Is There a Strongest Die in a Set of Dice with the Same Mean Pips?

Shang Lu, Shuji Kijima

Graduate School of Information Science and Electrical Engineering, Kyushu University
744 Motooka, Nishi-ku, Fukuoka, Japan
lu.shang.811@s.kyushu-u.ac.jp, kijima@inf.kyushu-u.ac.jp

Abstract

Jan-ken, a.k.a. rock-paper-scissors, is a cerebrated example of a non-transitive game with three (pure) strategies, rock, paper and scissors. Interestingly, any Jan-ken generalized to four strategies contains at least one useless strategy unless it allows a tie between distinct pure strategies. Non-transitive dice could be a stochastic analogue of Jan-ken: the stochastic transitivity does not hold on some sets of dice, e.g., Efron's dice. Including the non-transitive dice, this paper is involved in dice sets which do not contain some useless dice.

In particular, we are concerned with the existence of a strongest (or weakest, symmetrically) die in a dice set under the two conditions that (1) any number appears on at most one die and at most one side, i.e., no tie break between two distinct dice, and (2) the mean pips of dice are the same. We firstly prove that a strongest die never exist if a set of n dice of m-sided is given as a partition of the set of numbers $\{1,\ldots,mn\}$. Next, we show some sufficient conditions that the strongest die exists in a dice set which is not a partition of a set of numbers. We also give some algorithms to find the strongest die in a dice set which includes given dice.

Introduction

Jan-ken: a model of deterministic win-lose relations *Jan-ken*, a.k.a. rock-paper-scissors, is a simple model of a deterministic win-lose relation. Jan-ken is a symmetric game consisting of three (pure) strategies, rock, paper and scissors: rock beats scissors, scissors beats paper and paper beats rock. Thus, the win-lose relation is non-transitive, and the unique Nash equilibrium is completely mixed, i.e., rational players choose every strategy uniformly at random.

Interestingly, any Jan-ken generalized to four strategies contains at least one *useless* strategy unless it allows a tie between distinct strategies (Ito 2012). For example, suppose we introduce the fourth strategy "well" which beats both rock and scissor and is beaten by paper. Then, rock is useless; because both well and rock beat scissors and are beaten by paper, but well beats rock, thus a rational player uses well instead of rock. Komatsu and Ono gave an game theoretical analysis on a generalized Jan-ken, and proved that any Jan-ken with even number of strategies without ties between distinct strategies contains at least one game-theoretically useless strategy, meaning that at least one strategy takes probability zero in any mixed Nash equilibrium (Komatsu and Ono 2015).

Non-transitive dice While the win-lose relation is deterministic in Jan-ken, win-lose relations appearing in real life are often stochastic. Considering it, a dice set may provide a simple model of stochastic win-lose relation, as a stochastic counter part of Jan-ken. In fact, it is known that the transitivity does not hold on some dice set, such as *Efron's dice*.

For a pair of dice D and D', let a pair of random variables X_D and $X_{D'}$ denote independent roles of them, respectively. Then, we say the die D is *stronger* than D' (resp. D is *strictly* stronger than D') if $\Pr[X_D > X_{D'}] \geq 1/2$ (resp. $\Pr[X_D > X_{D'}] > 1/2$) holds, and write it by $D \succcurlyeq D'$ (resp. $D \succ D'$).

Efron's dice is a set of dice $A = (0,0,4,4,4,4)$, $B = (3,3,3,3,3,3)$, $C = (2,2,2,2,6,6)$ and $D = (5,5,5,1,1,1)$[1]. Let X_A, X_B, X_C and X_D be independent roles of dice A, B, C and D, respectively. Then, $\Pr[X_A \geq X_B] = \Pr[X_B \geq X_C] = \Pr[X_C \geq X_D] = \Pr[X_D \geq X_A] = 2/3$ holds, meaning that $A \succ B \succ C \succ D \succ A$ holds. In other words, $\succ$ on the dice set $\{A,B,C,D\}$ is NOT transitive. Such a dice set is called *non-transitive dice*.

Related works Research on non-transitive dice have been developed in the context of applied probabilities. Non-transitive relations of stochastic events were studied in the community voting problem (Black 1958). Upper bounds of each other's winning probabilities in some non-transitive cases were given in (Usiskin 1964). Gardner linked non-transitivity to Efron's dice, and began the research of non-transitive dice (Gardner 1970).

Concerning non-transitive dice, Buhler et al. showed that a magnitude relationship consisting of repetitive dice could cover all tournament graphs (Buhler, Graham, and Hales 2018). Then, a dice set given by a *regular partition* (see Section , appearing later) was introduced in (Savage 1994), and it was used in (Schaefer and Schweig 2017) and (Schaefer 2017) which showed for an arbitrarily given tournament that there exists a corresponding regular partition dice set. Conrey et al. used statistical methods to generate a large number

[1] The notation of dice here is different from the following sections because Efron's dice allow the same number appears on different faces.

of dice groups to estimate the proportion of non-transitive dice set (Conrey et al. 2016). Ethan showed the result, there is a set of dice of regular partition that could be obtained in any tournament, in another way (Akin 2019).

Contribution For a set of dice $\mathcal{D} = \{A_1, \ldots, A_n\}$, we say a die $D \in \mathcal{D}$ is the *strongest* (resp. the *strictly* strongest) if $D \succcurlyeq D'$ (resp. $D \succ D'$) holds for any $D' \in \mathcal{D} \setminus \{D\}$. The existence of a strongest die makes any other dice useless, which makes the game trivial: a rational player always chooses the strongest die. In this paper, we are mainly involved in conditions that a dice set does not contain a strongest die.

In particular, we are concerned with the existence of a strongest die in a dice set under the two conditions that (1) any number appears on at most one die and at most one side, i.e., no tie break between any pair of distinct dice, and (2) the mean pips of dice are the same. We firstly show that the strongest die never exist if a set of n dice of m-sided is given as a *partition* of the set of numbers $\{1, \ldots, mn\}$ (Theorem 5). Its proof highly depends on the value of mean pips, which is determined by the condition that the set of dice is a partition. Thus, we next are concerned with more general setting on the mean pips, and give some sufficient conditions that the strongest die exists in a dice set which is not a partition of a set of numbers (Theorems 11–13). We also give some algorithms to find a strongest die in a dice set which includes some given dice (Section), and we demonstrate some computational results on the number of strongest dice with respect to the value of mean pips (Section).

Preliminary

In this paper, an *m-sided die* (or simply a die) is a subset of $\{1, \ldots, k\}$ of order m, where m and k are positive integers satisfying $m \leq k$. For convenience, we define a set function $W \colon 2^{\{1,\ldots,k\}} \to \mathbb{Z}$ by $W(X) = \sum_{x \in X} x$ for any subset $X \subseteq \{1, \ldots, k\}$.

In this paper, we are concerned with dice sets parameterized by a 4 tuple of positive integers (k, m, n, w). Let $A_1, A_2, \ldots, A_n \subseteq \{1, \ldots, k\}$ be a set of m-sided dice satisfying the following two conditions:

1. Dice are *disjoint*, i.e., $A_i \cap A_j = \emptyset$ for any i, j $(i \neq j)$.
2. Every die has the same sum of pips w, i.e., $W(A_i) = w$ for any $i \in \{1, \ldots, n\}$.

If a dice set satisfies the above conditions, then we call it a (k, m, n, w) *dice set*. We particularly call the dice set a *regular partition* if $k = mn$, i.e., $A_1 \cup A_2 \cup \cdots \cup A_n = \{1, \ldots, k\}$. Notice that $w = \frac{1+\cdots+mn}{n} = \frac{m(mn+1)}{2}$ holds for a regular partition, by the above condition 2. If $k \geq mn$, we say the dice set is a *regular packing*, i.e., $A_1 \cup A_2 \cup \cdots \cup A_n \subseteq \{1, \ldots, k\}$.

For convenience, we define a set function $S \colon 2^{\{1,\ldots,k\}} \times 2^{\{1,\ldots,k\}} \to \mathbb{Z}$ by

$$S(X, Y) = |\{(x_i, y_j) \in X \times Y \mid x_i > y_j\}|$$

for any *disjoint* subsets $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$ of $\{1, \ldots, k\}$. Since X and Y are disjoint, we observe the following.

Observation 1. $S(X, Y) + S(Y, X) = st.$

Suppose A and B are disjoint m-sided dice. Let X_A and X_B be independent rolls of dice A and B, respectively, then $\Pr[X_A > X_B] = \frac{S(A,B)}{m^2}$ holds. We say A is *strictly stronger* than B (resp. A is "stronger than" and "draws to" B) if $S(A, B) > \frac{m^2}{2}$ (resp., $S(A, B) \geq \frac{m^2}{2}$ and $S(A, B) = \frac{m^2}{2}$), and write $A \succ B$ (resp., $A \succcurlyeq B$ and $A \sim B$). Clearly, it corresponds to $\Pr[X_A > X_B] > \frac{1}{2}$ (resp. $\Pr[X_A > X_B] \geq \frac{1}{2}$ and $\Pr[X_A > X_B] = \frac{1}{2}$). A (k, m, n, w) dice set $\{A_1, \ldots, A_n\}$ is *transitive* if $\forall i, j, k$, if $A_i \succcurlyeq A_j$ and $A_j \succcurlyeq A_k$ then $A_i \succcurlyeq A_k$, otherwise the set is *non-transitive*.

Schaefer and Schweig showed the following facts, where Theorem 4 is a generalization of Theorem 2.

Theorem 2 ((Schaefer and Schweig 2017))**.** *For any $m \geq 3$, there exists a $(3m, m, 3, w)$ dice set which is non-transitive.*

Theorem 3 ((Schaefer and Schweig 2017))**.** *Let $m \geq 3$. Suppose $A, B, C \subset \{1, \ldots, 3m\}$ are m-sided* disjoint *dice. Then, $S(A, B) = S(B, C) = S(C, A)$ hold if and only if $W(A) = W(B) = W(C)$ hold.*

Theorem 4 ((Schaefer 2017))**.** *For any $n \geq 3$ and $m \geq 3$, there exists a (mn, m, n, w) dice set which is non-transitive.*

In the proofs of Theorems 2–4, they represented disjoint dice $A, B, C \subseteq \{1, \ldots, k\}$ by a string σ of length k with alphabets a, b, c and x, where the positions of a, b, c in σ corresponds to the elements of A, B, C while the positions of x represent that the corresponding elements does not appear in A, B, C.

Non-transitivity in a Regular Partition

This section establishes the following theorem.

Theorem 5. *Let $n \geq 3$. Suppose $\mathcal{D} = \{D_1, \ldots, D_n\}$ is a regular partition, i.e., a $(mn, m, n, \frac{m(mn+1)}{2})$ dice set. If there exists a distinct pair of dice $A, B \in \mathcal{D}$ satisfying $A \succ B$ then there exists $C \in \mathcal{D}$ such that $C \succ A$.*

Our proof technique is similar to (Schaefer and Schweig 2017; Schaefer 2017). As a preliminary step, we remark the following fact.

Lemma 6. *Let A, B, C be disjoint subsets of $\{1, \ldots, mn\}$ where $W(A) = W(B) = W(C) = \frac{m(mn+1)}{2}$. Then, $S(A, C) + S(B, C) = S(A \cup B, C)$ holds.*

Proof. Since A, B, C are disjoint, we see $S(A, C) + S(B, C) = |\{(a, c) \in A \times C \mid a > c\}| + |\{(b, c) \in B \times C \mid b > c\}| = |\{(x, c) \in (A \cup B) \times C \mid x > c\}| = S(A \cup B, C)$. ☐

Now, we prove Theorem 5.

Proof of Theorem 5. Firstly, we claim that

$$S([mn] \setminus (A \cup B),\ A) > \frac{(n-2)m^2}{2} \tag{1}$$

holds, where $[mn]$ denotes $\{1, \ldots, mn\}$. By Lemma 6,

$$S([mn] \setminus (A \cup B), A) = S([mn] \setminus A,\ A) - S(B, A) \tag{2}$$

holds. For convenience, let $A = \{a_1, \dots, a_m\}$ where $a_1 < \cdots < a_m$ hold. Then,

$$\begin{aligned} S([mn] \setminus A, A) &= \left| \bigcup_{i=1}^{m} \{(x, a_i) \mid x > a_i,\ x \in [mn] \setminus A\} \right| \\ &= \sum_{i=1}^{m} (mn - a_i - (m - i)) \\ &= m^2 n - \sum_{i=1}^{m} a_i - \sum_{i'=0}^{m-1} i' \\ &= m^2 n - \frac{m(nm+1)}{2} - \frac{m(m-1)}{2} \\ &= \frac{(n-1)m^2}{2} \end{aligned} \quad (3)$$

holds. The hypothesis $A \succ B$ implies $S(A, B) > \frac{m^2}{2}$, and hence $S(B, A) < \frac{m^2}{2}$ holds by Observation 1. Thus

$$\begin{aligned} (2) &> \frac{(n-1)m^2}{2} - \frac{m^2}{2} \\ &= \frac{(n-2)m^2}{2} \end{aligned}$$

holds, and we obtain the claim (1).

Next, we assume for a contradiction that any $D \in \mathcal{D} \setminus \{A, B\}$ satisfies $A \not\succ D$. In the case, $S(D, A) \leq \frac{m^2}{2}$ holds for any $D \in \mathcal{D} \setminus \{A, B\}$. Since $\mathcal{D}$ is a partition of $[mn]$,

$$\begin{aligned} S([mn] \setminus (A \cup B), A) &= \sum_{D \in \mathcal{D} \setminus \{A, B\}} S(D, A) \\ &\leq \frac{(n-2)m^2}{2} \end{aligned}$$

holds, which contradicts to (1). □

We remark in case of $n = 2$.

Proposition 7. *Let $\mathcal{D} = \{A, B\}$ be a regular partition, i.e., a $(2m, m, 2, \frac{m(2m+1)}{2})$ dice set, where m is even. Then, $A \sim B$.*

Proof. For convenience, let $A = \{a_1, \dots, a_m\}$ where $a_1 < \cdots < a_m$ hold. Then,

$$\begin{aligned} S(A, B) &= \left| \bigcup_{i=1}^{m} \{(a_i, b) \mid a_i > b,\ b \in [2m] \setminus A\} \right| \\ &= \sum_{i=1}^{m} (a_i - i) \\ &= \frac{m(2m+1)}{2} - \frac{m(m+1)}{2} \\ &= \frac{m^2}{2} \end{aligned}$$

holds, and we obtain the claim (recall the definition of $S(A, B)$. □

Corollary 8. *For any $n \geq 2$, any regular partition, i.e., $(mn, m, n, \frac{m(mn+1)}{2})$ dice set, does not contain a strictly strongest die.*

Proof. It is trivial from Proposition 7 in case of $n = 2$. It is also easy from Theorem 5 in case of $n \geq 3$. □

Remarks Here, we briefly remark the following proposition, which is proved in a similar way as Theorem 5.

Proposition 9. *Let $\mathcal{D} = \{A, B, C\}$ be a regular partition, i.e., a $(3m, m, 3, \frac{m(3m+1)}{2})$ dice set, where m is even. If $A \sim B$ then $A \sim C$.*

Proof. By (3),

$$S(B \cup C, A) = m^2$$

holds. Since $A \sim B$, $S(A, B) = S(B, A) = \frac{m^2}{2}$. Thus

$$S(C, A) = S(B \cup C, A) - S(B, A) = m^2 - \frac{m^2}{2} = \frac{m^2}{2}$$

holds. This implies $A \sim C$. □

Note that Proposition 9 implies $A \sim B \sim C \sim A$ since "$\sim$" is transitive by the definition.

Strongest Die in a Regular Packing

This section is concerned with the existence of a strongest die in a *regular packing*, which is a generalization from a regular partition so that k and w are no longer fixed to mn and $\frac{m(mn+1)}{2}$, respectively. In Section , we give some sufficient conditions of the existence of a strongest die with respect to w. In Section , we give an algorithm to decide whether a given die is the strongest in any regular packing. In Section , we demonstrate some results of computer search of the existence of a strongest die for some k and w, by an exhaustive search using the algorithm in Section as a subroutine.

Sufficient Conditions of the Existence of a (Strictly) Strongest Die

As a preliminary step, we remark the following fact.

Proposition 10 (condition of the existence of a disjoint pair of dice). *Suppose m is a positive even number, and $k \geq 2m$. At least two distinct m-sided dice A and B exist such that $W(A) = W(B) = w$ if and only if w satisfies $m^2 + \frac{m}{2} \leq w \leq m(k - m) + \frac{m}{2}$.*

Proof. It is not difficult to see that the minimum w_* is $\frac{1+\cdots+2m}{2} = m^2 + \frac{m}{2}$, achieved when A and B is a partition of $\{1, \dots, 2m\}$. Similarly, the maximum w^* is $\frac{(k-2m+1)+\cdots+k}{2} = m(k - m) + \frac{m}{2}$, achieved when A and B is a partition of $\{k - 2m + 1, \dots, k\}$.

Next, we prove a dice pair A, B exists whenever w satisfies the condition. For the minimum $w_* = m^2 + \frac{m}{2}$, a pair A, B is represented by $\sigma(\Omega) = ab \dots abba \dots baxx \dots xx$. For any string corresponding to $W(A) = W(B) = w$, if we replace abx by xab, or bax by xba, then we obtain a dice set A', B' such that $W(A') = W(B') = w + 1$. This operation will be ended by $\sigma(\Omega) = xx \dots xxab \dots abba \dots ba$ with $W(A) = W(B) = w^*$. □

In the following, we assume for (k, m, n, w) that m is positive even, $k \geq 2m$ and w satisfies the condition in Proposition 10. Then, we establish the following Theorems 11, 12 and 13.

Theorem 11 (existence of the strictly strongest die for large w)**.** *If w satisfies $\frac{m(k-3)}{2} + k < w < m(k-m) + \frac{m}{2}$ then there exists an m-sided die $A \subset \{1, \ldots, k\}$ with $W(A) = w$ such that $A \succ D$ holds for any m-sided die $D \subseteq \{1, \ldots, k\} \setminus A$ with $W(D) = w$, i.e., A is the* strictly *strongest die in any (k, m, n, w) dice set containing A, in the case of w.*

Proof. Firstly, we are concerned with the case that $\frac{m(k-3)}{2} + k < w \leq \frac{m(k-1)}{2} + k$. Let

$$A = \left\{1, \ldots, \tfrac{m}{2} - 1, q, k - (\tfrac{m}{2} - 1), \ldots, k\right\}$$

where q satisfies $k - \frac{3}{2}m < q \leq k - \frac{1}{2}m$. Notice that $w = \frac{mk}{2} + q$. It is not difficult to see that A is the strictly strongest.

Next, we are concerned with the case $\frac{m(k-1)}{2} < w < m(k-m) + \frac{m}{2}$. Similarly, if

$$A \supset \left\{k - \tfrac{m}{2}, \ldots, k\right\}$$

then it is not difficult to see that A is the strictly strongest. We can design such a die A with any w satisfying $\frac{m(k-1)}{2} < w < m(k-m) + \frac{m}{2}$ in a similar way as Proposition 10. $\square$

Theorem 12 (existence of the strictly strongest die for small w)**.** *Suppose $k \geq \frac{5}{2}m$ and $m \geq 4$. If w satisfies $m^2 + \frac{m}{2} < w \leq m^2 + 2m$ then there exists an m-sided die $A \subset \{1, \ldots, k\}$ with $W(A) = w$ such that $A \succ D$ holds for any m-sided die $D \subseteq \{1, \ldots, k\} \setminus A$ with $W(D) = w$, i.e., A is the* strictly *strongest die in any (k, m, n, w) dice set containing A, in the case of w.*

Proof. Firstly, we consider the case that $m^2 + \frac{m}{2} < w \leq m^2 + \frac{3}{2}m - 1$. Let $w = m^2 + \frac{m}{2} + q$ for $q = 1, \ldots, m-1$. We prove that

$$A = \left\{\tfrac{m}{2} + 1, \ldots, \tfrac{m}{2} + m - 1, \tfrac{m}{2} + m + q\right\} \tag{4}$$

is the strictly strongest. For convenience, let $A = \{a_1, \ldots, a_m\}$ and $a_1 < \cdots < a_m$, i.e., $a_1 = \frac{m}{2} + 1, a_2 = \frac{m}{2} + 2, \ldots, a_{m-1} = \frac{3m}{2} - 1, a_m = \frac{3m}{2} + q$. We can verify that

$$\begin{aligned} W(A) &= \sum_{i=1}^{m-1} \left(\frac{m}{2} + i\right) + \left(\frac{3m}{2} + q\right) \\ &= \frac{m^2}{2} + \frac{m(m+1)}{2} + q \\ &= m^2 + \frac{m}{2} + q = w \end{aligned} \tag{5}$$

holds.

Assume for a contradiction that there exists a die $D \subseteq [k] \setminus A$ such that $D \succcurlyeq A$ and $W(D) = w$. For convenience, let $D = \{d_1, \ldots, d_m\}$ where $d_1 < \cdots < d_m$. Then, we consider the following cases on the die D:

Case 1 $a_1 > d_{\frac{m}{2}}$.
Case 2 $d_{\frac{m}{2}-1} < a_1 < d_{\frac{m}{2}}$.
Case 3 $a_1 < d_{\frac{m}{2}-1}$.

In Case 1, $D \succcurlyeq A$ requires $d_{\frac{m}{2}+1} > a_m$. Then,

$$\begin{aligned} W(D) &\geq \sum_{i=1}^{\frac{m}{2}} i + \sum_{j=1}^{\frac{m}{2}} \left(\frac{3m}{2} + q + j\right) \\ &= \frac{m}{2}\left(\frac{m}{2} + 1\right) + \frac{m}{2}\left(\frac{3m}{2} + q\right) + \frac{m}{2}\left(\frac{m}{2} + 1\right) \\ &= \frac{m}{2}\left(\frac{5m}{2} + q + 2\right) \\ &> (5) \end{aligned}$$

which contradicts to $W(D) = w$.

In Case 2, $D \succcurlyeq A$ requires $d_{\frac{m}{2}} > a_{m-1}$. Then,

$$\begin{aligned} W(D) &\geq \sum_{i=1}^{\frac{m}{2}-1} i + \sum_{j=0}^{\frac{m}{2}} \left(\frac{3m}{2} + j\right) \\ &= \frac{1}{2}\left(\frac{m}{2} - 1\right)\frac{m}{2} + \left(\frac{m}{2} + 1\right)\frac{3m}{2} \\ &\quad + \frac{1}{2} \cdot \frac{m}{2}\left(\frac{m}{2} + 1\right) \\ &= \frac{m}{4}(4m + 6) \\ &= m^2 + \frac{m}{2} + \frac{m}{2}\left(\frac{m}{2} + 1\right) \\ &\geq m^2 + \frac{m}{2} + m \\ &> (5) \end{aligned}$$

where the last inequality follows $q < m$. This contradicts to $W(D) = w$.

In Case 3, $D \succcurlyeq A$ requires $d_{\frac{m}{2}-1} > a_{m-1}$. Then, we can prove $W(D) > w$ in a similar way as Case 2, and obtain a contradiction. This proved the claim in the case of $m^2 + \frac{m}{2} < w \leq m^2 + \frac{3m}{2} - 1$.

Next, we consider the case that $w = m^2 + \frac{3m}{2} + q$ for $q = 0, \ldots, \frac{m}{2}$. We prove that

$$A = \left\{\tfrac{m}{2} + 2, \ldots, \tfrac{3m}{2}, \tfrac{3m}{2} + 1 + q\right\} \tag{6}$$

is the strictly strongest, For convenience, let $A = \{a_1, \ldots, a_m\}$ and $a_1 < \cdots < a_m$, i.e., $a_1 = \frac{m}{2} + 2, a_2 = \frac{m}{2} + 3, \ldots, a_{m-1} = \frac{3m}{2}, a_m = \frac{3m}{2} + 1 + q$. We can verify that

$$\begin{aligned} W(A) &= \sum_{i=1}^{m-1} \left(\frac{m}{2} + 1 + i\right) + \left(\frac{3m}{2} + 1 + q\right) \\ &= \frac{m^2}{2} + \frac{m(m+1)}{2} + q + m \\ &= m^2 + \frac{3m}{2} + q = w \end{aligned} \tag{7}$$

holds.

Assume for a contradiction that there exists a die $D \subseteq [k] \setminus A$ such that $D \succcurlyeq A$ and $W(D) = w$. For convenience, let $D = \{d_1, \dots, d_m\}$ where $d_1 < \cdots < d_m$. Then, we consider the following cases:

Case 0 $a_1 > d_{\frac{m}{2}+1}$.
Case 1 $d_{\frac{m}{2}} < a_1 < d_{\frac{m}{2}+1}$.
Case 2 $d_{\frac{m}{2}-1} < a_1 < d_{\frac{m}{2}}$.
Case 3 $a_1 < d_{\frac{m}{2}-1}$.

In Case 0, it is not difficult to see that $D \prec A$ holds.

In Case 1, $D \succcurlyeq A$ requires $d_{\frac{m}{2}+1} > a_m$. Then,

$$\begin{aligned}
W(D) &\geq \sum_{i=1}^{\frac{m}{2}} i + \sum_{j=1}^{\frac{m}{2}} \left(\frac{3m}{2} + 1 + q + j\right) \\
&= \frac{m}{2}\left(\frac{m}{2}+1\right) + \frac{m}{2}\left(\frac{3m}{2}+q\right) + \frac{m}{2}\left(\frac{m}{2}+1\right) \\
&\quad + \frac{m}{2} \\
&= \frac{m}{2}\left(\frac{5m}{2} + q + 3\right) \\
&> (7)
\end{aligned}$$

which contradicts to $W(D) = w$.

In Case 2, $D \succcurlyeq A$ requires $d_{\frac{m}{2}} > a_{m-1}$. Then,

$$\begin{aligned}
W(D) &> \sum_{i=1}^{\frac{m}{2}-1} i + \sum_{j=0}^{\frac{m}{2}} \left(\frac{3m}{2} + 1 + j\right) \\
&= \frac{1}{2}\left(\frac{m}{2}-1\right)\frac{m}{2} + \left(\frac{m}{2}+1\right)\frac{3m}{2} \\
&\quad + \frac{1}{2}\cdot\frac{m}{2}\left(\frac{m}{2}+1\right) + \left(\frac{m}{2}+1\right) \\
&= \frac{m}{4}(4m+8) + 1 \\
&= m^2 + \frac{3m}{2} + \frac{m}{2} + 1 \\
&> (7)
\end{aligned}$$

where the last inequality follows $q \leq \frac{m}{2}$. This contradicts to $W(D) = w$.

In Case 3, $D \succcurlyeq A$ requires $d_{\frac{m}{2}-1} > a_{m-1}$. Then, we can prove $W(D) > w$ in a similar way as Case 2, and obtain a contradiction. Now we obtain the claim. $\square$

One may feel the bound on w is too loose, from the proof, but our some computational results suggest that Theorem 12 may be tight (see Section).

Theorem 13. *If $w = \frac{m}{2}(1+k)$ then there exists an m-sided die $A \subset \{1, \dots, k\}$ with $W(A) = w$ such that $A \sim D$ holds for any m-sided die $D \subseteq \{1, \dots, k\} \setminus A$ with $W(D) = w$, i.e., A is the strongest die*[2] *in any (k, m, n, w) dice set containing A, in the case of w.*

Proof. Set die A as $a_1 = 1, a_2 = 2, \dots, a_{\frac{m}{2}} = \frac{m}{2}, a_{\frac{m}{2}+1} = t - \frac{m}{2} + 1, \dots, a_m = t$. Let $D \subset [k] \setminus A$ be arbitrary. Then we can see $\forall i, b_i > a_j$ when $j \leq \frac{m}{2}$, and $\forall i, b_i < a_j$ when $j \geq \frac{m}{2}$. This implies $S(A, D) = \frac{m^2}{2}$, meaning that D draws with die A. $\square$

[2]Recall: a die A is strongest if $A \succcurlyeq D$ for any $D \in \mathcal{D}$.

Deciding Whether a Die Is Strongest

This section discusses an algorithm to decide whether a die D exists stronger than a given die A such that D and A are disjoint and $W(D) = W(A)$. We give an algorithm based on a dynamic programming which runs in $\mathrm{O}(kmw)$ time. The basic idea is to maximize $S(D, A)$ over $D \in \binom{\{1,\dots,k\}\setminus A}{m}$, and a die D exists strictly stronger than (resp. draws to) A if $S(D, A) > \frac{m^2}{2}$ (resp. $S(D, A) = \frac{m^2}{2}$).

Given a m-sided die $A \subseteq \{1, \dots, k\}$, we define $v_i^A \in \mathbb{Z} \cup \{-\infty\}$ for any $i = 1, \dots, k$ by

$$v_i^A = \begin{cases} |\{a_j \in A | a_j < i\}|, & i \in \{1, \dots, k\} \setminus A \\ -\infty, & i \in A. \end{cases} \tag{8}$$

Then, we define a function $F : \mathbb{Z}^3 \to \mathbb{Z}$ by

$$\begin{aligned}
F[t, q, p] = \text{max.} \ & \sum_{i=1}^{t} x_i v_i^A \\
\text{s. t.} \ & \sum_{i=1}^{t} x_i w_i = q \\
& \sum_{i=1}^{t} x_i = p \\
& x_i \in \{0, 1\} \quad i = 1, \dots, t
\end{aligned} \tag{9}$$

for $1 \leq t \leq k$, $0 \leq q \leq w$, $0 \leq p \leq m$, where we define $F[t, q, p] = -\infty$ if (9) is infeasible. Clearly, $F[k, w, m]$ provides the maximum of $S(D, A)$.

Lemma 14. *F satisfies*

$$\begin{aligned}
F[t, q, p] = \max\{&F[t-1, q, p], \\
&F[t-1, q - w_t, p-1] + v_t^A\}.
\end{aligned} \tag{10}$$

Proof. Firstly, consider the case that (9) is feasible. Let $x_t^* = [x_1^*, x_2^*, \dots, x_t^*]$ be the optimal solution of (9), then $F[t, q, p] = x_1^* v_1^A + x_2^* v_2^A + \cdots + x_t^* v_t^A$. We consider the two cases $x_t^* = 0$ or $x_t^* = 1$. Suppose $x_t^* = 0$, which means t is not chosen. Then, it is not difficult to see that $x_{t-1}^* = [x_1^*, x_2^*, \dots, x_{t-1}^*]$ is a feasible solution and hence $F[t-1, q, p] \geq x_1^* v_1^A + x_2^* v_2^A + \cdots + x_{t-1}^* v_{t-1}^A = F[t, q, p]$. Now we clam $F[t-1, q, p] = x_1^* v_1^A + x_2^* v_2^A + \cdots + x_{t-1}^* v_{t-1}^A$. Assume for a contradiction $F[t-1, q, p] > x_1^* v_1^A + x_2^* v_2^A + \cdots + x_{t-1}^* v_{t-1}^A$. Then there is a solution y^* such that $F[t-1, q, p] = y_1^* v_1^A + y_2^* v_2^A + \cdots + y_{t-1}^* v_{t-1}^A$. It is not difficult to see that $(y_1^*, \dots, y_{t-1}^*, 0)$ is a feasible solution and $F[t, q, p] < y_1^* v_1^A + y_2^* v_2^A + \cdots + y_{t-1}^* v_{t-1}^A + 0 v_t^A$. It contradicts to the assumption that x^* is a optimal solution. Thus we obtain $F[t, q, p] = F[t-1, q, p]$ when $x_t^* = 0$.

Suppose $x_t^* = 1$, which means t is chosen. Then, it is not difficult to see that $x_{t-1}^* = [x_1^*, x_2^*, \dots, x_{t-1}^*]$ is a feasible solution and hence $F[t-1, q - w_t, p-1] + v_t^A \geq x_1^* v_1^A + x_2^* v_2^A + \cdots + x_{t-1}^* v_{t-1}^A + x_t^* v_t^A = F[t, q, p]$. Now we clam $F[t-1, q - w_t, p-1] = x_1^* v_1^A + x_2^* v_2^A + \cdots + x_{t-1}^* v_{t-1}^A$. Assume for a contradiction $F[t-1, q - w_t, p-1] > x_1^* v_1^A + x_2^* v_2^A + \cdots + x_{t-1}^* v_{t-1}^A$. Then there is a solution y^* such that $F[t-1, q - w_t, p-1] = y_1^* v_1^A + y_2^* v_2^A + \cdots + y_{t-1}^* v_{t-1}^A$.

Algorithm 1: JUDGEMENT-ONE(A, k)

Input:
int k
int array A $\triangleright$ a die $A \subset \{1, \ldots, k\}$
Output:
$\max\{S(D, A) \mid D \subseteq \{1, \ldots, k\} \setminus A,\ |D| = m\}$
1: $m \leftarrow$ Length$[A]$
2: $w \leftarrow$ Sum$[A]$
3: $F[0, q, p] \leftarrow -\infty, (q = 0, \ldots, w; p = 0, \ldots, m)$
4: $F[0, 0, 0] \leftarrow 0$
5: Calculate v_i^A for $i = 1, \ldots, k$.

6: **for** $t \leftarrow 0$ to k, $q \leftarrow w_t$ to w, and $p \leftarrow 1$ to m **do**
7: $F[t, q, p] = \max\{F[t-1, q, p], F[t-1, q-w_t, p-1] + v_i^A\}$
8: **end for**

9: return $F[k, w, m]$

Algorithm 2: OUTPUT-DICE(G)

Input:
function G $\triangleright$ see (11)
Output:
die D

for $t \leftarrow k$ downto 1 **do**
if $G[t, w, m] = 1$ **then**
output t,
$w \leftarrow w - w_t, m \leftarrow m - 1$
end if
end for

It is not difficult to see that $(y_1^*, \ldots, y_{t-1}^*, 1)$ is a feasible solution and $F[t, q, p] < y_1^* v_1^A + y_2^* v_2^A + \cdots + y_{t-1}^* v_{t-1}^A + v_t^A$. It contradicts to the assumption that x^* is a optimal solution. Thus we obtain $F[t, q, p] = F[t-1, q-w_t, p-1] + v_t^A$ when $x_t^* = 1$.

By the above argument, it is not difficult to see that (10) holds when (9) is feasible.

In case that (9) is infeasible, both $P[t-1, q, p]$ and $P[t-1, q-w_t, p-1]$ are infeasible. Otherwise if $P[t-1, q, p]$ is feasible and $(y_1^*, \ldots, y_{t-1}^*)$ is a feasible solution, then $(y_1^*, \ldots, y_{t-1}^*, 0)$ is a feasible solution for $P[t, q, p]$, which contradicts to the assumption that $P(t, q, p)$ is infeasible. Similarly, $P[t-1, q-w_t, p-1]$ is feasible, then $P(t, q, p)$ is also feasible, that leads a contradiction. $\square$

The function F is efficiently calculated by the following algorithm based on a dynamic programming.

Lemma 15. *An m-sided die $C \subseteq [k] \setminus A$ with $W(C) = w$ exists if and only if $F[k, w, m] \geq 0$. If $F[k, w, m] > \frac{m^2}{2}$ then there exists such a die C satisfying $C \succ A$. If $F[k, w, m] \leq \frac{m^2}{2}$ then $A \succeq C$ for any such a die C. Particularly, if $F[k, w, m] = \frac{m^2}{2}$ there exists such a die C satisfying $C \sim A$.*

Proof. In case of $F[k, w, m] > 0$, problem (9) has a feasible solution. We write the optimal solution as $x^* = [x_1^*, x_2^*, \ldots, x_k^*]$, then, we get $F[k, w, m] = x_1^* v_1^A + x_2^* v_2^A + \cdots + x_k^* v_k^A$ satisfying $\sum_{i=1}^k x_i^* w_i = w$ and $\sum_{i=1}^k x_i^* = m$. Then, we set $C = \{c_i | x_i^* = 1, for i \in [1, k]\}$. Obviously, it satisfies that $|C| = m$ and $\sum_{c_i \in C} c_i = w$. At the same time, $S(C, A) = \sum_{c_i \in C} v_{c_i}^A = F[k, w, m]$, and this C is the optimal correspondence for A.

Accordingly, in the case of $F[k, w, m] > \frac{m^2}{2}$, there exist a die C owns $S(C, A) > \frac{m^2}{2}$ as the optimal correspondence for A. In case of $F[k, w, m] = \frac{m^2}{2}$, the optimal correspondence for A is $S(C, A) = \frac{m^2}{2}$, meaning that A beats any C. In the case of $0 < F[k, w, m] < \frac{m^2}{2}$, the optimal correspondence for A is $S(C, A) < \frac{m^2}{2}$, and A can beat anyone else. Lastly, if $F[k, w, m] \leq 0$, the problem (9) is infeasible, and a die C satisfying $|C| = m$ and $\sum_{c_i \in C} c_i = w$ does not exist. $\square$

Theorem 16. *By Algorithm 1, we can correctly decide whether an m-sided die $D \subseteq \{1, \ldots, k\} \setminus A$ exists such that $S \succ A$. The time complexity is $O(kmw)$.*

Algorithm 1 only *decides* the existence of a die stronger than A. With an extra operation, we can find a die D in Lemma 15. We define a function G by

$$G[t, q, p] = \begin{cases} 0 & \text{if } F[t, q, p] = \{F[t-1, q, p] \\ 1 & \text{otherwise} \end{cases} \tag{11}$$

for $1 \leq t \leq k$, $0 \leq q \leq w$, $0 \leq p \leq m$. This G is easily computed just below line 7 in Algorithm 1. Then, we can find a desired die by the following algorithm.

Computer Search of Strongest Dice

As discussed in Section , Theorems 11 and 12 imply the existence a strongest die in *any* (k, m, n, w) dice set when $w \leq m^2 + 2m$ or $w > \frac{m(k-3)}{2} + k$. That is, if a player is allowed to *arbitrarily* choose an m-sided die $D \subseteq \{1, \ldots, k\}$ under the condition that $W(D) = w$ then a rational player should choose the strongest die. To design a game avoiding such a situation, what w is appropriate in $m^2 + 2m < w \leq \frac{m(k-3)}{2} + k$? We do not know the answer, but this section demonstrates some results by a computational search.

Precisely, we employ an (exhaustive) depth-first-search by the following recursive algorithm, where Algorithm 1 (JUDGEMENT-ONE(A, k)) given in Section is used as a subroutine. The time complexity is $O(kwm2^k)$.

Figures 1 and 2 respectively show the results for $(k, m) = (32, 8)$ and $(k, m) = (36, 8)$. By Proposition 10, respectively $w \in [68, 196]$ and $w \in [68, 228]$ are the range where at least two distinct dice can coexist. We implemented Algorithm 3 in C++, and ran it on a machine with GPU/CPU models: Apple M1, amount of memory: 16G operating system: macOS 11.5, The running times were respectively 2,768 sec. ($\approx$ 46 min.) for $k = 32$, $m = 8$, $w \in [68, 196]$, and 16,601 sec. ($\approx$ 4.6 hours) for $k = 36$, $m = 8$, $w \in [68, 228]$.

Algorithm 3: JUDGEMENT-W(w, k, m)

Input:
    int m
    int k
    int w
Output:
    all lists A satisfying JUDGEMENT-ONE$(A, k) > \frac{m^2}{2}$
        ▷ output all dice which are strictly strongest.
        ▷ replace line 7 for the strongest but not strictly.

```
 1: list<int>A;
 2: if k ≤ 0 or w ≤ 0 then
 3:     return
 4: end if
 5:
 6: if w = 0 and A.size = m then
 7:     if JUDGEMENT-ONE(A, k) > 1/2 then
                    ▷ replace '>' by '=' for purely "≽."
 8:         output A
 9:     end if
10: end if
11:
12: A.push(k)
13: JUDGEMENT-W(w − k, k − 1, m)
14: A.pop()
15: JUDGEMENT-W(w, k − 1, m)
```

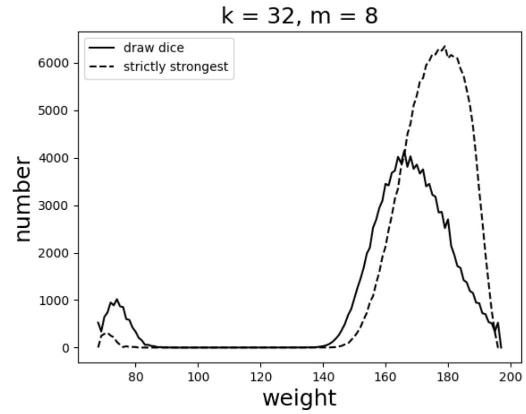

Figure 1: the numbers of 8-sided draw dice (solid line) and strictly strongest dice (dash line) in the range of $w \in [68, 196]$ with $k = 32$.

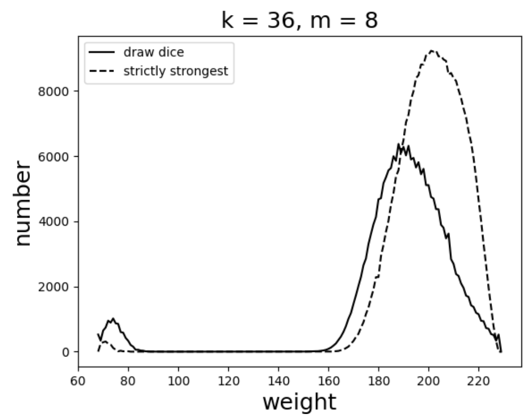

Figure 2: the numbers of 8-sided draw dice (solid line) and strictly strongest dice (dash line) in the range of $w \in [68, 228]$ with $k = 36$.

In Figure 1, we observe that $w \in [81, 141] \setminus \{83, 87\}$ does not allow any *strictly* strongest die (by dash line[3]), while a strongest die A, meaning that $A \succcurlyeq D$ for any $D \subseteq [k] \setminus A$, exists for $w = 81, \dots 93, 100, 132, \dots, 141$ in the range $[81, 141]$ (by solid line[4]). In this case of $k = 32$ and $m = 8$, Theorem 11 implies that at least one of strictly strongest dice exist for $w \in (148, 196)$ and Theorem 12 implies that at least one of strictly strongest dice exist for $w \in (68, 80]$. Theorem 13 implies the existence of a strongest die for $w = 132$, and our computational result shows that the strongest die is unique (solid line).

Similarly, in Figure 2, we observe that $w \in [81, 159] \setminus \{83, 87\}$ does not allow any *strictly* strongest die (by dash line), while a strongest die A, meaning that $A \succcurlyeq D$ for any $D \subseteq [k] \setminus A$, exists for $w = 81, \dots 92, 100, 148, \dots, 159$ in the range $[81, 159]$ (by solid line). In this case of $k = 36$ and $m = 8$, Theorem 11 implies that at least one of strictly strongest dice exists for $w \in (168, 228)$ and Theorem 12 implies that at least one of strictly strongest dice exists for $w \in (68, 80]$. Theorem 13 implies the existence of a strongest die for $w = 148$, and our computational result shows that the strongest die is unique (solid line).

Concluding Remarks

This paper has investigated the existence of a strongest die in a (k, m, n, w) dice set. A complete characterization with respect to w remains as an open question. Another question is when a directed graph provided by $\succ$ becomes strongly connected. Game theoretical analysis of non-transitive dice, like (Rump 2001; Hulko and Whitmeyer 2019) for non-transitive dice or (Komatsu and Ono 2015) for generalized Jan-ken is another future work.

In an algorithmic aspect, we gave in Section an algorithm to find a die D satisfying $D \succ A$ for a given die A. In Section B in appendix (appearing in supplemental pdf file), we give some extensions of the algorithm such as to find a die D satisfying $B \succ D \succ A$ for some given dice A and B. It is an open question whether a polynomial-time algorithm exists to decide the existence of a strongest die for given k and w.

[3]the dash line shows the number of dice which are the strictly strongest.

[4]the solid line shows the number of dice which are the strongest but are not the strictly strongest, we here call them "draw dice."

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