A Model of Winners Allocation

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Abstract

We propose a model of winners allocation. In this model, we are given are two elections where the sets of candidates may intersect. The goal is to find two disjoint winning committees from respectively the two elections that are subjected to certain reasonable restrictions. For our model, we first propose several desirable properties. Then, we investigate the implication relationships among these properties. Finally, we study the complexity of computing winners allocations providing these properties. For hardness results, we also study some fixed-parameter algorithms.

Introduction

Multiwinner voting has attracted a considerable amount of attention of AI community very recently, due to its significant and wide applications in many areas. Standard multiwinner voting aims to select exactly k winning candidates based on the preferences of voters over candidates, where k is a given integer. In this paper, we propose a model of *winners allocation*. In particular, in our model there are two elections (C_1, V_1) and (C_2, V_2), where C_1 and C_2 are two sets of candidates, and V_1 and V_2 are two multisets of votes over C_1 and C_2 respectively. Given two nonnegative integers k_1 and k_2, our model aims to select (allocate) two disjoint subsets $w_1 \subseteq C_1$ and $w_2 \subseteq C_2$ of cardinalities k_1 and k_2 respectively and of high quality, based on the votes in V_1 and V_2. The quality of the allocated candidates are measured via certain utility functions. For simplicity, we study only approval-based elections.

To see the applicability of our model, let us first consider two special cases. When $C_1 \cap C_2 = \emptyset$, our model is equivalent to performing two independent standard multiwinner voting. Additionally, when one of k_1 and k_2 is 0, our model is equivalent to performing one standard multiwinner voting. Besides these two special cases, in general our model is relevant for several real-world applications.

Awards (grants) assignment. Assume that a community is going to allocate two types of awards/grants to a number of applicants (nominees), where each type of awards is given to a respectively fixed number of applicants. Each applicant may be eligible for one or two types of awards,

which results in two sets of candidates. However, for some kind of fairness, every applicant is only allowed to receive at most one award, corresponding to that the two selected winning committees are disjoint in our model. As each type of awards has its own feature, the organizer may invite two committees to valuate the applicants.

Sister conferences organization. Suppose two sister conferences are going to be held soon and they have the same submission deadlines. As the program committees of the conferences share many common scholars, they want to organize a cooperative review procedure to reduce their workload. In this case, a meaningful review procedure would be as follows. When submitting a paper, the authors are free to decide whether they are only interested in one particular conference or they are indifferent of the two conferences. Therefore, submitted papers are classified into two sets C_1 and C_2 which may intersect. However, the copyright policy allows each paper to be published in at most one of the conference proceedings.

Committee evaluations. Criticisms over the peer review processes of many AI conferences have been posted a number of times recently (see, e.g., (Brezis and Birukou 2020)), mainly triggered by the existence of high-quality papers being rejected. It is speculated that a paper may receive different results when evaluated by two independent committee members. To test this speculation, NIPS 2014 organizers select two independent program committees to handle 10 percent of the submissions. The result more or less confirms the speculation. Such a review procedure is related to a special case of our model where $C_1 = C_2$ (each C_i, $i \in [2]$, consists of the 10% of the submissions).

For our model, we study winners allocations that provide the maximum utilitarian/egalitarian/Nash social welfare (USW/ESW/NSW) or some envy-freeness properties, and investigate their implication relationships. In addition, we study the complexity of calculating a winners allocation belonging to these properties. Some of our results are quite interesting. For instance, we prove that computing a (k_1, k_2)-winners allocation of maximum USW is polynomial-time solvable as long as the utility functions adopted by the two voting communities V_1 and V_2 are (weakly) additive. The complexity is consistent with the polynomial-time solvability of computing winning com-

mittees with respect to additive functions. However, we show that the complexity of computing a winners allocation with the maximum ESW/NSW can be significantly different: there are both natural additive functions which lead to polynomial-time solvability results and natural additive functions which lead to NP-hardness results. It is also notable that the hardness results hold even when the given elections are both **voters interval** (VI) and **candidates interval** (CI). Recall that an election is VI (CI) if the candidates (votes) can be linearly ordered so that each vote approves only consecutive candidates (each candidate is approved in consecutive votes). It is known that when restricted to only one of these two domains, many NP-hard problems become already polynomial-time solvable (Brandt et al. 2015; Elkind and Lackner 2015; Liu and Guo 2016; Peters 2018b; Faliszewski et al. 2011).

Related Works

As an extension of approval-based multiwinner voting (ABMV), our study is clearly related to the large body of literature on ABMV (see, e.g., (Aziz et al. 2018; Aziz and Lee 2020; Cygan et al. 2017; Kocot et al. 2019; Peters 2018a; Yang 2019, 2020)). Recently, extensions of standard ABMV have been proposed in the literature, including for instance ABMV with a variable number of winners (Kilgour 2016; Faliszewski, Slinko, and Talmon 2020), ABMV with labeled candidates (Bredereck et al. 2018; Celis, Huang, and Vishnoi 2018), ABMV with graph structured candidate set (Yang and Wang 2018), etc. However, to the best of our knowledge, our extension of ABMV has not been considered in the literature heretofore. It is also worth mentioning that a model which captures parallel single-winner elections with a common set of candidates has been studied recently (Boehmer et al. 2020). Particularly, this model aims to identify a high-quality allocation of a set of candidates to a number of positions so that every position receives exactly one candidate and every candidate is allocated to at most one position.

Our model is also related to the model of allocating indivisible goods. In this model, a set of indivisible goods are supposed to be allocated to several agents so that every good is allocated to exactly one agent and all goods are allocated. Each agent has a utility function to valuate the goods allocated to her. The goal is to compute an allocation that achieves some optimization objectives or fulfills some desirable properties. We refer to Chapters 12–13 of (Brandt et al. 2016) for a comprehensive survey on this topic.

Preliminaries

We assume the reader is familiar with the basic notions of (parameterized) complexity such as NP-hardness and fixed-parameter tractability (FPT). For readers without these background, we refer to (Tovey 2002; Downey 2012). For an integer i, let $[i] = \{j \in \mathbb{N} : 1 \leq j \leq i\}$.

Multiwinner voting. An approval-based election is a tuple (C, V) where C is a set of candidates and V is a multiset of votes cast by a set of voters. Each vote $v \in V$ is defined as a subset of C consisting of all the candidates approved by the corresponding voter. A committee is a subset

of candidates, and a k-committee is a committee of cardinality k. A k-committee selection rule (k-CSR) is a function that maps each election (C, V) such that $|C| \geq k$ to a class of k-committees of C, called the winning k-committees. We focus on k-CSRs each of which determines the k-winning committees via a particular utility function. Precisely, a utility function maps each tuple (w, V) of a committee $w \subseteq C$ and a multiset V of votes over C to a nonnegative rational number, which is the utility of V derived from w. We also call $f(w, V)$ the f score of w from V. The rule selects all k-committees that yield the maximum possible utility among all k-committees. For notational brevity, throughout this paper we write x for a singleton $\{x\}$. For instance, $f(c, V)$ is exactly $f(\{c\}, V)$. Table 1 summarizes utility functions widely studied in the literature. In this paper, we consider only utility functions f such that $f(\emptyset, V) = f(w, \emptyset) = 0$, which is the case for all utility functions in Table 1.

utility $f(w, V)$	
AV	$\sum_{v \in V} |v \cap w|$
SAV	$\sum_{v \in V, v \neq \emptyset} \frac{|v \cap w|}{|v|}$
CSAV	$\sum_{v \in V} \frac{|v \cap w|}{|w|}$
MSAV	$\sum_{v \in V, v \neq \emptyset} \frac{|v \cap w|}{\min\{|v|, |w|\}}$
PAV	$\sum_{v \in V, v \cap w \neq \emptyset} \sum_{i=1}^{|v \cap w|} \frac{1}{i}$
WAV	$\sum_{v \in V, v \cap w \neq \emptyset} \sum_{i=1}^{|v \cap w|} \frac{1}{2i-1}$
CCAV	$|\{v \in V : v \cap w \neq \emptyset\}|$

Table 1: Where a value is not well-defined, it is 0.

A utility function f is additive if for every nonempty $w \subseteq C$, it holds that $f(w, V) = \sum_{c \in w} f(c, V)$. Moreover, f is weakly additive if for every integer k there is a utility function g_k such that $f(w, V) = \sum_{c \in w} g_k(c, V)$ for every k-committee $w \subseteq C$. We call g_k an additive realization of f of order k. Apparently, additive functions must be weakly additive. It is fairly easy to check that AV and SAV are additive, but other functions given in Table 1 are not. However, CSAV and MSAV are weakly additive.

We assume that all utility functions are given as oracles. In addition, for weakly additive utility functions, we assume that their additive realizations of all possible orders are available when needed.

Winners allocation. Let C be a set of candidates. In addition, let $C_1 \subseteq C$ and $C_2 \subseteq C$ be two subsets of C such that $C_1 \cup C_2 = C$, and let V_1 and V_2 be two multisets of votes over C_1 and C_2, respectively. Let $E_1 = (C_1, V_1)$ and $E_2 = (C_2, V_2)$. We call (E_1, E_2) a double-election, and call each of V_1 and V_2 a voting community. For two nonnegative integers k_1 and k_2 such that $k_1 \leq |C_1|$, $k_2 \leq |C_2|$, and $k_1 + k_2 \leq |C_1 \cup C_2|$, a (k_1, k_2)-winners allocation of (E_1, E_2) is a tuple (w_1, w_2) such that

- $w_1 \subseteq C_1$ and $w_2 \subseteq C_2$;

- $w_1 \cap w_2 = \emptyset$; and

- $|w_1| = k_1$ and $|w_2| = k_2$.

In other words, $w_1 \subseteq C_1$ and $w_2 \subseteq C_2$ are the subsets of candidates allocated to the two communities V_1 and V_2 respectively so that no candidate is allocated to two subsets. As in the standard setting, we assume that each V_i, $i \in [2]$, uses a utility function f_i to valuate the committees. The goal of the model is to find a winners allocation that provides certain desirable properties with respect to f_1 and f_2. These properties are defined in the next section.

Social Welfare and Envy-freeness

Needless to say, not all winners allocations of a double election are equally important. We start our exploration by studying numerous notions and properties which are helpful for us to distinguish important winners allocations from less important ones. Here, each property is defined as a subset of winners allocations. We first give the definitions of the properties, and then we study their implication relationships.

Properties of Winners Allocations

First, we study three notions of social welfare. In the following notions, let (w_1, w_2) be a (k_1, k_2)-winners allocation of a double-election $\mathcal{E} = (E_1 = (C_1, V_1), E_2 = (C_2, V_2))$. Let f_1 and f_2 be two utility functions.

Definition 1. *Three types of social welfare with respect to (f_1, f_2) are defined as follows.*

- *Utilitarian social welfare (USW):*

$$f_1(w_1, V_1) + f_2(w_2, V_2).$$

- *Egalitarian social welfare (ESW):*

$$\min\{f_1(w_1, V_1), f_2(w_2, V_2)\}.$$

- *Nash social welfare (NSW):* $(f_1(w_1, V_1) \cdot f_2(w_2, V_2))^{\frac{1}{2}}$.

For each $X \in \{\mathsf{USW}, \mathsf{ESW}, \mathsf{NSW}\}$, we use $\mathsf{XO}(\mathcal{E}, (k_1, k_2), (f_1, f_2))$ to denote the property consisting of all optimal (k_1, k_2)-winners allocations of the double election $\mathcal{E}$ with respect to the utility functions f_1 and f_2. (In the notions, "O" stands for "optimal"). By optimal, we meant a (k_1, k_2)-winners allocation of $\mathcal{E}$ with the maximum possible value of X among all (k_1, k_2)-winners allocations. It should be pointed out that the above social welfare have been also applied in other settings such as resource allocations (see, e.g., (Nguyen et al. 2014)). Each of the above notions has its own merits and which one is the best to apply tightly depends on the concrete settings. We refer to (Nguyen, Roos, and Rothe 2013) for a nice discussion on this issue.

Next, we study three notions of envy-freeness aimed to identify winners allocations such that each of the two voting communities is happier with its own allocated winners than with those allocated to the other community. In particular, the first notion states that it is impossible to make any of the voting communities better off by replacing some winners allocated to the community with the same number of winners allocated to the other community.

Definition 2 (envy-freeness (EF)). *For each $i \in [2]$, we say that V_i envy w_{3-i} with respect to f_i if there are $w \subseteq (w_{3-i} \cap$ $C_i)$ and $w' \subseteq w_i$ such that $|w| = |w'|$ and $f_i((w_i \setminus w') \cup w, V_i) > f_i(w_i, V_i)$.*

The allocation (w_1, w_2) is EF with respect to (f_1, f_2) if the community V_i does not envy w_{3-i} for both $i \in [2]$.

The above notion concerns only whether one of V_1 and V_2 can be made better off by replacing some winners but ignores whether after the replacement the other community is guaranteed to have a committee not worse than the previously allocated one. This motivates us to study the protective envy-freeness (PEF) defined as follows.

Definition 3 (PEF). *For each $i \in [2]$, we say that V_i protectively envy w_{3-i} with respect to (f_1, f_2) if there are $w \subseteq (w_{3-i} \cap C_i)$, $w' \subseteq w_i$, and $w^* \subseteq ((C_{3-i} \setminus (w_i \cup w_{3-i})) \cup (w' \cap C_{3-i}))$ such that*

(1) $|w| = |w'| = |w^*|$;

(2) $f_i((w_i \setminus w') \cup w, V_i) > f_i(w_i, V_i)$; and

(3) the lose of the community V_{3-i} after removing w can be compensated by the addition of w^*, i.e.,

$$f_{3-i}((w_{3-i} \setminus w) \cup w^*, V_{3-i}) \geq f_{3-i}(w_{3-i}, V_{3-i}).$$

The allocation (w_1, w_2) is PEF with respect to (f_1, f_2) if V_i does not protectively envy w_{3-i} for both $i \in [2]$.

The next notion, named replacement envy-freeness (REF), is stronger than the above one in the sense that replacements only take place inside $w_1 \cup w_2$.

Definition 4 (REF). *We sat that V_1 and V_2 replacement envy each other with respect to (f_1, f_2) if there are $w'_1 \subseteq (w_1 \cap C_2)$ and $w'_2 \subseteq (w_2 \cap C_1)$ such that*

(1) $|w'_1| = |w'_2|$;

(2) $f_1((w_1 \setminus w'_1) \cup w'_2, V_1) \geq f_1(w_1, V_1)$;

(3) $f_2((w_2 \setminus w'_2) \cup w'_1, V_2) \geq f_2(w_2, V_2)$; and

(4) at least one of the inequalities in (2) and (3) strictly holds.

The allocation (w_1, w_2) is REF with respect to (f_1, f_2) if V_1 and V_2 do not replacement envy each other.

Finally, we study the notion of Pareto optimality. A (k_1, k_2)-winners allocation is Pareto optimal (PO) if there are no other (k_1, k_2)-winners allocations which strictly improve the utility of one community without hurting the other.

Definition 5 (PO). *The (k_1, k_2)-winners allocation (w_1, w_2) is PO if there does not exist a different (k_1, k_2)-winners allocation (w'_1, w'_2) such that $f_1(w'_1, V_1) \geq f_1(w_1, V_1)$, $f_2(w'_2, V_2) \geq f_2(w_2, V_2)$, and at least one of the two inequalities holds strictly.*

For a notion X being one of EF, PEF, REF, and PO, let $\mathsf{X}(\mathcal{E}, (k_1, k_2), (f_1, f_2))$ be the property consisting of all (k_1, k_2)-winners allocations of the double election $\mathcal{E}$ that are X with respect to (f_1, f_2).

Implication Relationships

Now we study the implication relationships among the notions introduced above. We say that a property (strictly) implies another if for every double election $\mathcal{E}$, every two integers k_1 and k_2, and every two utility functions f_1 and f_2, the former with the arguments $\mathcal{E}$, (k_1, k_2), and (f_1, f_2) is a
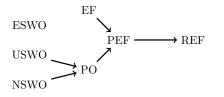
Figure 1: An arc from a property X to another property Y means that X implies Y.

(proper) subset of the latter with the same arguments. Two properties are independent if neither implies the other.

The following theorem offers the implication relationships among the properties. The relationships hold for all utility functions and, moreover, for many natural utility functions the implication relationships are strict. In the general case, two different utility functions value k-committees differently. However, when every voter approves exactly one candidate and we want to select only one winner, almost all natural utility functions degenerate to the AV function. Let F_{AV} be the set of all such utility functions. Precisely, a utility function f belongs to F_{AV} if for every election (C, V) where every vote in V is a singleton, it holds that $f(c, V) = \text{AV}(c, V) = \sum_{v \in V} |v \cap \{c\}|$ for every $c \in C$. Clearly, all utility functions listed in Table 1 belong to F_{AV}.

Theorem 1. *The implication relationships among the properties USWO, ESWO, NSWO, EF, PEF, REF, and PO shown in Figure 1 hold. Moreover, if the two utility functions in the properties belong to F_{AV}, the relationships are complete, in the sense that if it is not shown in the figure that a property X implies another property Y, then X does not imply Y.*

One may wonder why we state the completeness only for utility functions in F_{AV}. In fact, if this is not the case, certain implications won't be strict. Consider for instance the AV function and a constant function f such that $f(w, V) = 1$ for all possible w and V. Then, it is easy to see that, with respect to these two utility functions, PO implies also USWO, ESWO, and NSWO.

It is easy to find a double election where EF winners allocations do not exist (e.g., $C_1 = C_2 = \{a, b\}$, all votes in V_1 and V_2 approve only a but not b, and $k_1 = k_2 = 1$). However, as there are always winners allocations with the maximum USW, Theorem 1 gives us the following result.

Corollary 1. *Every double election admits nonempty PEF, REF, and PO (k_1, k_2)-winners allocations.*

Complexity of Winners Allocations

Given the desirable properties defined in the previous section, a natural question is then how efficiently can we compute a winners allocation that provides these properties. This is the focus of this section. In particular, we study the complexity of the following problem.

Let $X \in \{\text{USWO}, \text{ESWO}, \text{NSWO}, \text{EF}, \text{PEF}, \text{REF}, \text{PO}\}$, and let f_1 and f_2 be two utility functions.

X WINNERS ALLOCATION W.R.T. (f_1, f_2) (X-WA-(f_1, f_2))	
Given:	A double election $\mathcal{E} = ((C_1, V_1), (C_2, V_2))$ and two nonnegative integers k_1, k_2 such that $k_1 \le |C_1|$, $k_2 \le |C_2|$, and $k_1 + k_2 \le |C_1 \cup C_2|$.
Task:	If $X(\mathcal{E}, (k_1, k_2), (f_1, f_2)) \ne \emptyset$, return a (k_1, k_2)-winners allocation from $X(\mathcal{E}, (k_1, k_2), (f_1, f_2))$; otherwise, return $(\emptyset, \emptyset)$.

As shown in the previous section, except for $X = \text{EF}$, it holds that $X(\mathcal{E}, (k_1, k_2), (f_1, f_2)) \ne \emptyset$.

Winners Allocations with Maximum Social Welfare

For a utility function f, the COMMITTEE SELECTION problem is to compute a k-committee $w \subseteq C$ such that $f(w, V) \ge f(w', V)$ for all k-committees $w' \subseteq C$, where C is a given set of candidates, $k \le |C|$ is a given nonnegative integer, and V is a given multiset of votes over C. COMMITTEE SELECTION is a special case of our problem USWO-WA-(f_1, f_2) where one of k_1 and k_2 is zero. It is known that if f is additive, then COMMITTEE SELECTION is polynomial-time solvable (see, e.g., (Yang and Wang 2019)): one first ranks the candidates according to a nonincreasing order of their f scores, from the largest to the smallest, and then the first k candidates constitute a solution. It is easy to see that this algorithm is extendable to weakly additive functions. In the following, we further extend the result to USWO-WA-(f_1, f_2).

Theorem 2. *If both f_1 and f_2 are weakly additive, then USWO-WA-(f_1, f_2) is polynomial-time solvable.*

Now we consider winners allocations that provide the maximum egalitarian/Nash social welfare w.r.t. utility functions that are weakly additive. Somewhat interestingly, in this setting the complexity varies. We first show that if one of the utility functions is AV or CSAV, and the other is weakly additive, we still have polynomial-time solvability results. To this end, we first show the polynomial-time solvability of an intermediate problem defined below.

WINNERS COMPUTATION FOR f_1 AND f_2 (WC-(f_1, f_2))	
Given:	A double election $\mathcal{E} = ((C_1, V_1), (C_2, V_2))$, two nonnegative integers k_1 and k_2 such that $k_1 \le |C_1|$, $k_2 \le |C_2|$, and $k_1 + k_2 \le |C_1 \cup C_2|$, and one nonnegative rational number r_1.
Task:	Return a (k_1, k_2)-winners allocation (w_1, w_2) of $\mathcal{E}$ with the maximum value of $f_2(w_2, V_2)$ under the restriction that $f_1(w_1, V_1) = r_1$, or return $(\emptyset, \emptyset)$ if such a winners allocation does not exist.

We also study a decision version of WC-(f_1, f_2), where we have an additional rational number r_2 in the input, and the question is whether $\mathcal{E}$ admits a (k_1, k_2)-winners allocation (w_1, w_2) such that $f_1(w_1, V_1) = r_1$ and $f_2(w_2, V_2) \ge r_2$. We denote this problem by WD-(f_1, f_2).

Theorem 3. *Let f be a weakly additive utility function. Then, WC-(AV, f) is polynomial-time solvable.*

Proof (sketch). Let $I = ((E_1, E_2), k_1, k_2, r)$ be an instance of WC-(AV, f) where $E_1 = (C_1, V_1)$ and $E_2 = (C_2, V_2)$. For ease of exposition, let f_1 be the AV function and let f_2

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be an additive realization of f of order k_2. We assume that $r \leq |V_1| \cdot |C_1|$ (otherwise we directly return $(\emptyset, \emptyset)$).

Our algorithm first splits the given instance into polynomially many subinstances and then solves each subinstance in polynomial time via a dynamic programming algorithm. In particular, each subinstance is specified by three nonnegative integers $k_1' \leq \min\{k_1, |C_1 \setminus C_2|\}$, $k_2' \leq \min\{k_2, |C_2 \setminus C_1|\}$, and $s_1 \leq r$, and the goal is to compute a (k_1, k_2)-winners allocation (w_1, w_2) of (E_1, E_2) that has the maximum possible $f_2(w_2, V_2)$ under the restrictions that $|w_1 \cap (C_1 \setminus C_2)| = k_1'$, $|w_2 \cap (C_2 \setminus C_1)| = k_2'$, and $f_1(w_1 \cap (C_1 \setminus C_2), V_1) = s_1$, if it exists. We solve the subinstance as follows.

To compute a k_1'-committee of $C_1 \setminus C_2$ of AV score s_1 from V_1, we resort to pseudopolynomial-time algorithms for the classic κ-SUBSET SUM problem (see, e.g., (Koiliaris and Xu 2019)). Particularly, we solve a κ-SUBSET SUM instance $(A = \{f_1(c, V_1) : c \in C_1 \setminus C_2\}, k_1', s_1)$, where the goal is to find k_1' integers in A whose sum is s_1. As the input size of I is at least $|C_1| + |V_1|$ and every number in A is at most $|V_1| \cdot |C_1|$, the algorithm in fact runs in polynomial time in the input size of I. If the κ-SUBSET SUM instance is a NO-instance, we discard this subinstance of I. Otherwise, Let w_1' be the set of the k_1' candidates corresponding to k_1' numbers returned by the κ-SUBSET SUM algorithm.

Next, we order candidates in $C_2 \setminus C_1$ according to their f_2 scores received from V_2, from the highest to the lowest. Let w_2' be the set of the first k_2' candidates in the order and let s_2 denote the f_2 score of these k_2' candidates received from V_2, i.e., $s_2 = f_2(w_2', V_2)$.

The task now is to identify, for every $i \in [2]$, $k_i - k_i'$ candidates from $C_1 \cap C_2$ so that putting them into w_i' provides us a solution. Obviously, if $k_1 - k_1' + k_2 - k_2' > |C_1 \cap C_2|$, we can immediately discard the subinstance. So, we assume that this is not the case. The algorithm continues as follows. If $C_1 \cap C_2 = \emptyset$, we do the following: if $k_1' = k_1$, $k_2' = k_2$, and $s_1 = r$, we return (w_1', w_2') for the subinstance of I; otherwise, we discard the subinstance. We may assume now that $C_1 \cap C_2 \neq \emptyset$. Let $(c_1, c_2, \ldots, c_t)$ be a fixed order of $C_1 \cap C_2$. For each $i \in [t]$, let $C^{\leq i} = \{c_j : j \in [i]\}$. We maintain a 4-dimensional table $\mathsf{T}(\ell_1, \ell_2, i, r')$ of four nonnegative integers such that $i \in [t]$, $\ell_1 \leq \min\{i, k_1 - k_1'\}$, $\ell_2 \leq \min\{i, k_2 - k_2'\}$, and $r' \leq r - s_1$. The entry is supposed to store the maximum possible f_2 score of an ℓ_2-committee $w_2^i \subseteq C^{\leq i}$ from V_2, under the restriction that there is an ℓ_1-committee of $C^{\leq i} \setminus w_2^i$ of f_1 score exactly r' from V_1 (if such an ℓ_1-committee under this restriction does not exist, $-\infty$ is stored in the entry). The values of base entries can be calculated in polynomial time based on the definition of the table. The remaining entries are computed via the following recursive formula. In particular, if $f_1(c_i, V_1) \leq r'$, we define

$$\mathsf{T}(\ell_1, \ell_2, i, r') = \max\{\mathsf{T}(\ell_1 - 1, \ell_2, i - 1, r' - f_1(c_i, V_1)),$$
$$\mathsf{T}(\ell_1, \ell_2 - 1, i - 1, r') + f_2(c_i, V_2), \mathsf{T}(\ell_1, \ell_2, i - 1, r')\}.$$

The three entries in the max function respectively correspond to that c_i is allocated to the community V_1, to the community V_2, or none of the communities V_1 and V_2. If, however, $f_1(c_i, V_1) > r'$, c_i cannot be allocated to V_1, and

hence in this case we define $\mathsf{T}(\ell_1, \ell_2, i, r') =$

$$\max\{\mathsf{T}(\ell_1, \ell_2 - 1, i - 1, r') + f_2(c_i, V_2), \mathsf{T}(\ell_1, \ell_2, i - 1, r')\}.$$

The output of the above dynamic programming algorithm is a 3-tuple whose components are, respectively, denoted by $s(k_1', k_2', s_1)$, $\mathsf{w}_1(k_1', k_2', s_1)$ and $\mathsf{w}_2(k_1', k_2', s_1)$. Precisely,

$$s(k_1', k_2', s_1) = \mathsf{T}(k_1 - k_1', k_2 - k_2', t, r - s_1) + s_2.$$

If $s(k_1', k_2', s_1) \geq 0$, then using standard backtracking technique of dynamic programming algorithms, we can compute two disjoint subsets $w_1^*, w_2^* \subseteq C_1 \cap C_2$ such that $f_2(w_2^*, V_2)$ is maximized under the restrictions that $|w_1^*| = k_1 - k_1'$, $|w_2^*| = k_2 - k_2'$, and $f_1(w_1^*, V_1) = r - s_1$. In this case, we define $\mathsf{w}_1(k_1', k_2', s_1) = w_1' \cup w_1^*$ and $\mathsf{w}_2(k_1', k_2', s_1) = w_2' \cup w_2^*$. However, if $s(k_1', k_2', s_1) < 0$, we discard this subinstance.

For the original instance I, if all subinstances are discarded, we return $(\emptyset, \emptyset)$. Otherwise, let $\hat{k_1}$, $\hat{k_2}$, and $\hat{s_1}$ be such that $s(\hat{k_1}, \hat{k_2}, \hat{s_1}) = \max_{k_1', k_2', s_1}\{s(k_1', k_2', s_1)\}$, where k_1', k_2', and s_1 run over all integers such that the subinstance specified by k_1', k_2', and s_1 is not discarded. Then, we return $(\mathsf{w}_1(\hat{k_1}, \hat{k_2}, \hat{s_1}), \mathsf{w}_2(\hat{k_1}, \hat{k_2}, \hat{s_1}))$ for I. □

The algorithm in the proof of Theorem 3 can be adapted for the case where f_1 is CSAV. Furthermore, observe that for $f_1 \in \{\text{AV}, \text{CSAV}\}$ and f_2 being weakly additive, NSWO/ESWO-WA-(f_1, f_2) is polynomial-time reducible to WC-(f_1, f_2). We then have the following result.

Theorem 4. *Let $f_1 \in \{AV, CSAV\}$ and f_2 be a weakly additive utility function. Then, ESWO/NSWO-WA-(f_1, f_2) is polynomial-time solvable.*

One may wonder whether the polynomial-time solvability in Theorem 3 remains if we replace AV with some other additive functions like SAV. Somewhat surprisingly, this cannot be the case unless P=NP, as we can show that in this case the problem becomes NP-hard. More interestingly, our NP-hardness result holds even in a very special case, namely, the case where the two elections in the given double election are both CI and VI. (see Introduction for the definitions of CI and VI, and related references)

It should be noticed that the algorithm in the proof of Theorem 3 can be easily adapted to apply for all weakly additive functions f_1 and f_2, by changing the fourth component in the dynamic programming table into rational numbers which run over all possible scores of committees. However, for f_1 being some other additive functions like SAV, we could not expect the size of the table to be polynomially bounded in the input size, because of the fourth component.

Theorem 5. *WD-(SAV, SAV) is NP-hard even when $C_1 = C_2 = C$, $V_1 = V_2$ and, moreover, (C, V_1) is both VI and CI.*

Proof. We prove the theorem by a reduction from the κ-RATIONAL $(0, 1)$-SUBSET problem, where given a multiset $A = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\}$ of n rational numbers, an integer k, and a rational number r, the question is whether there are k numbers in A whose sum is r. Very recently, Wojtczak (2018) proved that this problem is strongly NP-hard (note that if A contains only integers, the problem is weakly

NP-hard). We assume that $0 < \frac{a_i}{b_i} < 1$ for all $i \in [n]$. [1] Let $s = \sum_{i \in [n]} \frac{a_i}{b_i}$. We construct a WD-(SAV, SAV) instance $((C, V_1), (C, V_2), k_1, k_2, r_1, r_2)$ as follows.

For each $\frac{a_i}{b_i} \in A$, $i \in [n]$, we create a set $C(i)$ of b_i candidates. For each $i \in [n]$, we choose an arbitrary but fixed candidate in $C(i)$, and let $c(i) \in C(i)$ denote this candidate. Let $C^* = \{c(i) : i \in [n]\}$ denote the set of the n selected candidates. Let $C = \bigcup_{i \in [n]} C(i)$. Clearly, C consists of $\sum_{i \in [n]} b_i$ candidates. This completes the construction of the candidates. The following votes are created in both V_1 and V_2. For each $\frac{a_i}{b_i} \in A$, $i \in [n]$, we create a_i votes each of which approves exactly the candidates in $C(i)$. In addition, for every $i \in [n]$ we create $k + 1$ votes each of which approves exactly the candidate $c(i)$. In total, each of V_1 and V_2 consists of $(k+1) \cdot n + \sum_{i \in [n]} a_i$ votes. We complete the reduction by setting $k_1 = k$, $k_2 = n - k$, $r_1 = r + k \cdot (k+1)$ and $r_2 = s - r + (n - k) \cdot (k + 1)$. Now we prove the correctness of the reduction.

($\Rightarrow$) Let $H \subseteq [n]$ be a set of k integers such that $\sum_{i \in H} \frac{a_i}{b_i} = r$. Let $w_1 = \{c(i) \in C^* : i \in H\}$ and $w_2 = \{c(i) \in C^* : i \in [n] \setminus H\}$. Due to the construction, each candidate $c(i) \in C^*$ has SAV score $\frac{a_i}{b_i} + k + 1$ from both V_1 and V_2, i.e.,

$$f_1(c(i), V_1) = f_2(c(i), V_2) = \frac{a_i}{b_i} + k + 1. \quad (1)$$

Then, $f_1(w_1, V_1) = \sum_{i \in H} \left(\frac{a_i}{b_i} + k + 1 \right) = r + k \cdot (k+1) = r_1$ and $f_2(w_2, V_2) = \sum_{i \in [n] \setminus H} \left(\frac{a_i}{b_i} + k + 1 \right) = s - r + (n - k) \cdot (k + 1) = r_2$. So, (w_1, w_2) is a desired winners allocation.

($\Leftarrow$) Assume that the constructed WD-(SAV, SAV) instance is a YES-instance. We first prove the following claim.

Claim 1. For every k-committee $w \subseteq C$ such that $f_1(w, V_1) = r_1$, it holds that $w \subseteq C^*$.

Proof of Claim 1. Assume for the sake of contradiction that there is a k-committee $w \subseteq C$ such that $f_1(w, V_1) = r_1$ but $w \setminus C^* \neq \emptyset$. W.l.o.g., let $x = |w \cap C^*|$. Hence, we have $x \leq k - 1$. From the construction of the votes, for every $i \in [n]$, it holds that every candidate in $C(i) \setminus \{c(i)\}$ has SAV score exactly $\frac{a_i}{b_i}$, which is smaller than 1 as we assumed. In addition, as pointed out above, every $c(i)$ where $i \in [n]$ has SAV score exactly $\frac{a_i}{b_i} + k + 1$. It follows that

$$f_1(w, V_1) \leq (k - x) + \sum_{c(i) \in w \cap C^*, i \in [n]} \left(\frac{a_i}{b_i} + k + 1 \right)$$
$$\leq (k - x) + x \cdot (k + 1) + x \leq k^2 + k - 1 < r_1.$$

a contradiction. This completes the proof of the claim.

As $k_1 + k_2 = n$, and the SAV score of every candidate in $\bigcup_{i \in [n]} (C(i) \setminus \{c(i)\})$ is strictly smaller than that of anyone in C^*, we know that if the WD-(SAV, SAV) instance

[1] In the strongly NP-hardness proof of k-RATIONAL $(0, 1)$-SUBSET in (Wojtczak 2018), each $\frac{a_i}{b_i}$ is a positive rational number smaller than 2. We can transform this instance into an equivalent instance by replacing each $\frac{a_i}{b_i}$ with $\frac{a_i}{2b_i}$ and replacing r with $\frac{r}{2}$.

is a YES-instance, it must admit a solution (w_1, w_2) such that $w_2 \subseteq C^*$. Furthermore, by Claim 1, we know that $w_1 \cup w_2 = C^*$ holds. According to the analysis for the other direction (see Equation (1)), the SAV score of the committee C^* from each of V_1 and V_2 is $\sum_{i \in [n]} \left(\frac{a_i}{b_i} + k + 1 \right) = s + n \cdot (k + 1)$, which is exactly $r_1 + r_2$. It follows that the SAV score of w_1 from V_1 and the SAV score of w_2 from V_2 are $r_1 = r + k \cdot (k + 1)$ and $r_2 = s - r + (n - k) \cdot (k + 1)$, respectively, which further implies that $\sum_{c(i) \in w_1, i \in [n]} \frac{a_i}{b_i} = r$.

Finally, we show that (C_1, V_1) is both VI and CI. To this end, we define a linear order over C where, for every $i \in [n]$, candidates in $C(i)$ are consecutive. Analogously, we define a linear order over V_1 where for every $i \in [n]$, the a_i votes approving $C(i)$ and the $k+1$ votes approving only the candidate $c(i)$ are consecutive, with the latter $k + 1$ votes ordered after the a_i votes. It is easy to check that the candidates approved in every vote constructed above are consecutive in the candidates-order, and for every candidate c all votes approving c are consecutive in the votes-order. $\square$

Based on the hardness reduction in the proof of Theorem 5, we establish the NP-hardness of ESWO/NSWO-WA-(f_1, f_2) for f_1 and f_2 being SAV or MSAV.

Theorem 6. *ESWO/NSWO-WA-(f_1, f_2) where $f_1, f_2 \in \{SAV, MSAV\}$ are NP-hard. This holds even when the two elections in the given double election are both VI and CI.*

Envy-Free Winners Allocations

This section is devoted to the computation of envy-free winners allocations. The following corollary is a consequence of Theorems 1 and 2.

Corollary 2. *Let $X \in \{PEF, REF, PO\}$, and let f_1 and f_2 be two utility functions that are weakly additive. Then, X-WA-(f_1, f_2) is polynomial-time solvable.*

As shown earlier, EF winners allocations do not always exist. Nevertheless, we show below that once they exist, we could compute in polynomial time an EF (k_1, k_2)-winners allocation with the maximum USW, when the utility functions used are weakly additive.

Theorem 7. *If f_1 and f_2 are both weakly additive, determining if an EF (k_1, k_2)-winners allocation exists can be done in polynomial time. Moreover, if such an allocation exists, an EF (k_1, k_2)-winners allocation with the maximum USW can be computed in polynomial time.*

Now we study utility functions f_1 and f_2 for at least one of which the COMMITTEE SELECTION problem is NP-hard.

Theorem 8. *If for at least one of f_1 and f_2 COMMITTEE SELECTION is NP-hard, then REF-WA-(f_1, f_2) is NP-hard.*

Theorems 1 and 8 bring to us the following corollary.

Corollary 3. *If for at least one of f_1 and f_2 COMMITTEE SELECTION is NP-hard, then for each $X \in \{USWO, NSWO, EF, PEF, PO\}$, X-WA-$(f_1, f_2)$ is NP-hard.*

The above corollary, however, does not extend to computation of winners allocations of maximum ESW. To verify

	USWO	ESWO	NSWO	EF	PEF/REF/PO
(AV/CSAV, wadd)	P	P (Thm. 4)		P	P
(wadd, wadd)	P (Thm. 2)	NP-h (Thm. 6)		P (Thm. 7)	P
		FPT ($|C_1 \cap C_2|$, $k_1 + k_2$ [Thms. 10, 12])			
(PAV/CCAV, f)	NP-h	NP-h (Thm. 9)	NP-h	NP-h	NP-h (Thm. 8)
	FPT ($|V_1 \cup V_2|$ [Thm. 11])				FPT: $|V_1 \cup V_2|$

Table 2: In the table, "wadd" stands for "weakly additive", and f can be any function in Table 1. Results in gray are implied by other results in the table. The NP-hardness holds even when the two elections in the given instance are both CI and VI.

this, consider the case where f_1 is CCAV for which COM-MITTEE SELECTION is NP-hard (Procaccia, Rosenschein, and Zohar 2008), and f_2 is a constant utility function such that $f_2(w, V_2) = 0$ for all w and V_2. Then, all winners allocations have the same ESW 0, implying that ESWO-WA-(f_1, f_2) for these specific functions f_1 and f_2 is polynomial-time solvable. Therefore, to derive an analogous result for ESWO, we need further restrictions. We say that a utility function f' is a *polynomial bound* of another utility function f if for every multiset V of votes over a set C of candidates and every nonnegative integer $k \leq |C|$, we can construct in polynomial time in $|V| + |C|$ a multiset V' of votes over C such that $f'(c, V') \geq f(w, V)$ holds for all k-committees $w \subseteq C$ and all candidates $c \in C$. It is not hard to see that every utility function shown in Table 1 is a polynomial bound of every function including itself in the table. For instance, SAV is a polynomial bound of PAV because for every C, V, and k as above, we can construct a multiset V' of $|C| \cdot |V| \cdot k$ votes each of which approves all candidates. Let $m = |C|$ and $n = |V|$. Then, every candidate has SAV score $m \cdot n \cdot k \cdot \frac{1}{m} = n \cdot k$ from V', while every k-committee has PAV score at most $n \cdot \sum_{i=1}^{k} \frac{1}{i} \leq n \cdot k$ from V. With the above notion, we have the following theorem.

Theorem 9. *Let f_1 and f_2 be two utility functions. If* COM-MITTEE SELECTION *for f_1 is NP-hard, and f_2 is a polynomial bound of f_1, then ESWO-WA-(f_1, f_2) is NP-hard.*

Some Fixed-Parameter Tractability Results

Given the NP-hardness results established in previous sections, one could be interested in fixed-parameter algorithms for these problems with respect to some natural parameters. The number of candidates and the number of votes are two natural parameters. It is trivial to check that with respect to the parameter $|C_1 \cup C_2|$, all the NP-hardness results established in this paper are FPT. This motivates us to consider a generally smaller parameter $|C_1 \cap C_2|$.

Theorem 10. *For weakly additive utility functions f_1 and f_2, ESWO-WA-(f_1, f_2) and NSWO-WA-(f_1, f_2) are FPT with respect to the parameter $|C_1 \cap C_2|$, where C_1 and C_2 are the two sets of candidates in the given double election. In particular, they can be solved in time $O^*(4^{|C_1 \cap C_2|})$.*

Now we study the parameter the number of votes. We show that with respect to this parameter, computing winners allocations with the maximum USW is FPT for f_1 and f_2 being any functions listed in Table 1.

Theorem 11. *If both f_1 and f_2 are utility functions in Table 1, then USWO-WA-(f_1, f_2) is FPT with respect to the number of total votes.*

Finally, we discuss the parameterized complexity of winners allocation with respect to the number of candidates that are allocated in total, i.e., the parameter $k_1 + k_2$.

Theorem 12. *Let f_1 and f_2 be two weakly additive utility functions. Then, ESWO-WA-(f_1, f_2) and NSWO-WA-(f_1, f_2) are FPT with respect to the number of allocated candidates.*

Concluding Remarks

We proposed a model of winners allocation where given two elections whose candidate sets may intersect, the goal is to allocate k_1 and k_2 winners to two voting communities respectively. We studied several properties of this model, including three kinds of social welfare, three types of envy-freeness, and the Pareto optimality property. For these properties, we explored their implication relationships (Theorem 1 and Figure 1). Then, we investigated the complexity of calculating winners allocations that provide these properties (Theorems 2–9). If both utility functions used by the voting communities are weakly additive, we designed polynomial-time algorithms for almost all properties except for egalitarian/Nash social welfare, for which the complexity varies, depending on the utility functions. Particularly, when the utility functions are SAV or MSAV, computing a winners allocation with the maximum egalitarian/Nash social welfare is NP-hard and, moreover, this holds even when the two elections in the given double election are both VI and CI. When one of the utility functions is not additive, we often have hardness results for all properties. Finally, for hardness results, we also studied their parameterized complexity and derived several fixed-parameter algorithms (Theorems 10–12). Table 2 summarizes these results.

Recently, a set of axiomatic properties such as the justified representation (JR), extended JR, proportional JR have been advocated as desirable properties for approval-based multi-winner voting in certain circumstances. We refer to (Aziz et al. 2015, 2018; Sánchez-Fernández et al. 2017) for the definitions of these properties. It is easy to see that there are double elections where there are no (k_1, k_2)-winners allocations (w_1, w_2) such that w_2 and w_2 satisfy these properties for both elections simultaneously. In fact, the exemplary double election to show the nonexistence of EF winners allocations (after Theorem 1) directly works here.

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