Network Satisfaction for Symmetric Relation Algebras with a Flexible Atom

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Abstract

Robin Hirsch posed in 1996 the *Really Big Complexity Problem*: classify the computational complexity of the network satisfaction problem for all finite relation algebras **A**. We provide a complete classification for the case that **A** is symmetric and has a flexible atom; the problem is in this case NP-complete or in P. If a finite integral relation algebra has a flexible atom, then it has a normal representation $\mathfrak{B}$. We can then study the computational complexity of the network satisfaction problem of **A** using the universal-algebraic approach, via an analysis of the polymorphisms of $\mathfrak{B}$. We also use a Ramsey-type result of Nešetřil and Rödl and a complexity dichotomy result of Bulatov for conservative finite-domain constraint satisfaction problems.

Introduction

One of the earliest approaches to formalise constraint satisfaction problems over infinite domains is based on *relation algebras* (Ladkin and Maddux 1994; Hirsch 1997). We think about the elements of a relation algebra as binary relations; the algebra has operations for intersection, union, complement, converse, and composition of relations, and constants for the empty relation, the full relation, and equality, and is required to satisfy certain axioms. Important examples of relation algebras are the Point Algebra, the Left Linear Point Algebra, Allen's Interval Algebra, RCC5, and RCC8, just to name a few.

The so-called *network satisfaction problem* (NSP) for a finite relation algebra asks whether a given finite network of constraints is consistent with the relation algebra. NSPs can be used to model many computational problems in temporal and spatial reasoning (Düntsch 2005; Renz and Nebel 2007; Bodirsky and Jonsson 2017). In 1996, Robin Hirsch (1996) asked the *Really Big Complexity Problem (RBCP)*: can we classify the computational complexity of the network satisfaction problem for every finite relation algebra? For example, the complexity of the network satisfaction problem for the Point Algebra and the Left Linear Point Algebra is in P (Vilain, Kautz, and van Beek 1989; Bodirsky and Kutz 2007), while it is NP-complete for all of the other examples mentioned above (Allen 1983; Renz and Nebel 1999). There

also exist relation algebras where the complexity of the network satisfaction problem is not in NP: Hirsch gave an example of a finite relation algebra with an undecidable network satisfaction problem (Hirsch 1999). This result might be surprising at first sight; it is related to the fact that the representation of a finite relation algebra by concrete binary relations over some set can be quite complicated. We also mention that not every finite relation algebra has a representation (Lyndon 1950). There are even non-representable relation algebras that are symmetric (Maddux 2006); a relation algebra is *symmetric* if every element is its own converse.

A simple condition that implies that a finite relation algebra has a representation is the existence of a so-called *flexible atom* (Comer 1984; Maddux 1982), combined with the assumption that **A** is *integral*; formal definitions can be found in section "Preliminaries".Such relation algebras have been studied intensively, for example in the context of the so-called *flexible atoms conjecture* (Maddux 1994; Alm, Maddux, and Manske 2008). We will see that integral relation algebras with a flexible atom even have a *normal representation*, i.e., a representation which is fully universal, square, and homogeneous (Hirsch 1996). The network satisfaction problem for a relation algebra with a normal representation can be seen as a *constraint satisfaction problem* for an infinite structure $\mathfrak{B}$ that is well-behaved from a model-theoretic point of view; in particular, we may choose $\mathfrak{B}$ to be *homogeneous* and *finitely bounded*.

Constraint satisfaction problems over finite domains have been studied intensively in the past two decades, and tremendous progress has been made concerning systematic results about their computational complexity. In 2017, Bulatov (2017) and Zhuk (2017) announced proofs of the famous Feder-Vardi dichotomy conjecture which states that every finite-domain CSP is in P or NP-complete. Both proofs build on an important connection between the computational complexity of constraint satisfaction problems and central parts of universal algebra.

The universal-algebraic approach can also be applied to study the computational complexity of countably infinite homogeneous structures $\mathfrak{B}$ with finite relational signature (Bodirsky and Nešetřil 2006). If $\mathfrak{B}$ is finitely bounded, then CSP($\mathfrak{B}$) is contained in NP; see, e.g. (Bodirsky 2012). If $\mathfrak{B}$ is homogeneous and finitely bounded then a complexity dichotomy has been conjectured, along with a conjecture

about the boundary between NP-completeness and containment in P (Bodirsky, Pinsker, and Pongrácz 2019). We verify these conjectures for all normal representations of finite integral symmetric relation algebras with a flexible atom, and thereby also solve Hirsch's RBCP for symmetric relation algebras with a flexible atom.

The exact formulation of the conjecture from (Bodirsky, Pinsker, and Pongrácz 2019) in full generality requires concepts that we do not need to prove our results. Because of space restriction, we therefore only present the proof of the P versus NP-complete dichotomy in the present conference paper. The discussion why our proof also confirms the mentioned general conjecture about the border of NP-hardness and containment in P (unless P=NP) in the special case for normal representations of finite integral symmetric relation algebras with a flexible atom will appear in a journal version of the article. Phrased in the terminology of relation algebras, our result is the following.

Theorem 1. *Let* **A** *be a finite integral symmetric relation algebra with a flexible atom, and let* A_0 *be the set of atoms of* **A**. *Then either*

- *there exists an operation* $f \colon A_0^6 \to A_0$ *that preserves the allowed triples of* **A** *and satisfies the* Siggers identity

$$\forall x, y, z \in A_0 \colon f(x, x, y, y, z, z) = f(y, z, x, z, x, y);$$

in this case the network satisfaction problem for **A** *is in P, or*

- *the network satisfaction problem for* **A** *is NP-complete.*

This also implies a P versus NP-complete dichotomy theorem for network satisfaction problems of symmetric (not necessarily integral) relation algebras with a flexible atom, because for every finite relation algebra with a flexible atom there exists an integral relation algebra with a flexible atom and polynomial-time equivalent NSP (Proposition 22).

Proof Strategy

Every finite integral relation algebra **A** with a flexible atom has a normal representation $\mathfrak{B}$; for completeness, and since we are not aware of a reference for this fact, we include a proof in section "Relation Algebras with a Flexible Atom".It follows that the classification question about the complexity of the NSP of **A** can be translated into a question about the complexity of the constraint satisfaction problem for the relational structure $\mathfrak{B}$. It is well known that $\mathrm{CSP}(\mathfrak{B})$ is contained in NP; see, e.g. (Bodirsky 2018).

We associate a finite relational structure $\mathfrak{O}$ to $\mathfrak{B}$ and show that $\mathrm{CSP}(\mathfrak{B})$ can be reduced to $\mathrm{CSP}(\mathfrak{O})$ in polynomial-time (section "Polynomial-time Tractability"). If the structure $\mathfrak{O}$ satisfies the condition of the first case in Theorem 1, then known results about finite-domain CSPs imply that $\mathrm{CSP}(\mathfrak{O})$ can be solved in polynomial-time (Bulatov 2003, 2016; Barto 2011), and hence $\mathrm{CSP}(\mathfrak{B})$ is in P, too. If the first case in Theorem 1 does not apply, then known results about finite-domain algebras imply that there are $a, b \in A_0$ such that the *canonical polymorphisms* of $\mathfrak{B}$ act as a projection on $\{a, b\}$ (Bulatov 2003, 2016; Barto 2011). We first show NP-hardness of $\mathrm{CSP}(\mathfrak{B})$ if $\mathfrak{B}$ does not have a binary injective

polymorphism. If $\mathfrak{B}$ has a binary injective polymorphism, we use results from structural Ramsey theory to show that $\mathfrak{B}$ must even have a binary injective polymorphism which is canonical. This implies that none of a, b equals $\mathrm{Id} \in A$. We then prove that $\mathfrak{B}$ does not have a binary $\{a, b\}$-symmetric polymorphism; also in this step, we apply Ramsey theory. This in turn implies that all polymorphisms of $\mathfrak{B}$ must be canonical on $\{a, b\}$. Finally, we show that $\mathfrak{B}$ cannot have a polymorphism which acts as a majority or as a minority on $\{a, b\}$, and thus by Schaefer's theorem all polymorphisms of $\mathfrak{B}$ act as a projection on $\{a, b\}$. It follows that $\mathrm{CSP}(\mathfrak{B})$ is NP-hard. This concludes the proof of Theorem 1.

Detailed proofs of all theorems, propositions, and lemmas had to be omitted because of the space restrictions, but can be found in the upcoming journal version or on arXiv (Bodirsky and Knäuer 2020b).

Preliminaries

We recall some basic definitions and results about relation algebras, constraint satisfaction, model theory, universal algebra, and structural Ramsey theory.

Relation Algebras and Network Satisfaction Problems

For relation algebras that are not representable the set of yes-instances of the network satisfaction problem is empty (see Def. 5). We thus omit the definition of relation algebras and start immediately with the simpler definition of *representable relation algebras*; here we basically follow the textbook of Maddux (2006).

Definition 2. *Let* D *be a set and* $E \subseteq D^2$ *an equivalence relation. Let* $(\mathcal{P}(E); \cup, \bar{\ }, 0, 1, \mathrm{Id}, \check{\ }, \circ)$ *be an algebra with the following operations:*

1. $a \cup b := \{(x, y) \mid (x, y) \in a \vee (x, y) \in b\}$,
2. $\bar{a} := E \setminus a$,
3. $0 := \emptyset$,
4. $1 := E$,
5. $\mathrm{Id} := \{(x, x) \mid x \in D\}$,
6. $a^{\check{\ }} := \{(x, y) \mid (y, x) \in a\}$,
7. $a \circ b := \{(x, z) \mid \exists y \in D : (x, y) \in a \text{ and } (y, z) \in b\}$.

A subalgebra of $(\mathcal{P}(E); \cup, \bar{\ }, 0, 1, \mathrm{Id}, \check{\ }, \circ)$ *is called a* proper relation algebra.

The class of *representable relation algebras*, denoted by RRA, consists of all algebras such that the tuple of the operation-arities is $(2, 1, 0, 0, 0, 1, 2)$ and that are isomorphic to some proper relation algebra. We use bold letters (such as **A**) to denote algebras from RRA and the corresponding roman letter (such as A) to denote the domain of the algebra. An algebra is called *finite* if its domain is finite. We call **A** $\in$ RRA *symmetric* if all its elements are symmetric, i.e., $a^{\check{\ }} = a$ for every $a \in A$.

To link the theory of relation algebras with model theory, it will be convenient to view representations of algebras in RRA as relational structures.

Definition 3. *Let* **A** $\in$ RRA. *Then a* representation of **A** *is a relational structure* $\mathfrak{B}$ *such that*
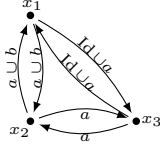
Figure 1: Multiplication table of the relation algebra #17 and an example instance.

- $\mathfrak{B}$ *is an A-structure, i.e., the elements of A are the relation symbols of* $\mathfrak{B}$*;*
- *The map* $a \mapsto a^{\mathfrak{B}}$ *is an isomorphism between* **A** *and the proper relation algebra induced by the relations of* $\mathfrak{B}$ *in* $(\mathcal{P}(1^{\mathfrak{B}}); \cup, \bar{\ }, 0, 1, \mathrm{Id}, \smile, \circ)$.

We write $\leq$ for the partial order on A defined by $x \leq y :\Leftrightarrow x \cup y = y$. Note that for proper relation algebras, this ordering coincides with the set-inclusion order. The minimal elements of this order in $A \setminus \{0\}$ are called *atoms*. The set of atoms of **A** is denoted by A_0. A tuple $(x, y, z) \in (A_0)^3$ is called an *allowed triple* if $z \leq x \circ y$. Otherwise, (x, y, z) is called a *forbidden triple*.

Definition 4. *Let* **A** $\in$ RRA. *An* **A**-network $(V; f)$ *is a finite set V together with a function $f : V^2 \to A$. An* **A**-network $(V; f)$ *is* satisfiable *in a representation* $\mathfrak{B}$ *of* **A** *if there exists an assignment $s : V \to B$ such that for all $x, y \in V^2$ the following holds:*

$$(s(x), s(y)) \in f(x, y)^{\mathfrak{B}}.$$

An **A**-network $(V; f)$ *is* satisfiable *if there exists a representation* $\mathfrak{B}$ *of* **A** *such that $(V; f)$ is satisfiable in* $\mathfrak{B}$.

We will in the following assume that for an **A**-network $(V; f)$ it holds that $f(V^2) \subseteq A \setminus \{0\}$. Otherwise, $(V; f)$ is not satisfiable. Note that every **A**-network $(V; f)$ can be viewed as an A-structure $\mathfrak{C}$ on the domain V: for all $x, y \in V$ and $a \in A$ the relation $a^{\mathfrak{C}}(x, y)$ holds if and only if $f(x, y) = a$.

Definition 5. *The* (general) network satisfaction problem *for a finite relation algebra* **A***, denoted by* NSP(**A**)*, is the problem of deciding whether a given* **A**-network is satisfiable.

We give an example of how an instance of a NSP for a relation algebra could look like. The numbering of the relation algebra is from (Andréka and Maddux 1994).

Example 6 (An instance of NSP of relation algebra #17). *Let* **A** *be the relation algebra with the set of atoms* $\{\mathrm{Id}, a, b\}$ *and the product rules given by Fig. 1. Note that the domain of* **A** *is the following set:*

$$A = \{\emptyset, \mathrm{Id}, a, b, \mathrm{Id} \cup a, \mathrm{Id} \cup b, a \cup b, \mathrm{Id} \cup a \cup b\}.$$

Let $V := \{x_1, x_2, x_3\}$ *be a set. Consider the directed, edge labelled graph in Fig. 1 which is a visualization of a map* $f : V^2 \to A$*. We skipped the three* Id*-labelled self-loops for better readability. The tuple $(V; f)$ is an example of an instance of* NSP(**A**)*. The representation of* **A** *considered in Ex. 28 witnesses that $(V; f)$ is satisfiable as an instance of* NSP(**A**)*.*

Normal Representations and CSPs

In this section we consider a subclass of RRA introduced by Hirsch in 1996. For relation algebras **A** from this class, NSP(**A**) corresponds naturally to a constraint satisfaction problem (CSP). In the last two decades a rich and fruitful theory emerged to analyse the computational complexity of CSPs. We use this theory to obtain results about the computational complexity of NSPs.

In the following let **A** be in RRA. An **A**-network $(V; f)$ is called *closed* (transitively closed in (Hirsch 1997)) if for all $x, y, z \in V$ it holds that $f(x, x) \leq \mathrm{Id}$, $f(x, y) = f(y, x)^{\smile}$, and $f(x, z) \leq f(x, y) \circ f(y, z)$. It is called *atomic* if the image of f only contains atoms from **A**.

Definition 7 (from (Hirsch 1996)). *Let* $\mathfrak{B}$ *be a representation of* **A***. Then* $\mathfrak{B}$ *is called*

- fully universal*, if every atomic closed* **A**-network is satisfiable in $\mathfrak{B}$*;*
- square*, if* $1^{\mathfrak{B}} = B^2$*;*
- homogeneous*, if for every isomorphism between finite substructures of* $\mathfrak{B}$ *there exists an automorphism of* $\mathfrak{B}$ *that extends this isomorphism;*
- normal*, if it is fully universal, square and homogeneous.*

Definition 8. *Let* τ *be a relational signature. A first-order formula* $\varphi(x_1, \ldots, x_n)$ *is called* primitive positive (pp) *if it has the form*

$$\exists x_{n+1}, \ldots, x_m (\varphi_1 \wedge \cdots \wedge \varphi_s)$$

where $\varphi_1, \ldots, \varphi_s$ *are atomic formulas, i.e., formulas of the form* $R(y_1, \ldots, y_l)$ *for* $R \in \tau$ *and* $y_i \in \{x_1, \ldots, x_m\}$*, of the form* $y = y'$ *for* $y, y' \in \{x_1, \ldots x_m\}$*, or of the form* false*.*

Formulas without free variables are called *sentences*.

Definition 9. *Let* τ *be a finite relational signature and let* $\mathfrak{B}$ *be a* τ-structure. Then the constraint satisfaction problem *of* $\mathfrak{B}$ (CSP($\mathfrak{B}$)) *is the computational problem of deciding whether a given primitive positive* τ-sentence holds in $\mathfrak{B}$.

If $\mathfrak{B}$ is a fully universal representation of **A** $\in$ RRA, then NSP(**A**) and CSP($\mathfrak{B}$) are the same problem (up to a straightforward translation between **A**-networks and A-sentences; see (Bodirsky and Jonsson 2017)).

Model Theory

Let τ be a finite relational signature. The class of finite τ-structures that embed into a τ-structure $\mathfrak{B}$ is called the *age* of $\mathfrak{B}$, denoted by $\mathrm{Age}(\mathfrak{B})$. If $\mathcal{F}$ is a class of finite τ-structures, then $\mathrm{Forb}(\mathcal{F})$ is the class of all finite τ-structures $\mathfrak{A}$ such that no structure from $\mathcal{F}$ embeds into $\mathfrak{A}$. A class $\mathcal{C}$ of finite τ structures is called *finitely bounded* if there exists a finite set of finite τ-structures $\mathcal{F}$ such that $\mathcal{C} = \mathrm{Forb}(\mathcal{F})$. A structure $\mathfrak{B}$ is called finitely bounded if $\mathrm{Age}(\mathfrak{B})$ is finitely bounded. We want to mention that normal representations are finitely bounded, by their property of being fully universal.

Definition 10. *A class $\mathcal{C}$ of finite τ-structures has the* amalgamation property *if for all structures* $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$ *with embeddings* $e_1 : \mathfrak{A} \to \mathfrak{B}_1$ *and* $e_2 : \mathfrak{A} \to \mathfrak{B}_2$ *there exist a structure* $\mathfrak{C} \in \mathcal{C}$ *and embeddings* $f_1 : \mathfrak{B}_1 \to \mathfrak{C}$ *and*

$f_2\colon \mathfrak{B}_2 \to \mathfrak{C}$ *such that* $f_1 \circ e_1 = f_2 \circ e_2$. *If additionally* $f_1(B_1) \cap f_2(B_2) = f_1(e_1(A)) = f_2(e_2(A))$, *then we say that* $\mathcal{C}$ *has the* strong amalgamation *property.*

Let $\mathfrak{B}_1, \mathfrak{B}_2$ be τ-structures. Then $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is the τ-structure on the domain $B_1 \cup B_2$ such that $R^{\mathfrak{B}_1 \cup \mathfrak{B}_2} := R_1^{\mathfrak{B}} \cup R_2^{\mathfrak{B}}$ for every $R \in \tau$. If Def. 10 holds with $\mathfrak{C} := \mathfrak{B}_1 \cup \mathfrak{B}_2$ then we say that $\mathcal{C}$ has the *free amalgamation property*; note that the free amalgamation property implies the strong amalgamation property.

Theorem 11 (Fraïssé; see, e.g., (Hodges 1997))**.** *Let τ be a finite relational signature and let $\mathcal{C}$ be a class of finite τ-structures that is closed under taking induced substructures and isomorphisms and has the amalgamation property. Then there exists an up to isomorphism unique countable homogeneous structure $\mathfrak{B}$ such that $\mathcal{C} = \mathrm{Age}(\mathfrak{B})$.*

The Universal-Algebraic Approach

In this section we present basic notions for the so-called universal-algebraic approach to the study of CSPs.

Definition 12. *Let B be some set. We denote by $O_B^{(n)}$ the set of all n-ary operations on B and by $O_B := \bigcup_{n \in \mathbb{N}} O_B^{(n)}$ the set of all operations on B. A set $\mathscr{C} \subset O_B$ is called an* operation clone *on B if it contains all projections of all arities and if it is closed under composition, i.e., for all $f \in \mathscr{C}^{(n)} := \mathscr{C} \cap O_B^{(n)}$ and $g_1, \ldots, g_n \in \mathscr{C} \cap O_B^{(s)}$ it holds that $f(g_1, \ldots, g_n) \in \mathscr{C}$, where $f(g_1, \ldots, g_n)$ is the s-ary function such that $f(g_1, \ldots, g_n)(x_1, \ldots, x_s)$ is defined as*

$$f(g_1(x_1, \ldots, x_s), \ldots, g_n(x_1, \ldots, x_s)).$$

An operation $f\colon B^n \to B$ is called *conservative* if for all $x_1, \ldots, x_n \in B$ it holds that $f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$. A clone is called *conservative* if all operations are conservative. We later need the following classical result for clones over a two-element set.

Theorem 13 (Post 1941)**.** *Let $\mathcal{C}$ be a conservative operation clone on $\{0, 1\}$. Then either $\mathcal{C}$ contains only projections, or at least one of the following operations:*

1. *the binary function* min,
2. *the binary function* max,
3. *the minority function,*
4. *the majority function.*

Operation clones occur naturally as polymorphism clones of relational structures. If $a_1, \ldots, a_n \in B^k$ and $f\colon B^n \to B$, then we write $f(a_1, \ldots, a_n)$ for the k-tuple obtained by applying f component-wise to the tuples $a_1, \ldots, a_n$.

Definition 14. *Let $\mathfrak{B}$ a structure with a finite relational signature τ and let $R \in \tau$. An n-ary operation f* preserves *the relation $R^{\mathfrak{B}}$ if for all $a_1, \ldots, a_n \in R^{\mathfrak{B}}$ it holds that*

$$f(a_1, \ldots, a_n) \in R^{\mathfrak{B}}.$$

If f preserves all relations from $\mathfrak{B}$ then f is called a polymorphism *of $\mathfrak{B}$.*

The set of all polymorphisms (of all arities) of a relational structure $\mathfrak{B}$ is an operation clone on B, which is denoted by $\mathrm{Pol}(\mathfrak{B})$. A *Siggers operation* is an operation that satisfies the Siggers identity (see Thm. 1).The following result can be obtained by combining known results from the literature.

Theorem 15 (Siggers 2010; Bulatov 2003; also see (Barto 2011; Bulatov 2016))**.** *Let $\mathfrak{B}$ be a finite structure with a finite relational signature such that $\mathrm{Pol}(\mathfrak{B})$ is conservative. Then either*

1. *there exist distinct $a, b \in B$ such that for every $f \in \mathrm{Pol}(\mathfrak{B})^{(n)}$ the restriction of f to $\{a, b\}^n$ is a projection. In this case, $\mathrm{CSP}(\mathfrak{B})$ is NP-complete.*
2. $\mathrm{Pol}(\mathfrak{B})$ *contains a Siggers operation; in this case, $\mathrm{CSP}(\mathfrak{B})$ is in P.*

We now discuss fundamental results about the universal-algebraic approach for constraint satisfaction problems of structures with an infinite domain.

Theorem 16 (Bodirsky and Nešetřil 2006)**.** *Let $\mathfrak{B}$ be a homogeneous structure with finite relational signature. Then a relation is preserved by $\mathrm{Pol}(\mathfrak{B})$ if and only if it is primitively positively definable (see Def. 8) in $\mathfrak{B}$.*

In the following let $\mathbf{A} \in \mathrm{RRA}$ be finite and with a normal representation $\mathfrak{B}$.

Definition 17. *Let $a_1, \ldots, a_n \in A_0$ be atoms of $\mathbf{A}$. Then a tuple $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is in the $2n$-ary relation $(a_1, \ldots, a_n)^{\mathfrak{B}}$ iff $a_i^{\mathfrak{B}}(x_i, y_i)$ holds for all $i \in \{1, \ldots, n\}$.*

An operation $f\colon B^n \to B$ is called *edge-conservative* if it satisfies for all $x, y \in B^n$ and all $a_1, \ldots, a_n \in A_0$

$$(a_1, \ldots, a_n)^{\mathfrak{B}}(x, y) \Rightarrow (f(x), f(y)) \in \bigcup_{i \in \{1, \ldots, n\}} a_i^{\mathfrak{B}}.$$

Note that for every $D \subseteq A_0$ the structure $\mathfrak{B}$ contains the relation $\bigcup_{a_i \in D} a_i^{\mathfrak{B}}$. Therefore the next proposition follows immediately since polymorphisms of $\mathfrak{B}$ preserve all relations of $\mathfrak{B}$.

Proposition 18. *All polymorphisms of $\mathfrak{B}$ are edge-conservative.*

Definition 19. *Let $X \subseteq A_0$. An operation $f\colon B^n \to B$ is called X-canonical if there exists a function $\bar{f}\colon X^n \to A_0$ such that for all $x, y \in B^n$ and $a_1, \ldots, a_n \in X$, if $(x_i, y_i) \in a_i^{\mathfrak{B}}$ for all $i \in \{1, \ldots, n\}$ then $(f(x), f(y)) \in \bar{f}(a_1, \ldots, a_n)^{\mathfrak{B}}$. An operation is called canonical if it is A_0-canonical. The function $\bar{f}$ is called the behaviour of f on X. If $X = A_0$ then $\bar{f}$ is just called the behaviour of f.*

It will always be clear from the context what the domain of a behaviour $\bar{f}$ is. An operation $f\colon S^2 \to S$ is called *symmetric* if for all $x, y \in S$ it holds that $f(x, y) = f(y, x)$. An X-canonical function f is called X-*symmetric* if the behaviour of f on X is symmetric.

Ramsey Theory and Canonisation

We avoid giving an introduction to Ramsey theory, since the only usage of the Ramsey property is via Thm. 21, and rather refer to (Bodirsky 2015) for an introduction.

Let $\mathfrak{A}$ be a homogeneous τ-structure such that $\mathrm{Age}(\mathfrak{A})$ has the strong amalgamation property. Then the class of all $(\tau \cup \{<\})$-structures $\mathfrak{A}$ such that $<^{\mathfrak{A}}$ is a linear order and whose τ-reduct (i.e. the structure on the same domain, but only with the relations that are denoted by symbols from τ, see e.g. (Hodges 1997)) is from $\mathrm{Age}(\mathfrak{A})$ is a strong amalgamation class, too (see for example (Bodirsky 2015)). By Thm. 11 there exists an up to isomorphism unique countable homogeneous structure of that age, which we denote by $\mathfrak{A}_<$. It can be shown by a straightforward back-and-forth argument that $\mathfrak{A}_<$ is isomorphic to an expansion of $\mathfrak{A}$, so we identify the domain of $\mathfrak{A}$ and of $\mathfrak{A}_<$ along this isomorphism, and call $\mathfrak{A}_<$ the *expansion of $\mathfrak{A}$ by a generic linear order*.

Theorem 20 (Nešetřil and Rödl 1989; Hubička and Nešetřil 2019). *Let $\mathfrak{A}$ be a relational τ-structure such that $\mathrm{Age}(\mathfrak{A})$ has the free amalgamation property. Then the expansion of $\mathfrak{A}$ by a generic linear order has the Ramsey property.*

The following theorem gives a connection of the Ramsey property with the existence of canonical functions and plays a key role in our analysis.

Theorem 21 (Bodirsky and Pinsker 2016). *Let $\mathfrak{B}$ be a countable homogeneous structure with finite relational signature and the Ramsey property. Let $h: B^k \to B$ be an operation and let L be the set of all k-ary operations $\alpha(h(\beta_1, \ldots, \beta_k))$ where $\alpha, \beta_1 \ldots, \beta_k$ are automorphisms of $\mathfrak{B}$. Then there exists a canonical operation $g: B^k \to B$ such that for every finite $F \subset B$ there exists $g' \in L$ such that $g'|_{F^k} = g|_{F^k}$.*

Relation Algebras with a Flexible Atom

In this section we define relation algebras with a flexible atom and show how to reduce the classification problem for their network satisfaction problem to the situation where they are additionally *integral*. Then we show that integral relation algebras with a flexible atom have a normal representation. Therefore, the universal-algebraic approach is applicable; in particular, we make heavy use of Thm. 16 in the subsequent sections. Finally, we prove that every normal representation of a finite relation algebra with a flexible atom has a Ramsey expansion. Therefore, the tools from section "Ramsey Theory and Canonisation" can be applied, too.

Let $\mathbf{A} \in \mathrm{RRA}$ and let $I := \{a \in A \mid a \leq \mathrm{Id}\}$. An atom $s \in A_0 \setminus I$ is called *flexible* if for all $a, b \in A \setminus I$ it holds that $s \leq a \circ b$. A finite representable relation algebra $\mathbf{A}$ is called *integral* if the element Id is an atom of $\mathbf{A}$.

Proposition 22. *Let $\mathbf{A} \in \mathrm{RRA}$ be finite and with a flexible atom s. Then there exists a finite integral $\mathbf{A}' \in \mathrm{RRA}$ with a flexible atom such that $\mathrm{NSP}(\mathbf{A})$ and $\mathrm{NSP}(\mathbf{A}')$ are polynomial-time equivalent.*

We assume for the rest of the section that $\mathbf{A} \in \mathrm{RRA}$ is finite, integral, and with a flexible atom s.

We consider the set $A - s := \{a \in A \mid s \not\leq a\}$. Let $(V; f)$ be an $\mathbf{A}$-network and let $\mathfrak{C}$ be the corresponding A-structure. Let $\mathfrak{C} - s$ be the $(A - s)$-structure on the same domain V as $\mathfrak{C}$ such that for all $x, y \in V$ and $a \in (A - s) \setminus \{0\}$ we have

$$a^{\mathfrak{C}-s}(x,y) \text{ if and only if } (a^{\mathfrak{C}}(x,y) \vee (a \cup s)^{\mathfrak{C}}(x,y)).$$

$\circ$	Id	a	b
Id	Id	a	b
a	a	$\{\mathrm{Id}, a, b, \}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{\mathrm{Id}, a, b\}$

Figure 2: Multiplication table of relation algebras #18

We call $\mathfrak{C} - s$ the *s-free companion* of an $\mathbf{A}$-network $(V; f)$.

The next lemma follows directly from the definitions of flexible atoms and s-free companions.

Lemma 23. *Let $\mathcal{C}$ be the class of s-free companions of atomic closed $\mathbf{A}$-networks. Then $\mathcal{C}$ has the free amalgamation property.*

As a direct consequence we obtain the following.

Proposition 24. $\mathbf{A}$ *has a normal representation $\mathfrak{B}$.*

The next theorem follows by Thm. 20 and Lem. 23.

Theorem 25. *Let $\mathfrak{B}$ be a normal representation of $\mathbf{A}$. Let $\mathfrak{B}_<$ be the expansion of $\mathfrak{B}$ by a generic linear order. Then $\mathfrak{B}_<$ has the Ramsey property.*

Remark 26. *The binary first-order definable relations of $\mathfrak{B}_<$ build a proper relation algebra since $\mathfrak{B}_<$ has quantifier-elimination (see (Hodges 1997)). By the definition of the generic order the atoms of this proper relation algebra are the relations*

$$\{a^{\mathfrak{B}_<} \cap <^{\mathfrak{B}_<} \mid a \in A_0 \setminus \{\mathrm{Id}\}\}$$
$$\cup \{a^{\mathfrak{B}_<} \cap >^{\mathfrak{B}_<} \mid a \in A_0 \setminus \{\mathrm{Id}\}\}$$
$$\cup \{\mathrm{Id}\}.$$

We give two concrete examples of finite, integral, symmetric relation algebras with a flexible atom (Ex. 27 and 28). The numbering of the relation algebras in the examples is from (Andréka and Maddux 1994).

Example 27 (Relation algebra #18). *The relation algebra #18 has three atoms, namely the identity atom Id and two symmetric atoms a and b. The multiplication table for the atoms is given in Fig. 2. In this relation algebra the atoms a and b are flexible. Consider the countable, homogeneous, undirected graph $\mathfrak{R} = (V; E^{\mathfrak{R}})$, whose age is the class of all finite undirected graphs (see, e.g., (Hodges 1997)), also called the Random graph. The expansion of $\mathfrak{R}$ by all binary first-order definable relations is a normal representation of the relation algebra #18. In this representation the atoms a and b are interpreted as the relation $E^{\mathfrak{R}}$ and the relation $N^{\mathfrak{R}}$, where $N^{\mathfrak{R}}$ is defined as $\neg E(x, y) \wedge x \neq y$.*

Example 28 (Relation algebra #17). *The relation algebra #17 also consists of three symmetric atoms. The multiplication table in Fig. 1 shows that in this relation algebra b is a flexible atom. To see that a is not a flexible atom, note that $a \not\leq a \circ a = \{\mathrm{Id}, b\}$. Let $\mathfrak{N} = (V; E^{\mathfrak{N}})$ be the countable, homogeneous, undirected graph, whose age is the class of all finite undirected graphs that do not embed the complete graph on three vertices (see, e.g., (Hodges 1997)). If we expand $\mathfrak{N}$ by all binary first-order definable relations we get a normal representation of the relation algebra #17. To see*

this note that we interpret a as the relation $E^{\mathfrak{N}}$. That $\mathfrak{N}$ is triangle free, i.e. triangles of $E^{\mathfrak{N}}$ are forbidden, matches with the fact that $a \nleq a \circ a$ holds in the relation algebra.

Polynomial-time Tractability

In this section we introduce for every finite $\mathbf{A} \in \mathrm{RRA}$ an associated finite structure, called the *atom structure* of $\mathbf{A}$. Note that it is closely related, but not the same, as the type structure introduced in (Bodirsky and Mottet 2016). In the context of relation algebras the atom structure has the advantage that its domain is the set of atoms of $\mathbf{A}$, rather than the set of 3-types, which would be the domain of the type structure in (Bodirsky and Mottet 2016); hence, our domain is smaller and has some advantages that we discuss at the end of the section. Up to a minor difference of the signature, our atom structure is the same as the atom structure introduced in (Lyndon 1950) (which was used there for different purposes; also see (Maddux 1982; Hirsch and Hodkinson 2001; Hirsch, Jackson, and Kowalski 2019)).

We will reduce $\mathrm{CSP}(\mathfrak{B})$ to the CSP of the atom structure. This means that if the CSP of the atom structure is in P, then so are $\mathrm{CSP}(\mathfrak{B})$ and $\mathrm{NSP}(\mathbf{A})$. For our main result we will show later that every network satisfaction problem for a finite integral symmetric relation algebra with a flexible atom that cannot be solved in polynomial time by this method is NP-complete. Let $\mathfrak{B}$ be throughout this section a normal representation of a finite $\mathbf{A} \in \mathrm{RRA}$.

Definition 29. *The* atom structure *of $\mathbf{A}$ is the finite relational structure $\mathfrak{D}$ with domain A_0 and the following relations:*

- *for all $x \in A$ the unary relation $x^{\mathfrak{D}} := \{a \in A_0 \mid a \leq x\}$*
- *the binary relation $E^{\mathfrak{D}} := \{(a_1, a_2) \in A_0^2 \mid a_1^{\smile} = a_2\}$*
- *the ternary relation*

$$H^{\mathfrak{D}} := \{(a_1, a_2, a_3) \in A_0^3 \mid a_3 \leq a_1 \circ a_2\}.$$

Proposition 30. *There is a polynomial-time reduction from $\mathrm{CSP}(\mathfrak{B})$ to $\mathrm{CSP}(\mathfrak{D})$.*

Proof (Sketch). We state the reduction explicitly and skip the correctness proof.

Let Ψ be an instance of $\mathrm{CSP}(\mathfrak{B})$ with variable set $X = \{x_1, \ldots, x_n\}$. We construct an instance Φ of $\mathrm{CSP}(\mathfrak{D})$ as follows. The variable set Y of Φ is given by $Y := \{(x_i, x_j) \in X^2 \mid i \leq j\}$. The constraints of Φ are of the two kinds:

1. Let $a \in A$ be an element of the relation algebra and let $a(x_i, y_j)$ be an atomic formula of Ψ. If $i < j$ holds, then we add the atomic (unary) formula $a((x_i, x_j))$ to Φ; otherwise we add the atomic formula $a^{\smile}((x_i, x_j))$.
2. Let $x_i, x_j, x_l \in X$ be such that $i \leq j \leq l$. Then we add the atomic formula $H((x_i, x_j), (x_j, x_l), (x_i, x_l))$ to Φ.

One can show that the reduction from Ψ to Φ is correct. $\square$

We obtain another property of the atom structure which is fundamental for our result. Recall that every canonical polymorphism f induces a behaviour $\bar{f} : A_0^n \to A_0$. One can observe that $\bar{f}$ is a polymorphism of $\mathfrak{D}$. Moreover the other

direction also holds. Every $g \in \mathrm{Pol}(\mathfrak{D})$ is the behaviour of a canonical polymorphism of $\mathfrak{B}$.

Recall from Prop. 18 that polymorphisms of $\mathfrak{B}$ are edge-conservative. Note that this implies that polymorphisms of $\mathfrak{D}$ are conservative. In fact, Thm. 15 and the previous proposition imply the following.

Proposition 31. *If $\mathrm{Pol}(\mathfrak{B})$ contains a canonical polymorphism s such that its behaviour $\bar{s}$ is a Siggers operation in $\mathrm{Pol}(\mathfrak{D})$ then $\mathrm{CSP}(\mathfrak{B})$ is polynomial-time solvable.*

We demonstrate how this result can be used to prove polynomial-time tractability of $\mathrm{NSP}(\mathbf{A})$ for a symmetric, integral $\mathbf{A} \in \mathrm{RRA}$ with a flexible atom.

Example 32 (Polynomial-time tractability of relation algebra #18, see (Cristiani and Hirsch 2004), see also Section 8.4 in (Bodirsky and Pinsker 2015))**.** *We consider the following function $\bar{s} : \{\mathrm{Id}, a, b\}^6 \to \{\mathrm{Id}, a, b\}$.*

$$\bar{s}(x_1, \ldots, x_6) := \begin{cases} a & \text{if } a \in \{x_1, \ldots, x_6\}, \\ b & \text{if } b \in \{x_1, \ldots, x_6\} \text{ and} \\ & a \notin \{x_1, \ldots, x_6\}, \\ \mathrm{Id} & \text{otherwise.} \end{cases}$$

Let $\mathfrak{R}'$ be the normal representation of the relation algebra #18 given in Ex. 27. Note that $\bar{s}$ is the behaviour of an injective, canonical polymorphism of $\mathfrak{R}$. The injectivity follows from the last line of the definition; if $\bar{s}(x_1, \ldots, x_6) = \mathrm{Id}$ then $\{x_1, \ldots, x_6\} = \{\mathrm{Id}\}$. Therefore $\bar{s}$ preserves all allowed triples, since in the relation algebra #18 the only forbidden triples involve Id. One can check that $\bar{s}$ is a Siggers operation and therefore we get by Prop. 31 that $\mathrm{NSP}(\#18)$ is polynomial-time solvable.

NP-Hardness

Let $\mathfrak{B}$ be a normal representation of a finite integral symmetric $\mathbf{A} \in \mathrm{RRA}$ with a flexible atom s. The goal of this section is to sketch a proof of the remaining part of Thm. 1. We want to point out that the technical details of this proof will appear in the journal version of the present article. We want to show that if $\mathfrak{B}$ does not have a canonical polymorphism with a behaviour that satisfies the Siggers identity, then $\mathrm{CSP}(\mathfrak{B})$ is NP-hard. Let us start by stating the next lemma which is central in our proof. A lemma of a similar type appeared for example as Lem. 42 in (Bodirsky and Pinsker 2014) (see also (Bodirsky, Jonsson, and von Oertzen 2011)).

Lemma 33. *Let a and b be atoms of $\mathbf{A}$. Then the following are equivalent:*

1. *$\mathfrak{B}$ has an $\{a, b\}$-symmetric polymorphism g with $\bar{g}(a, b) = \bar{g}(b, a) = a$.*
2. *For every primitive positive formula φ such that $\varphi \wedge a(x_1, x_2) \wedge b(y_1, y_2)$ and $\varphi \wedge b(x_1, x_2) \wedge a(y_1, y_2)$ are satisfiable over $\mathfrak{B}$, the formula $\varphi \wedge a(x_1, x_2) \wedge a(y_1, y_2)$ is also satisfiable over $\mathfrak{B}$.*

We use this lemma to obtain the following statement:

Proposition 34. *Let f be a binary injective polymorphism of $\mathfrak{B}$ and let $a \nleq \mathrm{Id}$ and $b \nleq \mathrm{Id}$ be two atoms such that $\mathfrak{B}$ has*

no $\{a, b\}$-symmetric polymorphism. Then all polymorphisms are canonical on $\{a, b\}$.

The next step is to obtain a condition for NP-hardness.

Proposition 35. *If $\mathfrak{B}$ does not have a binary injective polymorphism, then $\mathrm{CSP}(\mathfrak{B})$ is NP-complete.*

In order to prove Prop. 35 we show the following lemma.

Lemma 36. *If $\mathfrak{B}$ does not have an injective binary polymorphism, then $\mathfrak{B}$ does not have an $\{s, \mathrm{Id}\}$-symmetric polymorphism.*

By a similar lemma to Lem. 34 (without assuming the existence of a binary polymorphism and without assuming $a \not\leq \mathrm{Id}$ and $b \not\leq \mathrm{Id}$) it follows that all polymorphisms are $\{s, \mathrm{Id}\}$-canonical. This means that $\mathrm{Pol}(\mathfrak{B})$ induces an operation clone $\mathscr{C}$ on a two-element set. Note that $\mathscr{C}$ cannot have a majority or a minority operation since one of the two atoms is Id (this can be proved by an analogous argument as in the proof of Lem. 36). We also know that $\mathscr{C}$ does not contain a symmetric operation and therefore by Thm. 13, $\mathscr{C}$ contains only projections. By a well-known result (see, e.g., (Bodirsky 2008)) this implies NP-hardness of $\mathrm{CSP}(\mathfrak{B})$ and proves Prop. 35.

Our assumption that $\mathfrak{B}$ does not have a canonical polymorphism with a behaviour that satisfies the Siggers identity implies by Thm. 15 and the connection of canonical polymorphisms and polymorphisms of the atom structure (see section "Polynomial-time Tractability") the existence of two distinct atoms $a, b \in A_0$ such that all canonical polymorphisms behave like projections on $\{a, b\}$. This means that on these atoms there exists no *canonical $\{a, b\}$-symmetric* polymorphism. In order to apply Prop. 34 on these atoms we show that there exists also no $\{a, b\}$-symmetric (and not necessarily canonical) polymorphism.

Assume for contradiction that there exists an $\{a, b\}$-symmetric polymorphism f with $\overline{f}(a, b) = a = \overline{f}(b, a)$. We first show that f is injective. Assume there exist $c \in A_0$ and $x, y \in B^2$ with $(c, \mathrm{Id})(x, y)$ such that $\mathrm{Id}(f(x), f(y))$ holds in $\mathfrak{B}$. Consider the case that $s \notin \{a, b\}$ holds. Then we choose $z \in B^2$ such that $(a, b)(z, x)$ and $(s, b)(z, y)$ hold. This is possible since s is a flexible atom. By the assumption on the polymorphism f we get $a(f(z), f(x))$ and $(s \cup b)(f(z), f(y))$. Therefore, the substructure induced on $f(x), f(y)$, and $f(z)$ would either imply that the triple (Id, s, a) holds or that the triple (Id, b, a) holds, which are both forbidden triples in $\mathbf{A}$ since $s \neq a$ and $b \neq a$. This is a contradiction because f is a polymorphism of $\mathfrak{B}$. The cases where $s \in \{a, b\}$ holds can be shown by an analogous argument. Therefore, $\mathrm{Id}(f(x), f(y))$ is not possible for our choice of x and y. Since $c \in A_0$ was arbitrary and we showed that $\overline{f}(c, \mathrm{Id}) = c = \overline{f}(\mathrm{Id}, c)$ this implies that f is injective.

For an injective polymorphism f there exists a polymorphism $f_<$ of the generic combination $\mathfrak{B}_<$ such that there exists an injective endomorphism e of $\mathfrak{B}$ with $f = e \circ f_<$ as mappings from $B^2 \to B$. Note that $\mathfrak{B}_<$ is by Thm. 25 an ordered Ramsey structure. Let $g_<$ be the canonical (with respect to $\mathfrak{B}_<$) operation that exists if we apply Thm. 21 on

$f_<$. Note that $g_<$ is an $\{a, b\}$-canonical polymorphism of $\mathfrak{B}$ (but not canonical).

Consider the proper relation algebra induced by the binary first-oder definable relations of $\mathfrak{B}_<$ (see Rem. 26). Let X be the subset of atoms with the following definition: $X := \{a^{\mathfrak{B}<} \cap <^{\mathfrak{B}<} \mid a \in A_0 \setminus \{\mathrm{Id}\}\} \cup \{\mathrm{Id}\}$. The behaviour of g on X induces a function $A_0^2 \to A_0$. This function is well-defined since all of the atoms of $\mathbf{A}$ are symmetric by our assumption. One can observe that this function is the behaviour of some canonical polymorphism of $\mathfrak{B}$. This polymorphism is also $\{a, b\}$-symmetric. Therefore we get that if no canonical $\{a, b\}$-symmetric polymorphism exists then no $\{a, b\}$-symmetric polymorphism exists.

By Prop. 34 we obtain that all polymorphisms of $\mathfrak{B}$ are $\{a, b\}$-canonical. This means again that $\mathrm{Pol}(\mathfrak{B})$ induces an operation clone on a two-element set. To complete our hardness proof one step is missing. $\mathrm{Pol}(\mathfrak{B})$ could possibly induce a majority or a minority operation f on $\{a, b\}$. To prove that this is impossible we again "extend" a partial canonical behaviour to a global one. This yields a contradiction, since our assumption was that all canonical polymorphism are like projection on $\{a, b\}$. Therefore, $\mathrm{Pol}(\mathfrak{B})$ induces by Thm. 13 a projection clone on $\{a, b\}$. We use the result from (Bodirsky 2008) and obtain NP-hardness of $\mathrm{CSP}(\mathfrak{B})$.

The following shows how to apply our hardness result to a concrete $\mathbf{A} \in \mathrm{RRA}$.

Example 37 (Hardness of relation algebra #17, see (Bodirsky et al. 2019; Bodirsky and Knäuer 2020a))**.** *To prove the NP-hardness of the NSP for the relation algebra from Ex. 28 we do not need the full power of our classification result. It is enough and easier to see that the hardness condition given in Prop. 35 applies. Let $\mathfrak{N}'$ be the normal representation of the relation algebra #17 mentioned in Ex. 28. The structure $\mathfrak{N}'$ does not have a binary injective polymorphism. To see this, consider a substructure of $\mathfrak{N}'^2$ on elements $x, y, z \in V^2$ such that $(E, =)(x, y)$, $(=, E)(y, x)$, and $(E, E)(x, z)$ hold in $\mathfrak{N}'$. Assume there exists an injective binary polymorphism f. This means that $\overline{f}(E, \mathrm{Id}) = E = \overline{f}(\mathrm{Id}, E)$ holds. Then we get that $E(f(x), f(y))$, $E(f(y), f(z))$ and $E(f(x), f(z))$ hold in $\mathfrak{N}'$, which is a contradiction, since in $\mathfrak{N}'$ triangles of this form are forbidden. Therefore, Prop. 35 implies NP-hardness of $\mathrm{NSP}(\#17)$.*

Conclusion

We classified the computational complexity of the network satisfaction problem for a finite symmetric $\mathbf{A} \in \mathrm{RRA}$ with a flexible atom and obtained a P versus NP-complete dichotomy. We gave decidable criteria for $\mathbf{A}$ that characterize both the containment in P and NP-hardness. We want to mention that if we drop the assumptions on $\mathbf{A}$ to be symmetric and to have a flexible atom the statement of Thm. 1 is false. An example for this is the Point Algebra; although the NSP of this relation algebra is in P, the first condition of Thm. 1 does not apply. On the other hand, if we only drop the symmetry assumption we conjecture that Thm. 1 still holds. Similarly, if we only drop the flexible atom assumption we conjecture that the statement also remains true.

Acknowledgments

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