Truthful and Fair Mechanisms for Matroid-Rank Valuations

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Abstract

We study the problem of allocating indivisible goods among strategic agents. We focus on settings wherein monetary transfers are not available and each agent's private valuation is a submodular function with binary marginals, i.e., the agents' valuations are matroid-rank functions. In this setup, we establish a notable dichotomy between two of the most well-studied fairness notions in discrete fair division; specifically, between envy-freeness up to one good (EF1) and maximin shares (MMS).

First, we show that a known Pareto-efficient mechanism is group strategy-proof for finding EF1 allocations, under matroid-rank valuations. The group strategy-proofness guarantee strengthens an existing result that establishes truthfulness (individually for each agent) in the same context. Our result also generalizes prior work from binary additive valuations to the matroid-rank case.

Next, we establish that an analogous positive result cannot be achieved for MMS, even when considering truthfulness on an individual level. Specifically, we prove that, for matroid-rank valuations, there does not exist a truthful mechanism that is *index oblivious*, Pareto efficient, and maximin fair.

For establishing our results, we develop a characterization of truthful mechanisms for matroid-rank functions. This characterization in fact holds for a broader class of valuations (specifically, holds for binary XOS functions) and might be of independent interest.

Introduction

The field of discrete fair division studies the allocation of indivisible goods (i.e., goods that cannot be fractionally divided) among agents with possibly different preferences. Such allocation problems arise naturally in many real-world settings, e.g., assignment of flats in public housing estates (Deng, Sing, and Ren 2013), allocating courses to students (Budish et al. 2017), or distributing computational resources. Motivated, in part, by such applications, in recent years a significant body of work has been directed towards the development of algorithms that find fair and economically efficient allocations (Brandt et al. 2016; Endriss 2017).

In the fair division literature, envy-freeness is one of the most prominent fairness criterion. An allocation is said to be envy-free iff every agent values her bundle (allocated goods) at least as much as the bundle of any other agent. Notably, when the goods are indivisible, the existence of envy-free allocations is ruled out, even in rather simple instances; consider a setting with a single indivisible good that is desired by two agents. In light of such non-existence results, meaningful relaxations of envy-freeness (and other classic fairness criteria) have been the focus of research in discrete fair division. In the context of indivisible goods, the two most prominent notions of fairness are envy-freeness up to one good (EF1) (Lipton et al. 2004; Budish 2011) and maximin shares (MMS) (Budish 2011).

An allocation is said to be envy-free up to one good (EF1) iff each agent values her bundle no less than the bundle of any other agent, subject to the removal of one good from the other agent's bundle. For indivisible items, EF1 allocations are guaranteed to exist under monotone valuations and can be computed efficiently (Lipton et al. 2004). In fact, under additive[1] valuations, there necessarily exist allocations that are both EF1 and Pareto efficient (i.e., economically efficient) (Caragiannis et al. 2019; Barman, Krishnamurthy, and Vaish 2018).

Maximin share is a threshold-based fairness notion. That is, every agent has a threshold—referred to as her *maximin share*—and an allocation is deemed to be maximin fair (MMS) iff in the allocation each agent receives a value at least as much as her maximin share. Conceptually, the threshold follows from executing a discrete version of the cut-and-choose protocol: among n participating agents, the maximin share for an agent i is defined as the maximum value that i can guarantee for herself by partitioning the set of (indivisible) goods into n subsets, and then receiving a minimum valued (according to i's valuation) one. In contrast to envy-freeness up to one good, MMS allocations are not guaranteed to exist under additive valuations (Kurokawa, Procaccia, and Wang 2016; Procaccia and Wang 2014). This fairness notion, however, admits strong approximation guarantees for additive valuations and beyond; see (Garg and Taki 2020; Ghodsi et al. 2018), and references therein.

Equipped with relevant fairness criteria (such as EF1 and MMS), research in discrete fair division has primar-

[1]An agent's valuation function v is said to be additive iff $v(S) = \sum_{g \in S} v(\{g\})$ for every subset of goods S.

ily focussed on existence and computational tractability of fairness notions, along with their impact on economic efficiency. Another key desiderata in this context—and resource-allocations settings, in general—is that of *truthfulness*, i.e., one wants mechanisms wherein the participating agents cannot gain by misreporting their valuations. However, in settings wherein monetary transfers are not available, these three central objectives (of fairness, economic efficiency, and truthfulness) cannot be achieved together, even under additive valuations; it is known that, under additive valuations, the only Pareto-efficient and truthful mechanism is serial dictatorship (Klaus and Miyagawa 2002), which is notably unfair.

Motivated by these considerations, an important thread of work is aimed at identifying expressive valuation classes that admit truthful and fair mechanisms, without money. Of note here are valuations with binary marginals, i.e., valuations that change by at most one upon the addition or removal of a good, from any bundle. Such dichotomous preferences have been extensively studied in fair division literature; see, e.g., (Bogomolnaia and Moulin 2004; Bouveret and Lemaître 2016; Kurokawa, Procaccia, and Shah 2015). Here, two well-studied function classes, in order of containment, are: binary additive valuations and binary submodular[2] valuations. Binary additive (respectively, submodular) valuations are additive (submodular) functions with binary marginals. We note that binary submodular functions characteristically correspond to matroid-rank functions (Schrijver 2003) and, hence, constitute a combinatorially expressive function class. Matroid-rank valuations capture preferences in many resource-allocation domains; see (Benabbou et al. 2020) for pragmatic examples.

Halpern et al. (2020) provide a *group strategy-proof* mechanism for binary additive valuations. Their mechanism is based on maximizing the Nash social welfare (i.e., the geometric mean of the agents' valuations) with a lexicographic tie-breaking rule. Nash optimal allocations are Pareto efficient. Also, under binary additive valuations, such allocations are known to be MMS as well as EF1. Hence, for binary additive valuations, the work of Halpern et al. (2020) achieves all the three desired properties; their mechanism in fact can be executed in polynomial time.

For the broader class of binary submodular valuations, Babaioff et al. (2021) obtain a truthful, Pareto efficient, and fair mechanism. This work considers Lorenz domination as a fairness criterion and, hence as implications, obtains EF1 and $1/2$-MMS guarantees.

Contributing to this thread of work, the current paper studies mechanism design, without money, for fairly allocating indivisible goods. We focus on settings wherein the agents' valuations are matroid-rank functions (i.e., are binary submodular functions) and establish a notable dichotomy between EF1 and MMS.

Our Results. First, we show that the Pareto-efficient mechanism of Babaioff et al. (2021) is group strategy-proof for finding EF1 allocations, under matroid-rank valuations (Theorem 3). The group strategy-proofness guarantee strengthens the result of Babaioff et al. (2021), that establishes truthfulness (individually for each agent). Our result also generalizes the work of Halpern et al. (2020), from binary additive valuations to the matroid-rank case.

Next, we establish that an analogous positive result cannot be achieved for MMS, even when considering truthfulness for individual agents. Specifically, we prove that, for matroid-rank valuations, there does not exist a truthful mechanism that is *index oblivious*, Pareto efficient, and maximin fair (Theorem 2). For establishing our results, we develop a characterization of truthful mechanisms under matroid-rank functions (Theorem 1). This characterization in fact holds for a broader class of valuations (specifically, holds for binary XOS functions) and might be of independent interest.

Additional Related Work. Barman and Verma (2021a) have shown that, under matroid-rank valuations, allocations that are MMS and Pareto efficient always exist. Hence, considering the existence of truthful mechanisms in this context is a well-posed question. Recall that in contrast to the matroid-rank case, under additive valuations, (exact) MMS allocations are not guaranteed to exist (Kurokawa, Procaccia, and Wang 2016; Procaccia and Wang 2014). Both positive and negative mechanism-design results have been obtained for various fairness notions under additive valuations; see (Amanatidis et al. 2017; Amanatidis, Birmpas, and Markakis 2016; Markakis and Psomas 2011), and references therein. These results are incomparable with the ones obtained in the current work, since they address additive valuations and, by contrast, we focus on matroid-rank functions.

Notation and Preliminaries

We study mechanisms, without money, for partitioning $[m] = \{1, 2, \ldots, m\}$ indivisible goods among $[n] = \{1, 2, \ldots, n\}$ agents in a fair and economically efficient manner. The cardinal preferences of the agents $i \in [n]$, over subsets of goods, are specified via valuation functions $v_i : 2^{[m]} \mapsto \mathbb{R}_+$; here, $v_i(S) \in \mathbb{R}_+$ denotes the value that agent $i \in [n]$ has for a subset of goods $S \subseteq [m]$. The valuation functions of all agents are collectively represented by a valuation profile $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. In this setup, an instance of the fair division problem corresponds to a tuple $\langle [m], [n], \mathbf{v} \rangle$. Our goal is to obtain fair and economically efficient allocations. Specifically, an *allocation* $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ is an n-partition of all the goods (i.e., $\cup_{i=1}^n A_i = [m]$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$) wherein subset A_i is assigned to agent $i \in [n]$. The assigned subsets will be referred to as bundles.

We will use the term *partial allocation* to refer to a collection of pairwise-disjoint subsets of goods $\mathcal{P} = (P_1, P_2, \ldots, P_n)$, in which subset P_i is assigned to agent i. In contrast to an allocation, for a partial allocation $\mathcal{P} = (P_1, \ldots, P_n)$ we may have $\cup_{i=1}^n P_i \subsetneq [m]$, i.e., it is not necessary that all the goods are assigned among the agents. Note that, for a partial allocation $\mathcal{P} = (P_1, \ldots, P_n)$, the set of goods $[m] \setminus \left(\bigcup_{i \in [n]} P_i \right)$ remain unallocated, and P is a

[2]Recall that a set function $v : 2^{[m]} \mapsto \mathbb{Z}_+$ is said to be submodular iff $v(S \cup \{g\}) - v(S) \geq v(T \cup \{g\}) - v(T)$, for all subsets $S \subseteq T \subseteq [m]$ and $g \notin T$

complete allocation iff $[m] \setminus \left(\bigcup_{i\in[n]} P_i\right) = \emptyset$.

We now define the notions of fairness and economic efficiency considered in this work.

Nash social welfare and Pareto optimality. The Nash social welfare $\text{NSW}(\cdot)$ of an (partial) allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is the geometric mean of the agents' values in $\mathcal{A}$, i.e., $\text{NSW}(\mathcal{A}) \coloneqq (\Pi_{i=1}^n v_i(A_i))^{1/n}$.

Given two (partial) allocations $\mathcal{A} = (A_1, A_2, \ldots A_n)$ and $\mathcal{A}' = (A'_1, A'_2, \ldots A'_n)$, we say that $\mathcal{A}$ *Pareto dominates* $\mathcal{A}'$ iff for every agent $i \in [n]$, we have $v_i(A_i) \geq v_i(A'_i)$ and this inequality is strict for at least one agent. An allocation is referred to as *Pareto optimal* (or PO) iff there is no other allocation that Pareto-dominates it. Throughout, we will consider Pareto optimality across all allocations. In particular, under the valuations considered in the current work, there could exist partial allocations that are Pareto optimal among all (complete) allocations.

Fairness notions. This paper studies two prominent fairness criteria: envy-freeness up to one good (EF1) and maximin fairness (MMS). An (partial) allocation $\mathcal{P} = (P_1, \ldots, P_n)$ is said to be EF1 iff for every pair of agents $i, j \in [n]$, with $P_j \neq \emptyset$, there exists $g \in P_j$ such that $v_i(P_i) \geq v_i(P_j \setminus \{g\})$.

In a fair division instance $\langle [m], [n], \mathbf{v}\rangle$, the maximin share of agent $i \in [n]$ is defined as

$$\mu_i \coloneqq \max_{(X_1,\ldots,X_n)} \min_{j\in[n]} v_i(X_j).$$

Here, the maximization is considered over all possible allocations. With these agent-specific thresholds in hand, we say that an (partial) allocation $\mathcal{P} = (P_1, \ldots, P_n)$ is maximin fair (MMS) iff $v_i(P_i) \geq \mu_i$ for all agents $i \in [n]$.

Truthful Mechanisms. In the current context, a mechanism, $f(\cdot)$, is a mapping from (reported) valuation profiles $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ to (partial) allocations $\mathcal{A}$. That is, the mechanism asks each agent i to report a valuation function v_i and assigns the bundles from the (partial) allocation $(A_1, \ldots, A_n) = f(v_1, \ldots, v_n)$. We will solely address deterministic mechanisms. A key desiderata is to identify mechanisms f wherein it is in the best interest of each agent to report her true valuation to f, i.e., no agent can gain by misreporting her valuation. This requirement is formally realized through *truthfulness* (also referred to as strategy-proofness), and its stronger variant, *group strategy-proofness*.

A mechanism f is said to be *truthful* iff for each agent $i \in [n]$, any valuation profile $(v_1, v_2, \ldots, v_n)$, and any function v'_i we have $v_i(A_i) \geq v_i(A'_i)$; where $(A_1, \ldots, A_n) = f(v_1, v_2, \ldots, v_n)$ and $(A'_1, \ldots, A'_n) = f(v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_n)$. Indeed, truthfulness ensures that that agent i does not receive a higher-valued (under her true valuation) bundle by misreporting her valuation to be v'_i. The notion can be strengthened by considering subsets of misreporting agents, instead of a single agent.

Definition 1 (Group Strategy-Proofness). *A mechanism f is said to be group strategy-proof iff for each subset of agents $C \subseteq [n]$ and any pair of valuation profiles $(v_1, v_2, \ldots, v_n)$ and $(v'_1, v'_2, \ldots, v'_n)$, with the property that $v_j = v'_j$ for all $j \notin C$, we necessarily have $v_i(A_i) \geq v_i(A'_i)$ for some agent $i \in C$.*

This definition equivalently asserts that for a group strategy-proof mechanism f there does not exist a subset of colluding agents C such that *all* of them gain (i.e., $v_i(A'_i) > v_i(A_i)$ for all $i \in C$) by misreporting together.

An even more demanding notion is that of *strong group strategy-proofness* that requires the nonexistence of any colluding subset of agents C wherein $v_i(A'_i) \geq v_i(A_i)$ for all $i \in C$, and the inequality has to be strict for at least one agent in C. Notably, with this stringent notion, one cannot achieve Pareto efficiency (let alone fairness) for the valuations considered in this work (Babaioff, Ezra, and Feige 2021; Bogomolnaia and Moulin 2004).

We will throughout say that a mechanism f is Pareto efficient iff, for any given valuation profile $\mathbf{v}$, the mechanism outputs an (partial) allocation that is Pareto optimal (with respect to $\mathbf{v}$). Similarly, a mechanism is said to be EF1 (respectively MMS) iff, for any given valuation profile, it outputs an (partial) allocation that is EF1 (respectively MMS).

The current work addresses fair division instances in which, for each agent $i \in [n]$, the valuation function v_i is the rank function of a matroid $\mathcal{M}_i = ([m], \mathcal{I}_i)$. Below we define rank functions and other relevant notions from matroid theory.

For subsets $X \subseteq [m]$ and goods $g \in [m]$, we will use the following shorthands: $X + g \coloneqq X \cup \{g\}$, $X - g \coloneqq X \setminus \{g\}$ and, $\overline{X} \coloneqq [m] \setminus X$. Also, for notational convenience, we will write $\overline{g}$ to denote the subset $[m] \setminus \{g\}$.

Matroid Preliminaries

A pair $([m], \mathcal{I})$ is said to be a matroid iff $\mathcal{I}$ is a nonempty collection of subsets of $[m]$ (i.e., $\mathcal{I} \subseteq 2^{[m]}$) that satisfies (a) Hereditary property: if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$, and (b) Augmentation property: if $X, Y \in \mathcal{I}$ and $|Y| < |X|$, then there exists $g \in X \setminus Y$ such that $Y + g \in \mathcal{I}$. Given a matroid $\mathcal{M} = ([m], \mathcal{I})$, a subset $I \subseteq [m]$ is said to be *independent* iff $I \in \mathcal{I}$.

For a matroid $\mathcal{M} = ([m], \mathcal{I})$, the rank function $r : 2^{[m]} \mapsto \mathbb{Z}_+$ captures, for each subset $X \subseteq [m]$, the cardinality of the largest independent subset contained in X, i.e., $r(X) \coloneqq \max\{|I| \ : \ I \in \mathcal{I} \text{ and } I \subseteq X\}$.

Rank functions bear binary marginals: $r(X \cup \{g\}) - r(X) \in \{0, 1\}$, for all subsets $X \subseteq [m]$ and $g \in [m]$. Also, by definition, rank functions are nonnegative ($r(X) \geq 0$ for all $X \subseteq [m]$) and monotone ($r(Y) \leq r(X)$ for all $Y \subseteq X$). Furthermore, the following characterization is well known (Schrijver 2003): any submodular function r with binary marginals is in fact a matroid-rank function.

Note that, if agent i's valuation v_i is the rank function of matroid $\mathcal{M}_i = ([m], \mathcal{I}_i)$, then, for any subset $S \subseteq [m]$, we have $v_i(S) \leq |S|$; here, equality holds iff S is an independent set in $\mathcal{M}_i$, i.e., $S \in \mathcal{I}_i$. With this observation, we next define non-wasteful allocations and mechanisms.

Non-Wasteful Mechanism. Under valuations $v_1, \ldots, v_n$, an (partial) allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is said to be *non-wasteful* iff, for each agent $i \in [n]$, the assigned bundle's value $v_i(A_i) = |A_i|$. Hence, for matroid-rank valuations, this defining condition corresponds to $A_i \in \mathcal{I}_i$, for each $i \in [n]$. Furthermore, a mechanism f is called *non-wasteful* if it yields non-wasteful allocations for all input valuation profiles. Note that any truthful mechanism f can be converted into one that is both truthful and non-wasteful: for every profile $\mathbf{v}$ and $(A_1, \ldots, A_n) = f(\mathbf{v})$, the corresponding mechanism returns a largest-cardinality independent subset $A'_i \subseteq A_i$, for each $i \in [n]$. Indeed, $(A'_1, \ldots, A'_n) \in \mathcal{I}_1 \times \ldots \times \mathcal{I}_n$ is a non-wasteful allocation and $v_i(A'_i) = v_i(A_i)$ for all agents i. We will establish a stronger result (Proposition **??**) showing that non-wastefulness can be achieved with additional properties and, hence, in relevant contexts it can be assumed without loss of generality.

Exchange Graph and Path Augmentation. We will use certain well-known constructs from matroid theory. In particular, *exchange graphs* and the related *path augmentation* operation will be utilized while establishing the results in Section 5.

Consider a setting wherein, for each agent $i \in [n]$, the valuation v_i is the rank function of matroid $\mathcal{M}_i = ([m], \mathcal{I}_i)$. Here, given a non-wasteful (partial) allocation $\mathcal{A} = (A_1, \ldots, A_n) \in \mathcal{I}_1 \times \ldots \times \mathcal{I}_n$, we define the *exchange graph*, $\mathcal{G}(\mathcal{A})$, to be a directed graph where the set of vertices is $[m]$ (i.e., each vertex corresponds to a good) and there is a directed edge (g, g') in the graph iff for some $i \in [n]$, the good $g \in A_i$, $g' \notin A_i$, and $A_i - g + g' \in \mathcal{I}_i$; hence, exchanging good g with g' in the bundle A_i maintains independence (with respect to $\mathcal{I}_i$).

Now we define the *path augmentation* operation. If $P = (g_1, g_2, \ldots, g_k)$ is a directed path in the exchange graph $\mathcal{G}(\mathcal{A})$, then we define bundle[3] $A_i \Delta P := A_i \ \Delta \ \{g_j, g_{j+1} : g_j \in A_i\}$ for all $i \in [n]$ [4]. Hence, $A_i \Delta P$ is obtained by exchanging goods along every edge of P that goes out of the set A_i.

Given an agent $i \in [n]$ and an independent set $X \in \mathcal{I}_i$, we define $F_i(X)$ to be the set of goods that can be added to X, while still maintaining its independence, i.e., $F_i(X) := \{g \in [m] \setminus X : X + g \in \mathcal{I}_i\}$.

In Section 5, we will use the following well-known result (stated in our notation) about the augmentation operation to prove the group strategy-proofness result. This lemma ensures that if the augmentation is performed along a shortest path[5] in the exchange graph, then independence of all bundles is maintained.

Lemma 1 ((Schrijver 2003))**.** *Let $\mathcal{X}' = (X'_1, \ldots, X'_n)$ be any non-wasteful (partial) allocation. Additionally, for a pair of agents $i, j \in [n]$, let $Q = (g_1, g_2, \ldots, g_t)$ be a shortest path in the exchange graph $\mathcal{G}(\mathcal{X}')$ between the vertex sets $F_i(X'_i)$ and X'_j (in particular, $g_1 \in F_i(X'_i)$ and $g_t \in X'_j$). Then, for all $k \in [n] \setminus \{i, j\}$, we have $X'_k \Delta Q \in \mathcal{I}_k$, along with $(X'_i \Delta Q) + g_1 \in \mathcal{I}_i$ and $X'_j - g_t \in \mathcal{I}_j$.*

As per the above lemma, if we perform augmentation along a shortest path between $F_i(X'_i)$ and X'_j, we get a new non-wasteful (partial) allocation in which the valuation of agent i increases by one and that of j decreases by one; the valuations of all other agents remain unchanged.

[3]Recall that the *symmetric difference* of any two subsets A and B is defined as $A \Delta B := (A \setminus B) \cup (B \setminus A)$.

[4]If path P is just a single vertex, then define $A_i \Delta P := A_i$

[5]Following standard terminology, a shortest path between two vertex sets is a path with the fewest number of edges among all paths that connect the two vertex sets.

Characterizing Truthfulness

This section develops a characterization (under matroid-rank valuations) of mechanisms that are truthful and non-wasteful. Recall that in the case of matroid-rank valuations, by definition, non-wasteful mechanisms—for all input valuation profiles—output allocations $(A_1, \ldots, A_n)$ comprised of independent bundles, $A_i \in \mathcal{I}_i$, for all $i \in [n]$.

As mentioned previously, from any truthful mechanism f, one can obtain a value-equivalent mechanism f' that is both truthful and non-wasteful. That is, given truthfulness, one can assume non-wastefulness without loss of generality. We will establish a stronger result (Proposition **??**) showing that non-wastefulness can in fact be achieved with additional properties and, hence, in relevant contexts it can be assumed without loss of generality.

We will use the following notation. Let $v : 2^{[m]} \mapsto \mathbb{R}_+$ be a valuation function and $X \subseteq [m]$ a subset of goods. Then, write $v^X(\cdot)$ to denote the function obtained by restricting v to the subset X, i.e., $v^X(S) := v(S \cap X)$, for each $S \subseteq [m]$. One can verify that if v is a matroid-rank function, then so is v^X, for any subset X.

Also, for notional convenience we will write v^{-g} for $v^{[m]\setminus\{g\}}$, i.e., v^{-g} is the valuation obtained by removing good g from consideration. Furthermore, for a valuation profile $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ along with agent $i \in [n]$ and function v'_i, we write (v'_i, v_{-i}) to denote the profile wherein the valuation of agent i is v'_i and the valuations of all the other agents remain unchanged.

Our characterization asserts that any non-wasteful mechanism f is truthful iff it is *gradual* (see Definition 2 below). Intuitively, this notion captures the idea that the output of the mechanism changes "gradually" under specific misreports: if an agent i misreports by excluding one good from her valuation (i.e., reports v_i^{-g} instead of v_i), then the number of goods assigned to her change by at most one. Also, if bundle A_i is assigned to an agent i, then (mis)reporting a valuation that is restricted to a superset $X \supseteq A_i$ does not change the number of goods assigned to i.

Definition 2 (Gradual Mechanism)**.** *A non-wasteful mechanism f is said to be gradual iff for any agent $i \in [n]$, any valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$, and allocation $(A_1, \ldots, A_n) = f(\mathbf{v})$ we have*
(C_1): $0 \leq |A_i| - |B_i| \leq 1$, for any good $g \in [m]$ and corresponding allocation $(B_1, \ldots, B_n) = f\left(v_i^{-g}, v_{-i}\right)$, and
(C_2): $|A_i| = |B_i|$, for any superset $X \supseteq A_i$ and corresponding allocation $(B_1, \ldots, B_n) = f\left(v_i^{X}, v_{-i}\right)$.

Note that this definition imposes conditions only on the sizes of bundles allocated by the mechanism and not on the valuations per se. Also, a repeated application of condition C_1 gives us
(C_1^*): $0 \leq |A_i| - |B_i| \leq |Y|$, for any set $Y \subseteq [m]$ and corresponding allocation $(B_1, \ldots, B_n) = f(v_i^{[m]\setminus Y}, v_{-i})$.

The following theorem is the main result of this section; see the full-version of this paper for the proof (Barman and Verma 2021b).

Theorem 1. *Under matroid-rank valuations, a non-wasteful mechanism f is truthful iff it is gradual.*

Remark: The above-mentioned characterization in fact holds for binary XOS valuations, a class of functions which encapsulates matroid-rank functions. We defer details of this observation to the full-version of this paper.

Impossibility Result

This section establishes a notable separation between EF1 and MMS in the current mechanism-design context. We show that, under matroid-rank valuations, truthful mechanisms (satisfying some additional, desirable properties) do not exist for maximin fairness. By contrast, EF1 admits such truthful mechanisms; in fact, for EF1 we have a stronger positive result guaranteeing group strategy-proofness.

We begin by defining the key concepts for our impossibility result. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be any valuation profile and $\pi : [m] \mapsto [m]$ be a permutation of the set of goods. Note that, since π is a bijection, for each good g, the inverse (pre-image) $\pi^{-1}(g)$ is unique. For each agent $i \in [n]$, we define the function

$$v_i^\pi(S) := v_i\left(\left\{\pi^{-1}(g)\right\}_{g \in S}\right) \quad \text{for all subsets } S \subseteq [m].$$

That is, after reindexing the goods via π, one would obtain the value—with respect to the initial valuation v_i—of any subset of goods by applying v_i^π; see, Figure 1 for an illustration.

Intuitively, the valuation profile $\mathbf{v}^\pi := (v_1^\pi, \ldots, v_n^\pi)$ represents the same set of preferences as in profile $\mathbf{v}$, the only difference is that the goods have been reindexed. The following definition aims to capture the idea that indexing of goods should *not* influence the values that the agents' receive. We will use $\pi(S)$ to denote the set $\{\pi(g) : g \in S\}$ and $\pi^{-1}(S)$ for $\{\pi^{-1}(g) : g \in S\}$.

Definition 3 (Index-Oblivious Mechanism)**.** *A mechanism f is said to be index-oblivious iff, given any valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ and any permutation $\pi : [m] \mapsto [m]$, for the (partial) allocations $(A_1, \ldots, A_n) = f(\mathbf{v})$ and $(A'_1, \ldots, A'_n) = f(\mathbf{v}^\pi)$, we have $v_i(A_i) = v_i^\pi(A'_i)$ for all agents $i \in [n]$.*

Note that if a mechanism is *not* index-oblivious, then certain ways of indexing the goods could be advantageous for some agents and disadvantageous for others. However, indexing of goods should ideally be irrelevant. This observation supports index-obliviousness as a reasonable robustness criterion for mechanisms.

Theorem **??** (proved in the full-version (Barman and Verma 2021b)) shows that for EF1 there exists an index-oblivious mechanism; in particular, we show that the Prioritized Egalitarian (PE) mechanism of Babaioff et al. (2021) is index-oblivious. It is known that, under matroid-rank valuations, PE outputs EF1 allocations and it is truthful as well as Pareto efficient (Babaioff, Ezra, and Feige 2021). The following theorem proves that an analogous result is not possible for MMS.

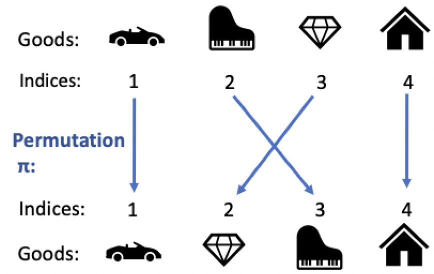

Figure 1: In this example permutation π reindexes the goods. Then, the value $v_i^\pi(\{1,2\}) = v_i(\pi^{-1}(\{1,2\})) = v_i(\{1,3\})$, which is equal to the value of car and diamond.

Theorem 2. *Under matroid-rank valuations, there does not exist a mechanism that is truthful, index-oblivious, Pareto efficient, and maximin fair.*

Proof. We assume, towards a contradiction, that there exists a mechanism f that is truthful, index-oblivious, and it outputs Pareto efficient and MMS allocations. We also assume that f is non-wasteful, this assumption can be made without loss of generality; see the full-version (Barman and Verma 2021b). To derive the desired contradiction, we will construct an instance wherein an agent can always benefit by misreporting her valuation. In particular, consider a setting with $n = 2$ agents and $m = 6$ goods, say $\{g_1, g_2, \ldots, g_6\}$. Fix subset $G := \{g_1, g_2\}$ and consider the following valuations for the two agents, respectively: $v_1(S) := |S \cap G|$ and $v_2(S) := \min\{1, |S \cap G|\} + \min\{2, |S \cap ([m] \setminus G)|\}$, for all subsets $S \subseteq [m]$. These valuations are rank functions of (partition) matroids. Note that under valuations v_1 and v_2, the maximin shares of the two agents are $\mu_1 = 1$ and $\mu_2 = 3$, respectively.

Furthermore, write allocation $(A_1, A_2) = f(v_1, v_2)$. Given the (assumed) properties of f, the (partial) allocation (A_1, A_2) is MMS. Hence, $v_1(A_1) \geq \mu_1 = 1$ and $v_2(A_2) \geq \mu_2 = 3$. By definition, the valuation v_2 is at most 3, and to achieve the above-mentioned MMS bound agent 2 must receive at least one good from $G = \{g_1, g_2\}$. The MMS guarantee for agent 1 implies that she also receives at least one good from G. These observations, along with the fact that (A_1, A_2) is a non-wasteful allocation ($|A_1| = v_1(A_1)$ and $|A_2| = v_2(A_2)$), ensures that agents 1 and 2 receive a bundle of size 1 and 3, respectively.

Write $a \in G = \{g_1, g_2\}$ to denote the good assigned to agent 1, and $b \in G$ be the other good in G allocated to agent 2, i.e., $A_1 = \{a\}$ and $A_2 = \{b, c, d\}$, for two goods $c, d \in [m] \setminus G$. Based on the three goods assigned to agent 2, we will define three valuations profiles $\mathbf{w}^b$, $\mathbf{w}^c$, $\mathbf{w}^d$, and show that agent 1 would benefit by misreporting from one at least of them. This will contradict the assumption that f is truthful and, hence, establish the theorem. In all the these three profiles, we set the second agent's valuation as $w_2(S) := |S \cap \{b, c, d\}|$, for all subsets $S \subseteq [m]$. For each $x \in \{b, c, d\}$, let function $w_1^x(S) := |S \cap \{a, x\}|$ (for all subsets S) and write profile $\mathbf{w}^x = (w_1^x, w_2)$.

A key technical step in the proof is to show that under all the three profiles $\mathbf{w}^x$ the two agents continue to receive the same bundles A_1 and A_2, respectively, i.e., for each $x \in \{b, c, d\}$ we have $(A_1, A_2) = f(\mathbf{w}^x)$; these three equalities are proved in the full version (Barman and Verma 2021b).

We now complete the proof by showing that agent 1 would benefit by misreporting in at least one of these three profiles. In particular, define the valuation (misreport) $w_1^*(S) := |S \cap \{a, b, c, d\}|$, for all subsets $S \subseteq [m]$. Under the valuation w_1^*, the maximin share of agent 1 is equal to two. Hence, given valuation profile (w_1^*, w_2), mechanism f (to maintain maximin fairness) must assign agent 1 a bundle of value 2. That is, for allocation $(A_1^*, A_2^*) = f(w_1^*, w_2)$, we have $w_1^*(A_1^*) \geq 2$. Furthermore, since the returned (partial) allocation must be Pareto optimal and good a is only desired by agent 1, good a must be allocated to the first agent, $a \in A_1^*$. These observation imply that A_1^* additionally contains at least one good from the set $\{b, c, d\}$; write $y \in A_1^* \cap \{b, c, d\}$ to denote that good. Now, consider the case wherein agent 1's true valuation is w_1^y (and that of agent 2 is w_2). Since $y \in \{b, c, d\}$, the above-mentioned claims imply that $(A_1, A_2) = f(w_1^y, w_2)$, i.e., reporting w_1^y truthfully agent 1 receives a bundle of size $|A_1| = 1$. However, misreporting her valuation to be w_1^*, agent 1 receives both the goods a and y (since $a, y \in A_1^*$). Therefore, $w_1^y(A_1^*) > w_1^y(A_1)$, and this contradicts the truthfulness of f. The theorem stands proved. ☐

Group Strategy-Proofness for EF1

This section establishes group strategy-proofness for a mechanism of Babaioff et al. (2021), called the prioritized egalitarian (PE) mechanism. This mechanism relies on finding *Lorenz dominating* allocations; we define this notion next. For any allocation $\mathcal{A}$ and valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$, write $\mathbf{s}_{\mathcal{A}} = (s_1, s_2, \ldots, s_n)$ to denote the vector wherein all the components of $(v_1(A_1), v_2(A_2) \ldots, v_n(A_n))$ appear in non-decreasing order, i.e., s_1 denotes the lowest valuation across the agents, s_2 is the second lowest, and so on. We say that allocation $\mathcal{A}$ Lorenz dominates another allocation $\mathcal{A}'$ iff, for every index $j \in [n]$, the sum of the first j components of $\mathbf{s}_{\mathcal{A}}$ is at least as large as the sum of the first j components of $\mathbf{s}_{\mathcal{A}'}$, i.e., iff $\mathbf{s}_{\mathcal{A}}$ majorizes $\mathbf{s}_{\mathcal{A}'}$ (see Marshall et al. (1994) for a detailed treatment of majorization). An (partial) allocation $\mathcal{A}$ is said to be *Lorenz dominating* iff $\mathcal{A}$ Lorenz dominates all other allocations. Notably, under matroid-rank valuations, Lorenz dominating allocations always exist (Babioff et al. 2021).

The PE mechanism is detailed below.[6] Our proof for the group strategy-proofness of this mechanism relies on the following technical lemma (see the full-version (Barman and Verma 2021b) for its proof). Here, we use the notation mentioned earlier; in particular, for any agent i and independent set $A_i \in \mathcal{I}_i$, we write $F_i(A_i) := \{g \in [m] \setminus A_i : A_i + g \in \mathcal{I}_i\}$. Also, given any two allocations $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{X} = (X_1, \ldots, X_n)$, define the subsets of agents $L(\mathcal{X}, \mathcal{A}) := \{i \in [n] : |X_i| < |A_i|\}$ and $H(\mathcal{X}, \mathcal{A}) := \{i \in [n] : |X_i| > |A_i|\}$.

[6]Babaioff et al. (2021) show that PE can be executed in polynomial time, in particular when the matroid-rank functions admit a succinct representation.

Mechanism 1: Prioritized Egalitarian (PE) (Babaioff, Ezra, and Feige 2021)

Input: Valuation profile $(v_1, v_2, \ldots, v_n)$ consisting of the reported (matroid-rank) valuations of all the agents.
Output: A non-wasteful Lorenz dominating allocation $\mathcal{A} = (A_1, A_2, \ldots, A_n)$.

1: For the given profile $(v_1, v_2, \ldots, v_n)$, compute a non-wasteful Lorenz dominating allocation $\mathcal{A} = (A_1, A_2, \ldots, A_n)$, breaking ties in favor of agents with lower indices. Equivalently, among all (non-wasteful) Lorenz dominating allocations, select one, $(A_1, A_2, \ldots, A_n)$, that lexicographically maximizes the vector $(v_1(A_1), v_2(A_2), \ldots, v_n(A_n))$ (i.e., the Lorenz dominating allocation maximizes $v_1(A_1)$, and then subject to that it maximizes $v_2(A_2)$, and so on).
2: **return** $\mathcal{A} = (A_1, A_2, \ldots, A_n)$

Lemma 2. *Let $\mathcal{X} = (X_1, \ldots, X_n)$ be a non-wasteful allocation and $\mathcal{A} = (A_1, \ldots, A_n)$ be a Pareto-efficient non-wasteful allocation such that there exists an agent $h \in H(\mathcal{X}, \mathcal{A})$. Then, there exists a simple directed path $P = (g_k, g_{k-1}, \ldots, g_2, g_1)$ in the exchange graph $\mathcal{G}(\mathcal{X})$ that satisfies the following two properties*

1. *For the path P, the source vertex $g_k \in A_\ell \cap F_\ell(X_\ell)$, for some agent $\ell \in L(\mathcal{X}, \mathcal{A})$ and its sink vertex, $g_1 \in X_h$ (with $h \in H(\mathcal{X}, \mathcal{A})$).*
2. *For each agent $i \in [n] \setminus \{h\}$, the subsets $A'_i := A_i \Delta \{g_{j+1}, g_j : g_j \in A_i\} \in \mathcal{I}_i$ and $A'_h := A_h \Delta \{g_{j+1}, g_j : g_j \in A_h\} + g_1 \in \mathcal{I}_h$.*[7]

Before presenting the main result of this section (Theorem 3), we will state another lemma which follows from prior work.

Lemma 3. *Under matroid-rank valuations, a non-wasteful (partial) allocation $\mathcal{A} = (A_1, \ldots, A_n)$ maximizes Nash social welfare iff $\mathcal{A}$ is Lorenz dominating.*

Theorem 3. *The* PE *mechanism is group strategy-proof.*

Proof. Assume, towards a contradiction, that the mechanism PE is not group strategy-proof. Specifically, let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a valuation profile wherein a subset of agents $C \subseteq [n]$ can benefit by misreporting to profile $\mathbf{v}' = (v'_1, v'_2, \ldots, v'_n)$; here $v'_j = v_j$ for all agents $j \notin C$. Also, write allocations $\mathcal{A} = (A_1, A_2, \ldots, A_n) = \text{PE}(\mathbf{v})$ and $\mathcal{X} = (X_1, X_2, \ldots, X_n) = \text{PE}(\mathbf{v}')$. Since all the misreporting agents $i \in C$ gain by misreporting, we have $v_i(X_i) > v_i(A_i)$, for each $i \in C$.

Let $\mathcal{X}' = (X'_1, X'_2, \ldots, X'_n)$ be the non-wasteful (partial) allocation within $\mathcal{X}$: for each agent $i \in [n]$, select bundle $X'_i := \arg\max_{S \subseteq X_i} \{|S| : S \in \mathcal{I}_i\}$ and note that $v_i(X_i) = v_i(X'_i) = |X'_i|$. Also, write $B \subseteq C$ to denote the subset

[7]In contrast to the augmentation Δ considered in Lemma 1, here we swap along edges that end in A_i-s. Also, note the indexing of vertices along path P.

of misreporting agents that receive the smallest-size bundle, $B \coloneqq \arg\min_{i \in C} |X_i|$. Considering agents in B, we write h to denote the one with the smallest index, i.e., agent h has the lowest value for $|X_h|$ among all agents $i \in C$, and subject to that, she has the smallest index.

The remainder of the proof will be from the perspective of the valuation profile $\mathbf{v}$, unless stated otherwise. Note that, the allocation $\mathcal{X}'$ is non-wasteful and allocation $\mathcal{A}$ is a Pareto efficient, since it is Lorenz dominating. Furthermore, for agent $h \in C$, we have $|X'_h| = v_h(X_h) > v_h(A_h) = |A_h|$; equivalently $h \in H(\mathcal{X}', \mathcal{A})$. Hence, Lemma 2 ensures the existence of a path $P = (g_k, \ldots, g_1)$ in $\mathcal{G}(\mathcal{X}')$ that satisfies conditions 1 and 2 mentioned in the lemma statement. From condition 1, we know that path P starts at $F_\ell(X'_\ell)$ (for some agent $\ell \in L(\mathcal{X}', \mathcal{A})$) and ends at X'_h.

For establishing the theorem, i.e., to arrive at a contradiction, we will prove two properties, $\mathcal{P}_1$ and $\mathcal{P}_2$, that directly contradict each other:

$$\text{Property } \mathcal{P}_1 : \begin{cases} |A_h| < |A_\ell| - 1, & \text{if } h > \ell \\ |A_h| \leq |A_\ell| - 1, & \text{otherwise, if } h < \ell. \end{cases}$$

$$\text{Property } \mathcal{P}_2 : \begin{cases} |A_h| \geq |A_\ell| - 1, & \text{if } h > \ell \\ |A_h| > |A_\ell| - 1, & \text{otherwise, if } h < \ell \end{cases}$$

We will derive $\mathcal{P}_1$ by using condition 1 (from Lemma 2) satisfied by path P and Lemma 1. Property $\mathcal{P}_2$ will be obtained by the fact that P satisfies condition 2 in Lemma 2. Hence, we complete the proof by establishing $\mathcal{P}_1$ and $\mathcal{P}_2$ next.

Property $\mathcal{P}_1$: Recall that path P starts at $F_\ell(X'_\ell)$, for some agent $\ell \in L(\mathcal{X}', \mathcal{A})$ and ends at X'_h with $h \in H(\mathcal{X}', \mathcal{A})$. Since all the agents in C gain (by misreporting), $C \subseteq H(\mathcal{X}', \mathcal{A})$. Therefore, the fact that $\ell \in L(\mathcal{X}', \mathcal{A})$ gives us $\ell \notin C$. Now, write Q to denote a shortest path from $F_\ell(X'_\ell)$ to $\cup_{i \in C} X'_i$; the existence of P guarantees that such a path exists. Further, assume that Q ends at X'_b for some $b \in C$. From the definition of agent h, we have[8]

$$|X_h| \leq |X_b| \text{ if } h \leq b \text{ and } |X_h| < |X_b| \text{ if } h > b \quad (1)$$

Furthermore, using the facts that $|X'_h| \geq |A_h| + 1$ (since $h \in H(\mathcal{X}', \mathcal{A})$) and $|X_h| \geq |X'_h|$ (since $X'_h \subseteq X_h$), equation (1) reduces to

$$|A_h| + 1 \leq |X_b| \text{ if } h \leq b \;\; \& \;\; |A_h| + 2 \leq |X_b| \text{ if } h > b \quad (2)$$

Note that the path Q in $\mathcal{G}(\mathcal{X}')$ is such that only its sink vertex lies in $\cup_{k \in C} X'_k$; all the other vertices in Q are present in bundles X'_j with $j \notin C$. For all agents $j \notin C$, the valuation functions v'_j and v_j are the same. Hence, the (non-wasteful) bundles X'_j and X_j are equal as well, for all $j \notin C$. These observations imply that the path Q also lies in the exchange graph $\mathcal{G}(\mathcal{X})$, where the graph is constructed with respect to matroids corresponding to the profile $\mathbf{v}'$. Therefore, allocation $\mathcal{X} = \mathrm{PE}(\mathbf{v}')$ can be augmented with path Q in $\mathcal{G}(\mathcal{X})$, which starts at $F_\ell(X_\ell)$ and ends at X_b (Lemma 1).

Indeed, performing path augmentation on $\mathcal{X}$, via Q, will increase $|X_\ell|$ by one, decrease $|X_b|$ by one, and the bundle sizes (and values) of other agents will remain unchanged. Also, since allocation $\mathcal{X}$ is optimal with respect to PE's selection criteria (and considering profile $\mathbf{v}'$), the distinct allocation obtained by path augmentation must be sub-optimal (again, with respect to PE's criteria). The Lorenz domination of $\mathcal{X}$ (equivalently its Nash optimality) ensures that $|X_b| \leq |X_\ell| + 1$; otherwise the resultant allocation will have higher Nash social welfare (under $\mathbf{v}'$). In fact, if $b > \ell$, then we must have $|X_b| \leq |X_\ell|$. Otherwise (i.e., in case $b > \ell$ and $|X_b| = |X_\ell| + 1$) , the resultant allocation will have the same Nash social welfare as $\mathcal{X}$ (i.e., the resultant allocation will also be Lorenz dominating) and would get preferred (over $\mathcal{X}$) under the lexicographic tie-breaking of PE. Therefore,

$$|X_b| \leq |X_\ell| \text{ if } b > \ell, \text{ and } |X_b| - 1 \leq |X_\ell| \text{ if } b < \ell \quad (3)$$

Using the bounds $|X'_\ell| \leq |A_\ell| - 1$ (since $\ell \in L(\mathcal{X}', \mathcal{A})$) and $|X_\ell| = |X'_\ell|$ (recall that $\ell \notin C$ and, hence, $v_\ell = v'_\ell$) along with equation (3), we obtain

$$|X_b| \leq |A_\ell| - 1 \text{ if } b > \ell, \text{ and } |X_b| \leq |A_\ell| \text{ if } b < \ell \quad (4)$$

Towards establishing property $\mathcal{P}_1$, we combine equations (2) and (4) by considering the following four cases

Case I. $h \leq b$ and $b > \ell$: $|A_h| + 1 \leq |X_b| \leq |A_\ell| - 1$

Case II. $h > b$ and $b < \ell$: $|A_h| + 2 \leq |X_b| \leq |A_\ell|$

Case III. $h > b$ and $b > \ell$: $|A_h| + 2 \leq |X_b| \leq |A_\ell| - 1$

Case IV. $h \leq b$ and $b < \ell$: $|A_h| + 1 \leq |X_b| \leq |A_\ell|$

Finally, we simplify the four cases above to obtain $\mathcal{P}_1$. Note that $h > \ell$ is possible only in *Cases* I, II, or III, and in all these cases we have $|A_h| + 2 \leq |A_\ell|$ or equivalently $|A_h| < |A_\ell| - 1$. Similarly, $h < \ell$ can happen only in *Cases* I, II, or IV and there we have $|A_h| + 1 \leq |A_\ell|$, which is same as $|A_h| \leq |A_\ell| - 1$. Therefore, we obtain property $\mathcal{P}_1$:

$$|A_h| < |A_\ell| - 1 \text{ if } h > \ell \;\; \& \;\; |A_h| \leq |A_\ell| - 1 \text{ if } h < \ell \quad (5)$$

Property $\mathcal{P}_2$: Here, we will use the fact that condition 2 in Lemma 2 is satisfied by the path $P = (g_k, \ldots, g_1)$. The condition implies that the bundles $\widehat{A}_h \coloneqq A_h \Delta \{g_{j+1}, g_j : g_j \in A_h\} + g_1 \in \mathcal{I}_h$ and $\widehat{A}_\ell \coloneqq A_\ell \Delta \{g_{j+1}, g_j : g_j \in A_\ell\} - g_k \in \mathcal{I}_\ell$ along with $\widehat{A}_i \coloneqq A_i \Delta \{g_{j+1}, g_j : g_j \in A_i\} \in \mathcal{I}_i$ for each $i \notin \{h, \ell\}$. Hence, the pairwise-disjoint bundles $\widehat{A}_1, \widehat{A}_2, \ldots, \widehat{A}_n$ form a non-wasteful allocation $\widehat{\mathcal{A}} = (\widehat{A}_1, \widehat{A}_2, \ldots, \widehat{A}_n)$ (with respect to profile $\mathbf{v}$).

Recall that $\mathcal{A} = \mathrm{PE}(\mathbf{v})$, and we have $|\widehat{A}_h| = |A_h| + 1$ along with $|\widehat{A}_\ell| = |A_\ell| - 1$; the bundle sizes of all the other agents remain unchanged. Furthermore, in contrast to $\mathcal{A}$, the distinct allocation $\widehat{A}$ must be sub-optimal under the selection criteria of PE (applied to valuation profile $\mathbf{v}$). Therefore, $|A_\ell| \leq |A_h| + 1$; otherwise $\widehat{\mathcal{A}}$ will have higher Nash social welfare (under $\mathbf{v}$) than $\mathcal{A}$, contradicting the optimality of $\mathcal{A}$ (see Lemma 3). In fact, if $h < \ell$, then we must have $|A_\ell| < |A_h| + 1$. Otherwise (i.e., in case $h < \ell$ and $|A_\ell| = |A_h| + 1$), the allocation $\widehat{\mathcal{A}}$ will have the same Nash social welfare as $\mathcal{A}$ (i.e., $\widehat{\mathcal{A}}$ will also be Lorenz dominating) and would get preferred (over $\mathcal{A}$) under the lexicographic tie-breaking of PE. These observations lead to property $\mathcal{P}_2$:

$$|A_h| \geq |A_\ell| - 1 \text{ if } h > \ell \;\; \& \;\; |A_h| > |A_\ell| - 1 \text{ if } h < \ell \quad (6)$$

This completes the proof, since properties $\mathcal{P}_1$ and $\mathcal{P}_2$ directly contradict each other. $\square$

[8]In fact, $|X_h| \leq |X_b|$ irrespective of the agents' indices.

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