eliminate envy; see Section 5 of Bilò et al. (2019) for details.

Instead, Bilò et al. (2019) followed the approach of Su (1999) that directly works on the simplex and rounds an elementary simplex, corresponding to a series of divisions. Like Su (1999), they encoded possible configurations of the $n - 1$ knives as the vertices of a triangulated simplex. Bilò et al. observed that if the configuration space only considers fully integral divisions, i.e., if knives move from one edge to another edge, the divisions corresponding to the vertices in each elementary simplex are too far apart from each other to ensure EF1: in their final rounding, one agent may get one additional item together with her desired bundle while another may lose one item, which makes it difficult to bound the envy up to one item. Thus, they consider a finer triangulation, allowing knives to move at both edges and vertices of a path. More precisely, they consider the following simplex:

$$S_m := \left\{ \boldsymbol{x} \in \mathbb{R}_+^{n-1} \,\middle|\, \frac{1}{2} \leqslant x^1 \leqslant x^2 \leqslant \cdots \leqslant x^{n-1} \leqslant m + \frac{1}{2} \right\},$$

and use a *Kuhn's triangulation* $\mathsf{T}_{\mathrm{half}}$ of S_m (Deng, Qi, and Saberi 2012) where the vertices $V(\mathsf{T}_{\mathrm{half}})$ are given by

$$V(\mathsf{T}_{\mathrm{half}}) = \left\{ \boldsymbol{x} \in \mathbb{R}_+^{n-1} \,\middle|\, x^i \in \{\frac{1}{2}, 1, \ldots, m + \frac{1}{2}\}, \forall i \in [n] \right\}$$

and it satisfies the property that each elementary simplex $S = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$ of $\mathsf{T}_{\mathrm{half}}$ is balanced, meaning that there exists a permutation $\phi \colon [n] \to [n]$ such that

$$\boldsymbol{x}_{\phi(i+1)} = \boldsymbol{x}_{\phi(i)} + \frac{1}{2}\boldsymbol{e}^{\phi(i)}, \text{ for each } i \in [n-1]. \quad (1)$$

The above property ensures that the vertices of the elementary simplex can be arranged in such a way that every distinct knife in this sequence moves always in half-step and in the same direction and such a movement happens exactly once over the sequence.

For each vertex $\boldsymbol{x} = (x^1, x^2, \ldots, x^{n-1}) \in V(\mathsf{T}_{\mathrm{half}})$, we call each x^j the j-th *knife position*; item $y \in \{1, 2, \ldots, m\}$ is *hidden* by a knife x^j if $y = x^j$. Here, $V(\mathsf{T}_{\mathrm{half}})$ encodes all the possible configurations $(x^1, x^2, \ldots, x^{n-1})$ of the $n - 1$ knives that move in half-steps. Specifically, each vertex $\boldsymbol{x} \in V(\mathsf{T}_{\mathrm{half}})$ induces a partial division $\mathcal{I}(\boldsymbol{x}) = (I_1(\boldsymbol{x}), I_2(\boldsymbol{x}), \ldots, I_n(\boldsymbol{x}))$ where each j-th bundle for $j \in [n]$ is given by

$$I_j(\boldsymbol{x}) = \{y \in \{1, 2, \ldots, m\} \mid x^{j-1} < y < x^j\},$$

setting $x^0 = \frac{1}{2}$ and $x^n = m + \frac{1}{2}$. For example, when $m = 12$, $n = 5$, and $\boldsymbol{x} = (3, \frac{9}{2}, 8, \frac{21}{2})$, $\mathcal{I}(\boldsymbol{x})$ corresponds to the partial division $(\{1, 2\}, \{4\}, \{5, 6, 7\}, \{9, 10\}, \{11, 12\})$. Items 3 and 8 are hidden by the two knives x^2 and x^3 and the other items are uncovered at $\boldsymbol{x}$. For each $\boldsymbol{x} \in V(\mathsf{T}_{\mathrm{half}})$ and each $j \in [n]$, we say that $\ell_j(\boldsymbol{x}) := \lfloor x^{j-1} + \frac{1}{2} \rfloor$ is the *left-most boundary item* of $I_j(\boldsymbol{x})$; similarly, we say that $r_j(\boldsymbol{x}) := \lceil x^j - \frac{1}{2} \rceil$ is the *right-most boundary item* of $I_j(\boldsymbol{x})$. Note that if $\ell_j(\boldsymbol{x})$ is hidden by the $(j-1)$-th knife, it is the
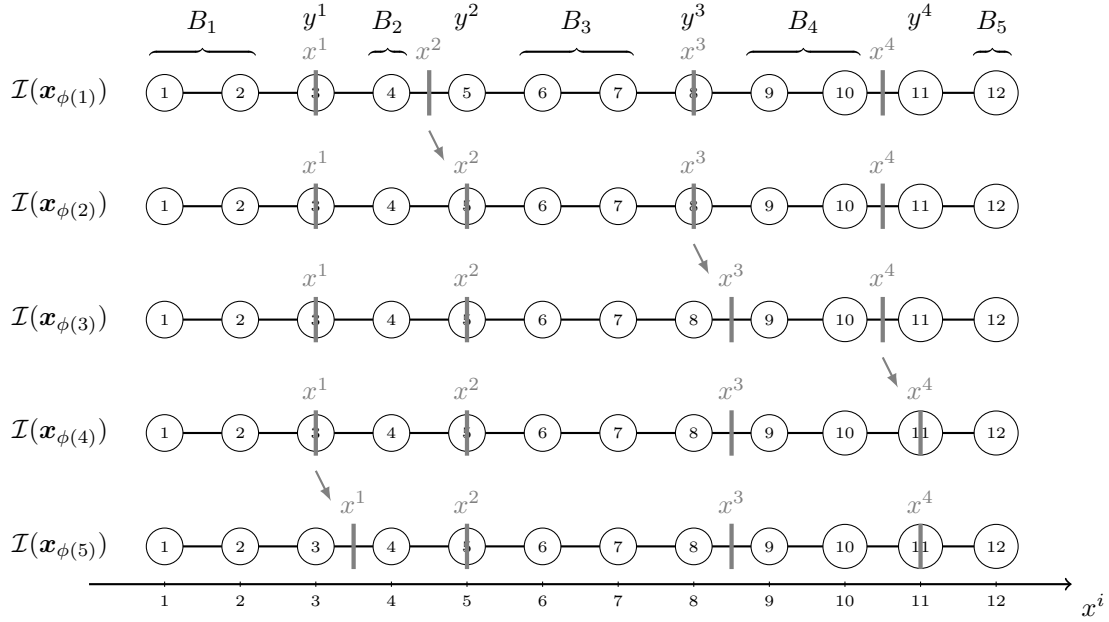
Figure 3: Example of a sequence of partial divisions represented by an elementary simplex of $\mathsf{T}_{\mathrm{half}}$. Each knife moves in half-step and the movement happens only once over the sequence.
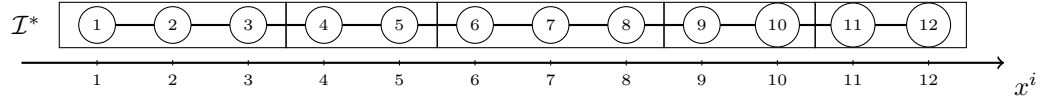

Figure 4: Division returned by Algorithm 1 for an elementary simplex whose vertices correspond to the partial divisions of Figure 3. Note that the second bundle I_2^* receives the boundary item 5 because it satisfies the condition (b) of Line 9 while the fourth bundle I_4^* does not receive item 11 because it violates the condition (b) of Line 9.

boundary item between $I_{j-1}(\boldsymbol{x})$ and $I_j(\boldsymbol{x})$ and if not, it is the leftmost item of $I_j(\boldsymbol{x})$; similarly, if $r_j(\boldsymbol{x})$ is hidden by the j-th knife, it is the boundary item between $I_j(\boldsymbol{x})$ and $I_{j+1}(\boldsymbol{x})$, and if not, it is the rightmost item of $I_j(\boldsymbol{x})$. We say that item $y \in \{1, 2, \ldots, m\}$ *fully appears* (or, is *fully visible*) in $I_j(\boldsymbol{x})$ if $y \in I_j(\boldsymbol{x})$.

How does each agent evaluate each of the partial divisions $\mathcal{I}(\boldsymbol{x})$? Bilò et al. introduced the following *virtual* valuations: for a bundle $I_j(\boldsymbol{x})$ in which at least one of the boundary items is fully visible, agents expect to obtain only those items that are fully visible in the bundle; for other bundles $I_j(\boldsymbol{x})$ in which no boundary item fully appears, agents expect to obtain the items that are fully visible in the bundle as well as at least one of the boundary items (choose one that is less valuable). Agents then color each of the vertices with the index of the favorite bundles based on virtual valuations. This coloring can be shown to be proper; thus, using a more general version of Sperner's lemma considering n colorings, we get a fully-colored simplex S^* that is fully-labeled, meaning that S^* receives different agent labels that like different bundles best.[4]

Note that each vertex in S^* only induces a partial division. How can we use S^* to obtain a full division? Bilò et al. showed that S^* can be rounded to yield an $\mathrm{EF1}_{outer}$ connected division for four or fewer agents as follows. First of all, recall that for each elementary simplex $S = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$ of $\mathsf{T}_{\mathrm{half}}$, there exists a permutation $\phi : [n] \to [n]$ satisfying (1). Thus, S corresponds to a sequence of partial divisions $(\mathcal{I}(\boldsymbol{x}_{\phi(1)}), \mathcal{I}(\boldsymbol{x}_{\phi(2)}), \ldots, \mathcal{I}(\boldsymbol{x}_{\phi(n)}))$, where each partial division $\mathcal{I}(\boldsymbol{x}_{\phi(k)})$ is obtained from $\mathcal{I}(\boldsymbol{x}_{\phi(k-1)})$ by moving one of the knives in half-step and the movement of each knife occurs only once across n partial divisions. An example of such sequences of partial divisions is shown in Figure 3. Using this, our path can be divided into $(B_1, y^1, B_2, \ldots, y^{n-1}, B_n)$ where each B_j is the set of items that fully appear in the j-th bundle of all partial divisions, and each y^j is the boundary item that can appear in both of the j-th and the $(j+1)$-th bundles over the sequence.

Building $(B_1, y^1, B_2, \ldots, y^{n-1}, B_n)$ for S^*, a division $\mathcal{I}^* = (I_1^*, I_2^*, I_3^*, I_4^*)$ for four agents can be constructed as follows:

1. Each I_j^* of interior bundles with $j \in \{2,3\}$ consists of B_j together with the boundary items y^{j-1} or y^j fully appearing in the j-th bundle of some partial division represented by the vertices in S^*. If none of the boundary items y^{j-1} or y^j fully appears in the j-th bundle, I_j^* additionally gets one boundary item that is adjacent to an exterior bundle I_j^* with $j \in \{1,4\}$.

2. Each exterior bundle I_1^* (respectively, I_4^*) consists of B_1 together with y_1 (respectively, y_3) if y_1 (respectively, y_3) is not allocated yet.

With this approach, Bilò et al. established an EF1_{outer} connected division for four or fewer agents. However, the four-agent proof in Bilò et al. does not extend to the general case. Their proof requires that each interior bundle receives an item traversed by a knife due to the left-right symmetry in their valuations. This requirement is met by four agents as the interior bundles (i.e., the second and third bundles) are adjacent to some exterior bundle (i.e., the first and fourth bundles). When the number of agents is five or more, any division includes a bundle that is not adjacent to any of exterior bundles.

3.2 Our Approach

Following the discretization approach developed in Bilò et al. (2019), we prove that an EF1 connected division exists for any number of agents with monotone valuations. When extending the result beyond four agents, the main difficulty is to find appropriate ways to evaluate bundles of partial divisions and to round the half-integral simplices. To this end, we create a left-right asymmetry both in the evaluation phase and in partial-division rounding. Intuitively, our rounding algorithm, formalized in Algorithm 1, prioritizes a left agent (i.e., an agent who is assigned to j-th bundle) over a right agent (i.e., an agent who is assigned to $(j+1)$-th bundle) when allocating each boundary item y^j. Our definition of virtual valuations allows us to do such rounding since each agent is pessimistic about obtaining the left-most boundary item and optimistic about obtaining the right-most boundary item. This then guarantees that the value of each agent's allocated bundle in the final division $\mathcal{I}^*$ is at least the value of the favorite bundle in their owned division, which is at least the value of another bundle in $\mathcal{I}^*$ after the removal of one outer-item. Below, we prove that our technique successfully ensures that the final division is EF1_{outer}.

Our proof is divided into the following two steps of coloring and rounding: first, we assign to each vertex an owner labeling and a color according to the preferences of owners; second, we round a fully-colored simplex into a full division. See Figure 2 for an illustration. We use the same simplex S_m and triangulation $V(\mathsf{T}_{\text{half}})$ as previously described.

Coloring We define the virtual valuation $\hat{v}_i(\boldsymbol{x}, j)$ of each vertex $\boldsymbol{x}$ of the triangulation. The virtual valuation determines how each owner agent assigns a color to his or her owned vertex. First, an agent who obtains the left exterior bundle $j = 1$ expects to obtain the items not hidden by the right-most knife:

$$\hat{v}_i(\boldsymbol{x}, 1) = v_i(I_1(\boldsymbol{x})).$$

For the interior bundles $2 \leqslant j \leqslant n - 1$, we set $\hat{v}_i(\boldsymbol{x}, j) = 0$ if $I_j(\boldsymbol{x}) = \emptyset$. Otherwise, the value $\hat{v}_i(\boldsymbol{x}, j)$ is given as follows:

$$\hat{v}_i(\boldsymbol{x}, j) = \begin{cases} v_i^-(I_j(\boldsymbol{x}) \cup \{\ell_j(\boldsymbol{x}), r_j(\boldsymbol{x})\}) \\ \quad \text{if } \ell_j(\boldsymbol{x}) \notin I_j(\boldsymbol{x}) \text{ and } r_j(\boldsymbol{x}) \notin I_j(\boldsymbol{x}), \\ v_i(I_j(\boldsymbol{x}) \setminus \{\ell_j(\boldsymbol{x})\}) \\ \quad \text{if } \ell_j(\boldsymbol{x}) \in I_j(\boldsymbol{x}) \text{ and } r_j(\boldsymbol{x}) \in I_j(\boldsymbol{x}), \\ v_i^-(I_j(\boldsymbol{x}) \cup \{r_j(\boldsymbol{x})\}) \\ \quad \text{if } \ell_j(\boldsymbol{x}) \in I_j(\boldsymbol{x}) \text{ and } r_j(\boldsymbol{x}) \notin I_j(\boldsymbol{x}), \\ v_i(I_j(\boldsymbol{x})) \\ \quad \text{if } \ell_j(\boldsymbol{x}) \notin I_j(\boldsymbol{x}) \text{ and } r_j(\boldsymbol{x}) \in I_j(\boldsymbol{x}). \end{cases}$$

For the right exterior bundle $j = n$, the item $\ell_n(\boldsymbol{x})$ is not expected in the final bundle I_j^*:

$$\hat{v}_i(\boldsymbol{x}, n) = v_i(I_n(\boldsymbol{x}) \setminus \{\ell_n(\boldsymbol{x})\})$$

Based on these virtual valuations, we define coloring functions $\lambda_i \colon V(\mathsf{T}_{\text{half}}) \to 2^{[n]}$ for each agent $i \in [n]$ where

$$\lambda_i(\boldsymbol{x}) = \operatorname{argmax}\{\, \hat{v}_i(\boldsymbol{x}, j) \mid j \in [n] \text{ such that } I_j(\boldsymbol{x}) \neq \emptyset \,\}.$$

It is not difficult to see that these colorings are proper. It is known that Kuhn's triangulation admits an owner labeling $a \colon V(\mathsf{T}_{\text{half}}) \to [n]$ where each elementary simplex has distinct owner labels (Deng, Qi, and Saberi 2012). We aggregate the colorings according to the preference of each owner as follows:

$$\lambda(\boldsymbol{x}) = \lambda_{a(\boldsymbol{x})}(\boldsymbol{x}).$$

Because each coloring λ_i is proper, so is λ.

Rounding Next, we present how to round each elementary simplex $S = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$ of T_{half} into a division $\mathcal{I}^* = (I_1^*, I_2^*, \ldots, I_n^*)$ as specified in Algorithm 1. Initially, in Line 2, each I_j^* is allocated to the set B_j of items that fully appear in all the j-th bundles $I_j(\boldsymbol{x}_1), \ldots, I_j(\boldsymbol{x}_n)$ of partial partitions represented by the elementary simplex S. In Lines $3 - 10$, Algorithm 1 allocates, from left to right, each boundary item y^j that appears in both jth and $j+1$st bundles on the sequence $I_j(\boldsymbol{x}_1), \ldots, I_j(\boldsymbol{x}_n)$. Specifically, we use the following left-right asymmetric rounding (Line 9): each bundle I_j^* obtains item y^j if

(a) y^j fully appears in some of the j-th bundles $I_j(\boldsymbol{x}_1), \ldots, I_j(\boldsymbol{x}_n)$, or

(b) I_j^* does not obtain item y^{j-1} in the previous step and none of the $I_j(\boldsymbol{x}_k)$ coincides with B_j for $k \in [n]$.

In this way, each interior bundle I_j^* must receive at least one of its adjacent boundary items, except when $I_j(\boldsymbol{x}_k)$ coincides with B_j for some $k \in [n]$. See Figure 4 for an example of the division returned by Algorithm 1.

In Lemma 3.2, we show that the estimate of each agent i on the j-th bundle of the output is neither overly optimistic nor overly pessimistic when applying Algorithm 1 to any elementary simplex S of T_{half}. The case distinction in the proof of Lemma 3.2 considers whether the bundle is an interior or exterior one, whether the bundle receives two boundary items, exactly one boundary item, or none of them, and

Algorithm 1: Rounding into a division

Input: an elementary simplex $S = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$
Output: a division $\mathcal{I}^* = (I_1^*, I_2^*, \ldots, I_n^*)$

1 for $j \in [n]$, let $B_j = I_j(\boldsymbol{x}_1) \cap I_j(\boldsymbol{x}_2) \cap \cdots \cap I_j(\boldsymbol{x}_n)$
   and let y^j be the item such that $y^j = x_k^j$ for some
   $k \in [n]$;
2 initialize $I_j^* \leftarrow B_j$ for each $j = 1, 2, \ldots, n$;
3 **if** y^1 *fully appears in the first bundle of some*
   division, i.e., $y^1 \in I_1(\boldsymbol{x}_k)$ *for some* k **then**
4     $\lfloor I_1^* \leftarrow I_1^* \cup \{y^1\}$;
5 **for** $j = 2, \ldots, n-1$ **do**
6   **if** y^{j-1} *is unallocated, i.e.,* $y^{j-1} \notin \bigcup_{k \leqslant j-1} I_k^*$
     then
7      $\lfloor I_j^* \leftarrow I_j^* \cup \{y^{j-1}\}$;
8   **if** $y^{j-1} \neq y^j$ **then**
9   **if** (a) $y^j \in I_j(\boldsymbol{x}_k)$ *for some* $k \in [n]$; *or* (b)
      y^{j-1} *is already allocated to some other agent,*
      i.e., $y^{j-1} \in \bigcup_{k \leqslant j-1} I_k^*$, *and there is no*
      $k \in [n]$ *with* $x_k^{j-1} = y^{j-1} + \frac{1}{2}$ *and*
      $x_k^j = y^j - \frac{1}{2}$ **then**
10        $\lfloor I_j^* \leftarrow I_j^* \cup \{y^j\}$;

11 **if** y^{n-1} *is unallocated, i.e.,* $y^{n-1} \notin \bigcup_{k \leqslant n-1} I_k^*$ **then**
12    $\lfloor I_n^* \leftarrow I_n^* \cup \{y^{n-1}\}$;
13 **return** $(I_1^*, I_2^*, \ldots, I_n^*)$;

whether each of the knives that induce the bundle is located left to the boundary item, at the boundary item, or right to the boundary item. The proof is presented in the full version (**?**).

Lemma 3.2. *Consider the triangulation* T_{half} *of* S_m. *Let* $S = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$ *be any elementary simplex of* T_{half} *and let* $(I_1^*, I_2^*, \ldots, I_n^*)$ *be a division returned by Algorithm 1. Then, for each* $i, j, k \in [n]$, *we have* $v_i(I_j^*) \geqslant \hat{v}_i(\boldsymbol{x}_k, j) \geqslant v_i^-(I_j^*)$.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Applying Theorem 2.2 to the triangulation T_{half} with the coloring function λ, we obtain an elementary simplex $S^* = \langle \boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_n^* \rangle$ of T_{half}. On this simplex, there exists a permutation $\pi \colon [n] \to [n]$ such that for each $i \in [n]$, $\pi(i) \in \lambda(\boldsymbol{x}_i^*)$. That is, for each $i \in [n]$, the bundle with index $\pi(i)$ in the division $\mathcal{I}(\boldsymbol{x}_i^*)$ is the bundle most preferred by the owner $a(\boldsymbol{x}_i^*)$:

$$\hat{v}_{a(\boldsymbol{x}_i^*)}(\boldsymbol{x}_i^*, \pi(i)) \geqslant \hat{v}_{a(\boldsymbol{x}_i^*)}(\boldsymbol{x}_i^*, j) \quad \text{for each } j \in [n]. \quad (2)$$

Without loss of generality, we assume that $a(\boldsymbol{x}_i^*) = i$ for each $i \in [n]$. Applying Algorithm 1 to S^*, we obtain a division $\mathcal{I}^* = (I_1^*, I_2^*, \ldots, I_n^*)$. For every pair of agents $i, j \in [n]$, we have

$$\begin{aligned} v_i(I_{\pi(i)}^*) &\geqslant \hat{v}_i(\boldsymbol{x}_i^*, \pi(i)) && \text{by Lemma 3.2,} \\ &\geqslant \hat{v}_i(\boldsymbol{x}_i^*, j) && \text{since } \pi(i) \in \lambda_i(\boldsymbol{x}_i^*), \\ &\geqslant v_i^-(I_j^*) && \text{by Lemma 3.2.} \end{aligned}$$

Thus, π certifies that $\mathcal{I}^*$ is a desired division. $\qquad \square$

4 Secretive and Extra Versions

In this section, we consider the secretive and extra versions of EF1 existence. A *secretive EF1$_{\text{outer}}$ division* for agent i^* is a division $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ of a path into n connected subsets where whichever part a secretive agent i^* selects, an EF1$_{\text{outer}}$ assignment of the remaining bundles can be made to the other agents, i.e., for every index $j \in [n]$, there exists a bijection $\pi \colon [n] \setminus \{i^*\} \to [n] \setminus \{j\}$ such that for every non-secretive agent $i \in [n] \setminus \{i^*\}$, $v_i(\mathcal{I}_{\pi(i)}) \geqslant \max_{i' \in [n]} v_i^-(\mathcal{I}_{i'})$.

For $n + 1$ agents with monotone valuations, an *extra EF1$_{\text{outer}}$ division* is a division $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ of a path into n connected subsets when any extra agent i^* leaves, an EF1$_{\text{outer}}$ assignment of the bundles can be made to the remaining agents, i.e., for every extra agent $i^* \in [n+1]$, there exists a bijection $\pi \colon [n+1] \setminus \{i^*\} \to [n]$ such that for every remaining agent $i \in [n+1] \setminus \{i^*\}$, $v_i(\mathcal{I}_{\pi(i)}) \geqslant \max_{j' \in [n]} v_i^-(\mathcal{I}_{j'})$.

By using a more general version of Sperner's lemma for multiple proper colorings (Meunier and Su 2019) and applying our rounding technique in the previous section, we obtain the existence of a secretive EF1$_{\text{outer}}$ connected division as well as that of an extra EF1$_{\text{outer}}$ connected division. The proofs are presented in the full version (**?**).

Theorem 4.1. *Suppose that there are n agents with monotone valuations over connected bundles of a path. Then, for any agent $i^* \in [n]$, there exists a secretive EF1$_{\text{outer}}$ division for i^*.*

Theorem 4.2. *Suppose that there are $n + 1$ agents with monotone valuations over connected bundles of a path. Then, there exists an extra EF1$_{\text{outer}}$ division.*

5 Conclusion and Future Work

We proved that under connectivity constraints, an EF1 division exists for any number of agents with monotone valuations, thereby resolving the open problem raised by Bilò et al. (2019). We further demonstrated the robustness of our rounding technique, by extending this existential result to the secretive and extra variants.

Recent research on fair division extensively investigates the setting where agents may have both positive and negative values for the items (Aziz et al. 2019; Meunier and Zerbib 2019; Segal-Halevi 2018; Bérczi et al. 2020; Jojić, Panina, and Živaljević 2021). In particular, Aziz et al. (2019) proposed an extension of EF1 to this more general setting, requiring agents' envy to disappear after removal of one chore from an envious bundle or that of one good from an envied bundle. It will be interesting to investigate whether such fairness notion can be achieved under connectivity constraints of a path.

Finally, this study highlights that the complexity of finding an EF1 connected division is an open problem. It would be interesting to settle the complexity question for a simple class of valuations, e.g. binary additive valuations.

Acknowledgments

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