

# Fair and Truthful Mechanisms for Dichotomous Valuations\*

Moshe Babaioff<sup>1</sup>, Tomer Ezra<sup>2</sup>, Uriel Feige<sup>3</sup>

<sup>1</sup> Microsoft Research

<sup>2</sup> Tel Aviv University

<sup>3</sup> Weizmann Institute

moshe@microsoft.com, tomer.ezra@gmail.com, uriel.feige@weizmann.ac.il

## Abstract

We consider the problem of allocating a set of indivisible items to players with private preferences in an efficient and fair way. We focus on valuations that have *dichotomous marginals*, in which the added value of any item to a set is either 0 or 1, and aim to design truthful allocation mechanisms (without money) that maximize welfare and are fair. For the case that players have submodular valuations with dichotomous marginals, we design such a deterministic truthful allocation mechanism. The allocation output by our mechanism is Lorenz dominating, and consequently satisfies many desired fairness properties, such as being envy-free up to any item (EFX), and maximizing the Nash Social Welfare (NSW). We then show that our mechanism with random priorities is envy-free ex-ante, while having all the above properties ex-post. Furthermore, we present several impossibility results precluding similar results for the larger class of XOS valuations.

## 1 Introduction

A central problem in Algorithmic Game Theory is the problem of allocating indivisible goods among players with private preferences. This problem is particularly challenging in settings in which utilities cannot be transferred between players (no money). One consideration in allocating the items is the economic efficiency of the allocation, as we want the best for society as a whole. Another consideration is fairness of the allocation, because in the absence of money, there is no other way for the players to evenly share the welfare generated by the efficient allocation.

In this work we design allocation mechanisms that enjoy desirable properties, related to their economic efficiency, to fairness of the allocation, and to incentive compatibility (truthfulness). Importantly, we consider only settings without money, so a mechanism defines an *allocation rule*, but does not involve a payment rule, as there are no payments. With general valuations, even without any fairness properties, the VCG mechanism is the unique truthful welfare-maximizing mechanism, and it requires payments. Consequently, the focus of our work is on instances in which the

valuation functions of the agents are restricted, and specifically, have the *dichotomous marginals* property. We say that a valuation function  $f$  has *dichotomous marginals* (or for brevity, we say that  $f$  is *dichotomous*) if for every set  $S$  of items and every item  $a$ , the marginal value of  $a$  relative to  $S$  is either 0 or 1. Namely,  $f(S \cup a) - f(S) \in \{0, 1\}$ .

The study of fairness with dichotomous preferences was initiated by Bogomolnaia and Moulin (2004), with additional extensive research of such preferences in various settings (see references in Section 1.2). The above references provide multiple examples of situations that can be modeled using dichotomous preferences. We provide another example that involves constraints not captured by prior work. Consider a setting where the agents are arts students seeking work as museum guides. The items are different shifts in which the students can work as guides in the local arts museum. Suppose that among the shifts (or combinations of shifts) that are feasible for a given student in a given month (for example, one student cannot work on weekends, another student can work at most two shifts a week, etc.), the student may wish to work for as many shifts as possible during the month, but other than that is indifferent to the exact choice of shifts (as long as the combination of shifts is feasible for the student). A model that first-order approximates this setting is one in which the valuation function of a student is modeled as being dichotomous.<sup>1</sup> The allocation problem is to assign students to shifts. Economic efficiency may correspond to filling as many shifts as possible. Fairness may correspond to trying to equalize the number of shifts that each student receives (subject to the feasibility constraints). Incentive compatibility means that it is a dominant strategy for a student to report her true valuation function to the museum – providing an incorrect report cannot lead to a situation in which she receives a bundle of shifts of higher value to her.

### 1.1 Our Contribution and Techniques

We now provide an overview of our main results. Some definitions and technicalities are omitted from this overview, but

\*A full version with additional results and the missing proofs appears in <https://arxiv.org/abs/2002.10704>.  
Copyright © 2021, Association for the Advancement of Artificial Intelligence (www.aaai.org). All rights reserved.

<sup>1</sup>In the full version we discuss a relaxation of the assumption that all desired shifts are *exactly* equivalent for the student, allowing strict preference, while still assuming *approximately* the same marginals.

can be found in Section 2.

We consider settings with a finite set  $M$  of  $m$  indivisible and non-identical items. There is a set of  $n \geq 2$  players (a.k.a. *agents*), denoted by  $V = [n]$ , with each player  $v \in V$  having a *valuation function*  $f_v$  over sets of items. The *value* (or *utility*) of player  $v$  for a set  $S \subseteq M$  is denoted by  $f_v(S)$ . We always assume that any valuation  $f$  is normalized ( $f(\emptyset) = 0$ ) and non-decreasing ( $f(S) \leq f(T)$  for  $S \subseteq T \subseteq M$ ). Given an allocation  $A$ , we use  $A_v$  to denote the set of items allocated to player  $v$ .

One question that we ask in this work is what is the largest class of dichotomous valuation functions for which one has a truthful deterministic allocation mechanism that enjoys good economic efficiency and fairness properties. Let us briefly discuss its various ingredients.

**Classes of valuation functions.** The dichotomous versions of some simple classes of valuation functions were considered in previous work (e.g., *unit demand* (matching) (Bogomolnaia and Moulin 2004), *additive* (Ortega 2020) and *0/1 valued sets* (Bouveret and Lang 2008)). We consider here the hierarchy of complement-free valuation functions introduced in Lehmann, Lehmann, and Nisan (2006), whose four highest classes (in order of containment) are *gross substitutes* (GS), *submodular*, *XOS*, and *subadditive* (recall that both *unit demand* and *additive* are gross substitutes). For valuations with dichotomous marginals, it can be shown that every submodular function is in fact a Matroid Rank Function (MRF), and hence also gross substitutes. We note that valuation functions may be used to express not only the preferences of the players, but also constraints imposed by the allocator. In the museum example above, the museum may impose a restriction that no student can work in two shifts in the same day, and another restriction that no student can work in five shifts in the same week. If a student has an additive valuation function, then incorporating these constraints into her valuation function makes it submodular.

**Economic efficiency.** We wish our allocations to maximize welfare, where the welfare of allocation  $A$  is defined as  $\sum_{v \in V} f_v(A_v)$ . Restricting attention to non-redundant allocations (no item can be removed from a set allocated to a player without decreasing its value), in the setting of dichotomous valuations, maximizing welfare is equivalent to allocating the maximum possible number of items. Hence maximizing welfare can serve as a measure of economic efficiency not only from the point of view of the players, but also from the point of view of the items (as in the museum guides example, where it is in the interest of the museum to fill as many shifts as possible).

**Fairness.** For allocation mechanisms without money it is customary to impose some fairness requirements. They come in many flavors. *Safety* guarantees (such as *proportionality*, *maximin share*<sup>2</sup>) promise the player a certain minimum value, based only on the valuation function of the given player and no matter what the valuation functions of other players are. *Envy-freeness* guarantees (*envy free up to*

*one good* (EF1), *envy free up to any good* (EFX)), ensure that every player  $v$  is at least as happy with her own bundle of goods as she would be with the bundle received by any other player (perhaps up to one good (EF1), or up to any good (EFX)). *Egalitarian* guarantees (*lexicographically maximal* allocations, *Lorenz-dominating* allocations, *maximizing Nash social welfare* (NSW)) attempt to equalize the utilities of all players (to the extent possible, given their valuation functions). Not all fairness notions are attainable in all settings, and in addition, there are settings in which two fairness notions that are attainable are not attainable simultaneously. For this reason, in our work we do not fix one particular fairness notion, but rather attempt to achieve a good mix of fairness properties.

**Truthfulness.** We wish our mechanisms to have the property that reporting her true valuation function is a (weakly) dominant strategy for every player. That is, for every player  $v$ , whatever the reports of other players are, if player  $v$  reports a valuation function different than  $f_v$ , the allocation she gets cannot have higher value to her, compared to the allocation when she reports  $f_v$ .

We now return to our question concerning the largest class of dichotomous valuation functions for which one has a truthful deterministic allocation mechanism that enjoys good economic efficiency and fairness properties. We address this question in the framework of the hierarchy of complement-free valuation functions defined in Lehmann, Lehmann, and Nisan (2006). Our main result shows that if the dichotomous valuation functions are submodular, then a deterministic mechanism that we refer to as *prioritized egalitarian* (PE) indeed satisfies the above requirements.

**Theorem 1** *The prioritized egalitarian (PE) mechanism has the following properties when players have submodular dichotomous valuations:*

1. *Being truthful is a dominant strategy.*
2. *When players are truthful the allocation is welfare maximizing.*
3. *When players are truthful, the allocation of the mechanism is a Lorenz dominating allocation, and consequently it enjoys additional fairness properties, including maximizing Nash social welfare, and being envy-free up to any item (EFX). If furthermore, the valuations are additive dichotomous, the allocation gives every player at least her maximin share.*
4. *If the valuations have succinct representations (that allow computation of function values in polynomial time), then the mechanism can be implemented in polynomial time<sup>3</sup>.*

In contrast, we show that if the valuation functions belong to the class XOS (one level higher than submodular in the hierarchy of Lehmann, Lehmann, and Nisan (2006)), then there is no truthful allocation mechanism (neither deterministic nor randomized) that maximizes welfare, even if one disregards all fairness considerations.

The PE mechanism is based on first proving that in the setting of submodular dichotomous valuation functions there

<sup>2</sup>The maximin share of player  $p$  is the maximum value that could be given to the least happy player if all players had the same valuation function as that of  $p$ .

<sup>3</sup>In the full version we also present an ex-post incentive compatible polynomial implementation in the value queries model

always is a *Lorenz dominating* allocation (exact definitions will follow, but at this point the reader may think a Lorenz dominating allocation as one that both maximizes welfare and equalizes as much as possible the number of items received by each player). The PE mechanism imposes a priority order  $\sigma$  among players, and chooses a non-redundant<sup>4</sup> Lorenz dominating allocation (namely, it does not allocate items that give 0 marginal value to the player receiving them), breaking ties among Lorenz dominating allocations in favor of higher priority players. Proving economic efficiency and fairness properties for this mechanism is straightforward, given the fact that the output allocation is Lorenz dominating. The main technical content in the proof of Theorem 1 (beyond the proof that a Lorenz dominating allocation exists) is to show that the PE mechanism is truthful (for players with submodular dichotomous valuations).

In Section 3.3 we consider a randomized variation of our PE allocation mechanism. This randomized mechanism first assigns the agents priorities uniformly at random, and then runs the PE allocation mechanism with the drawn priorities. We show that this mechanism achieves Envy-Freeness in expectation (ex-ante), is universally truthful and it obtains all the other good properties of the PE mechanism ex-post (a best-of-both-worlds result).

Due to space limitations, most proofs (including the statements of some lemmas) are deferred to the full version.

## 1.2 Previous Work

**Dichotomous preferences:** The study of dichotomous preferences was initiated by Bogomolnaia and Moulin (2004). They consider dichotomous matching problems (two-sided unit-demand preferences) and suggest the randomized Lorenz mechanism to get a probabilistic allocation that is fair in expectation. Within the setting of one-sided markets, the paper of (Bogomolnaia and Moulin 2004) addresses randomized mechanisms for unit-demand valuations. We consider the more general class of submodular valuations, and our main focus is on ex-post fairness. Dichotomous preferences have been further studied extensively in the literature for mechanisms without money (Bogomolnaia, Moulin, and Stong 2005; Freitas 2010; Bouveret and Lang 2008; Kurokawa, Procaccia, and Shah 2018; Ortega 2020), auction design (with private value scaling) (Babaioff, Lavi, and Pavlov 2009; Mishra and Roy 2013) and exchanges (Roth, Sonmez, and Utku Unver 2005; Aziz 2020b).

Maybe the most closely related to our paper is the work of Ortega (2020) which studies the Multi-unit assignment problem (MAP) with dichotomous valuations. MAP is a subclass of the submodular class that slightly extends additive (but does not contain unit demand, for example). The paper suggests picking a fractional “welfarist” solution (vector of fractional utilities) that is Lorenz dominating among those

<sup>4</sup>In the full version we discuss the issue of non-redundancy, showing that the result of Theorem 1 is impossible to obtain when one insists on allocating all items (even undesired ones). Specifically, we show that there is no truthful deterministic allocation mechanism that always allocates all items, maximizes welfare and is EFX. This holds even for additive dichotomous valuations, and even for only two agents.

that maximize welfare. Being fractional, this corresponds to a randomized allocation mechanism rather than a deterministic one. Consequently, the notion of truthfulness used is that of being truthful in expectation. Moreover, the notion of truthfulness is further restricted there, and only allows to conceal desired items in the report, but not to report undesirable items as desired. Under this notion, the solution is strongly group strategyproof. In contrast, the larger class of submodular dichotomous valuations considered in our work contains unit-demand dichotomous valuations, for which no Pareto optimal deterministic allocation mechanism is strongly group strategyproof (Bogomolnaia and Moulin 2004). Being Lorenz dominating, the fractional solution enjoys multiple fairness properties. The work of Ortega (2020) does not explicitly address the question of to what extent these fairness properties are preserved ex-post, after the fractional solution is rounded to an integer solution.

**Fairness:** The literature of fairness is too extensive to survey in this paper. For a general introduction see (Brandt et al. 2016; Brams and Taylor 1996; Moulin 2004). The notions of EF1 and EFX were defined by Budish (2011) and Caragiannis et al. (2019), respectively.

**Best-of-Both-Worlds:** Freeman, Shah, and Vaish (2020) presented a recursive probabilistic serial allocation mechanism for additive valuations. They showed that ex-ante envy-freeness can be achieved in combination with EF1 ex-post. Moreover, they showed that achieving EF ex-ante, and EF1 and PO ex-post is impossible. We, in contrast, are able to achieve all these properties (even for submodular valuations) as we consider dichotomous valuations. Aleksandrov et al. (2015) considered allocation mechanism of additive dichotomous agents when items arrive online that is both EF ex-ante and EF1 ex-post. We consider the offline setting, but for the more general submodular valuations case, and get stronger fairness guarantees (EFX, Lorentz domination).

## 1.3 Independent and Concurrent Work

Concurrent and independent of our work, Halpern et al. (2020) devise an allocation mechanism for the class of *additive* dichotomous valuations. They show that their MNW<sup>tie</sup> deterministic allocation mechanism is EF1, PO, and weakly group strategyproof. The additive dichotomous valuations setting is a special case of our more general setting of submodular dichotomous valuations, and our PE mechanism and MNW<sup>tie</sup> are identical for this special case. Halpern et al. (2020) also obtain a “best of both worlds” type result. They consider a randomized allocation mechanism based on rounding the fractional Nash Social Welfare maximizing allocation, and show that their mechanism is ex-ante weakly group strategyproof and ex-post PO and EF1. Aziz (2020a) reproves a similar result, using the same fractional allocation (and noting that it is in fact ex-ante strongly group strategy proof), but using a different rounding procedure. Our best-of-both-world result<sup>5</sup> (Theorem 6) holds for submodular dichotomous valuations, whereas the results of Halpern et al. (2020) and Aziz (2020a) hold only for dichotomous additive

<sup>5</sup>A previous version of our paper did not contain Theorem 6. It did contain a different best-of-both-worlds result.

valuations. Moreover, our randomized mechanism is *universally* truthful (agents have no regret even *after* they see the realized allocation), whereas it is not known whether this property holds for the mechanisms of Halpern et al. (2020) and Aziz (2020a). For submodular dichotomous valuations, no mechanism can be simultaneously ex-ante strongly group strategy proof and ex-post EF1 (see full version), and in our work we do not attempt to achieve ex-ante weak strategy proofness.

Another concurrent and independent work is of Benabbou et al. (2020), that shows how to find welfare-maximizing and EF1 allocations for dichotomous submodular valuations in a computationally efficient way. Their result is purely algorithmic and does not consider incentives, whereas we solve the harder problem of designing a truthful mechanism that obtains all desired fairness and economic efficiency properties (while being computationally efficient).

## 2 Model and Preliminaries

### 2.1 Valuations

In this paper we consider several classes of valuation functions. The *marginal value* of item  $a \in M$  given a set  $S \subseteq M$  is defined to be  $f(a|S) = f(S \cup \{a\}) - f(S)$ . Next we define some properties of valuation functions we will be using:

- A valuation function  $f$  is *dichotomous* if the marginal value of any item is either 0 or 1, that is,  $f(a|S) = f(S \cup \{a\}) - f(S) \in \{0, 1\}$  for every set  $S \subseteq M$  and item  $a \in M$ .
- A valuation function is *additive* if  $f(S) + f(T) = f(S \cup T)$  for any disjoint sets  $S, T \subseteq M$ .
- A valuation function  $f$  is *submodular* if  $f(S \cup \{a\}) - f(S) \geq f(T \cup \{a\}) - f(T)$  for every pair of sets  $S \subseteq T \subseteq M$  and every item  $a \in M$ .
- A valuation function  $f$  is a *Matroid Rank Function (MRF)* if there exists a matroid<sup>6</sup> for which for every set  $S$  it holds that  $f(S)$  is the rank of set  $S$  in the matroid.

A valuation function  $f$  is *submodular dichotomous* if it is both submodular and dichotomous. It is known that a function is submodular dichotomous if and only if it is a rank function of a matroid.<sup>7</sup> Thus, for brevity we will often refer to a submodular dichotomous valuation function as an MRF valuation.

<sup>6</sup>A *matroid*  $(\mathcal{U}, \mathcal{I})$  is constructed from a non-empty ground set  $\mathcal{U}$  and a nonempty family  $\mathcal{I}$  of subsets of  $\mathcal{U}$ , called the *independent* subsets of  $\mathcal{U}$ .  $\mathcal{I}$  must be downward-closed (if  $T \in \mathcal{I}$  and  $S \subseteq T$ , then  $S \in \mathcal{I}$ ) and satisfy the *exchange property* (if  $S, T \in \mathcal{I}$  and  $|S| < |T|$ , then there is some element  $x \in T \setminus S$  such that  $S \cup \{x\} \in \mathcal{I}$ ). The *rank* of a set  $S$  is the size of the largest independent set contained in  $S$ .

<sup>7</sup>(It is easy to see that an MRF is submodular and dichotomous. For the converse direction, given a submodular dichotomous function  $f$ , consider the family  $\mathcal{I}$  that contains those sets  $S$  for which  $f(S) = |S|$ . This family is downward closed, because  $f$  is dichotomous. Submodularity of  $f$  implies that if  $f(T) > f(S)$  there is an item  $x \in (T \setminus S)$  for which  $f(S \cup \{x\}) = f(S) + 1$ . This in turn implies that  $\mathcal{I}$  satisfies the exchange property. Hence  $\mathcal{I}$  defines a matroid, and  $f$  can be seen to be the rank function of this matroid.)

An interesting special case of submodular dichotomous valuations are such valuations that are additive. A valuation function  $f$  is *additive dichotomous* if it is both additive and dichotomous.

### 2.2 Allocations

We consider mechanisms to allocate items in  $M$  to the players. As we assume that utilities cannot be transferred and there is no money, a mechanism will only specify the allocation function, mapping valuation functions to allocations. We will mostly consider deterministic allocation functions.

An *allocation*  $(A_1, A_2, \dots, A_n)$  with  $A_v \subseteq M$  for every  $v \in V$  and  $\cup_v A_v \subseteq M$ , is an assignment of items to players, possibly leaving some items unallocated. We denote by  $A_v$  the set of items allocated to player  $v$  under allocation  $A$ . The *value* (or *utility*) of allocation  $A$  for player  $v$  that has valuation function  $f_v$  is  $f_v(A_v)$ .

Fix some valuation functions  $f = (f_1, f_2, \dots, f_n)$ . The *welfare of an allocation*  $A$  given  $f$  is  $\sum_v f_v(A_v)$  and an allocation is *welfare maximizing* if there is no other allocation with larger welfare. Note that a welfare maximizing allocation is Pareto optimal. An allocation  $A$  is called *non-redundant* for  $f$  if it does not give any player an item for which she has no marginal value, that is, for any player  $v$  and any item  $a \in A_v$  it holds that  $f_v(A_v) > f_v(A_v \setminus \{a\})$ , or equivalently, every strict subset of  $A_v$  has strictly lower value for  $v$ . We note that for MRF valuation  $f$ , for any non-redundant set  $S$  it holds that  $f(S) = |S|$ . A non-redundant allocation has *maximal size* with respect to  $f$ , if there is no other non-redundant allocation with respect to  $f$  that allocates more items. We say that an allocation is *reasonable* for  $f$  if it both non-redundant and has maximal size with respect to  $f$ . Note that if all players have dichotomous additive valuations, any reasonable allocation is welfare maximizing.

### 2.3 Mechanisms

An allocation mechanism (without money) maps profiles of valuations to an allocation. That is, given valuation functions  $f = (f_1, \dots, f_n)$  an *allocation mechanism*  $\mathcal{M}$  outputs an allocation  $A = \mathcal{M}(f) = \mathcal{M}(f_1, \dots, f_n)$ . We sometimes abbreviate and call an allocation mechanism simply a *mechanism*. A mechanism asks each player to report a valuation function, getting a report  $f'_v$  from each player  $v$ , and allocates the items by running the mechanism on the reported valuations  $(f'_1, \dots, f'_n)$ , that is, mechanism  $\mathcal{M}$  outputs  $A = \mathcal{M}(f'_1, \dots, f'_n)$ . We are interested in mechanisms that are truthful, that is, give players incentives to report their valuation function truthfully. A mechanism  $\mathcal{M}$  is *truthful* if for every player  $v$ , reporting  $f_v$  is a weakly dominant strategy (maximizes her value given any reports of the other players).

We say that a mechanism  $\mathcal{M}$  has property  $P$  if for any input  $f$ , its output allocation  $A = \mathcal{M}(f)$  has property  $P$ . For example, a mechanism is *reasonable* if for any  $f$  the allocation  $A = \mathcal{M}(f)$  is reasonable for  $f$ .

### 2.4 Fairness

The list below presents standard fairness conditions that one may desire.

1. The *maximin share* of a player  $i$  with valuation  $f_i$ , denoted by  $\text{maximin}(f_i)$ , is the maximum over all partitions of the items into  $n$  disjoint bundles  $S_1, \dots, S_n$  of the minimum value according to  $f_i$  of a bundle,  $\min_{j \in [n]} f_i(S_j)$ .
2. An allocation is *envy free* (EF) if for all  $i, j \in [n]$  it holds that  $f_i(A_i) \geq f_i(A_j)$ .  
*Envy free up to one good* (EF1). The envy free condition is relaxed as follows: for all  $i, j \in [n]$  either  $f_i(A_i) \geq f_i(A_j)$ , or there is an item  $e \in A_j$  such that  $f_i(A_i) \geq f_i(A_j \setminus \{e\})$ .  
*Envy free up to any good* (EFX). For all  $i, j \in [n]$  either  $f_i(A_i) \geq f_i(A_j)$ , or for every item  $e \in A_j$  it holds that  $f_i(A_i) \geq f_i(A_j \setminus \{e\})$ . EFX is stronger than EF1.
3. Given an allocation  $A = (A_1, \dots, A_n)$  and valuation functions  $f = \{f_1, \dots, f_n\}$ , the utility vector is  $u_{A,f} = (f_1(A_1), \dots, f_n(A_n))$ , and the sorted utility vector  $s_{A,f}$  is a vector whose entries are those of  $u_{A,f}$  sorted from smallest to largest. We impose a *lex-min* order among sorted vectors, where  $s_1 >_{\text{lexmin}} s_2$  if there is some  $k \in [n]$  such that  $s_1(k) > s_2(k)$  and for every  $1 \leq j < k$  we have that  $s_1(j) = s_2(j)$ . Given the valuation functions  $f$ , an allocation  $A$  is maximal in the lex-min order if for every other allocation  $A'$  we have that  $s_{A,f} \geq_{\text{lexmin}} s_{A',f}$ . We refer to such an allocation as a *lex-min allocation*. Given  $f$ , a lex-min allocation always exists (as the set of allocations is finite).
4. Using notation as above, we also impose a *Lorenz domination* partial order over sorted vectors, where  $s_1 \geq_{\text{Lorenz}} s_2$  if for every  $k \in [n]$ , the sum of first  $k$  entries in  $s_1$  is at least as large as the sum of first  $k$  entries in  $s_2$ . A *Lorenz dominating allocation* is an allocation that Lorenz dominates every other allocation. Given the valuation functions  $f$ , a Lorenz dominating allocation need not exist, but if it does, then it is also a lex-min allocation.
5. Given valuation functions  $f$ , an allocation  $A$  is said to maximize the *Nash Social Welfare* (NSW) if it maximizes the product  $\prod_i f_i(A_i)$ . (Formally, such an allocation maximizes NSW relative to the *disagreement point* of not allocating any item.) Given  $f$ , a maximum NSW allocation always exists, though it need not maximize welfare.

### 3 Submodular Dichotomous Valuations

In this section we prove Theorem 1.

One aspect of the proof of the theorem involves showing that a Lorenz dominating allocation exists. Lorenz domination can be shown to imply the desired welfare and fairness properties. However, simply picking an arbitrary Lorenz dominating allocation does not guarantee truthfulness (see full version). Hence a major part of the proof of the theorem is to show that a particular choice of a Lorenz dominating solution does ensure truthfulness.

#### 3.1 Lorenz Dominating Allocations

The following proposition puts together several observations regarding fairness properties of Lorenz dominating allocations, most (if not all) of which are known.

**Proposition 2** *Given any (normalized and monotone) valuation functions  $f = (f_1, \dots, f_n)$ , a Lorenz dominating allocation, if it exists, also maximizes welfare, is lex-min, and maximizes NSW. If moreover the valuation functions are MRFs, a Lorenz dominating allocation that is non-redundant (see definition of non-redundant in Section 2.2) is also EFX (and hence also EF1). If furthermore, the valuations are additive dichotomous, a Lorenz dominating allocation gives every agent at least his maximin share.*

For MRF valuations Lorenz dominating allocations might not give all agents their maximin share, but are approximately so.

**Proposition 3** *There are MRF valuation functions with respect to which no Lorenz dominating allocation is maximin fair. For every collection of MRF valuation functions, in every Lorenz dominating allocation every player gets at least half her maximin share.*

In view of Propositions 2 and 3, we choose Lorenz domination as our fairness requirement. As we shall see in Theorem 4, in our setting of MRF valuations, a Lorenz dominating allocation always exists, and often, more than one such allocation exists. For example, if there is only one item and all players desire it, then allocating the item to any of the players is a Lorenz dominating allocation. We now wish to address truthfulness of the allocation mechanism. This will be achieved by implementing a particular choice among Lorenz dominating allocations. This choice will be guided by two principles.

The first principle is that the allocation will be *non-redundant*. Namely, for every player, the allocation is such that the set of items given to the player does not contain redundant items that give the player no marginal value. In our setting, this is equivalent to requiring that the set of items received by a player forms an independent set in the matroid underlying the MRF of the player.

The second principle is that of imposing some arbitrary *priority* order among the players, fixed independently of their valuations. Among the possibly many Lorenz-dominating allocations that may exist, we choose one that favors the higher priority players as much as possible. Still, there may be several different allocations that satisfy this condition, but any two of them will be equivalent in terms of the utilities that the players (who have MRF valuation functions) derive from them.

W.l.o.g., let the priority order be such that player  $i$  has priority  $i$  (player 1 has highest priority, player  $n$  has lowest priority). A convenient mathematical way to reason about the priority order is as follows. Add to the instance  $n$  auxiliary items  $a_1, \dots, a_n$ . For every player  $i \in [n]$ , pretend that the marginal value of item  $a_i$  is  $\frac{i}{n^2}$  to player  $i$  (regardless of any other items that player  $i$  may hold), and the marginal value of  $a_j$  with  $j \neq i$  is 0. With the auxiliary items, the new valuations  $f' = (f'_1, \dots, f'_n)$  of players satisfy  $f'_i(S) = f_i(S \cap M) + \frac{i}{n^2} |S \cap \{a_i\}|$ . They are not MRFs (because the marginals of auxiliary items are not in  $\{0, 1\}$ ), but they are still gross substitutes (because each  $f'_i$  is a sum of a gross substitute function  $f_i$  on the original items and a gross substitute function on the auxiliary items). In every

welfare maximizing allocation, for every  $i \in [n]$ , item  $a_i$  is given to player  $i$ . Given the auxiliary item, when allocating the original items, a Lorenz dominating allocation will break ties in favor of higher priority players, as they derive less value from the auxiliary items.

**Theorem 4** *Given MRF valuations  $f = (f_1, \dots, f_n)$  and the auxiliary items (giving rise to new valuations  $f' = (f'_1, \dots, f'_n)$ ), there is a Lorenz dominating allocation  $A'$ . Moreover there is a unique vector of utilities  $(u_{A', f'} = (f'_1(A'_1), \dots, f'_n(A'_n)))$  shared by all Lorenz dominating allocations. Removing the auxiliary items from the Lorenz dominating allocation gives an allocation  $A$  that is Lorenz dominating with respect to the original MRF valuations.*

**Proof.** Consider the following function  $W$  that we shall refer to as a *welfare function*. Given a set  $M$  of indivisible items, a set  $V$  of  $n$  agents, and valuation functions  $f_1, \dots, f_n$ , the function  $W$  is a set function defined over the players. Given a set  $S \subseteq V$ ,  $W(S)$  is the maximum welfare attainable by the set  $S$ . Namely,  $W(S) = \max_{A=(A_1, \dots, A_n)} [\sum_{i \in S} f_i(A_i)]$ . In the full version, we show that given our valuation functions  $f' = (f'_1, \dots, f'_n)$  (the MRFs, augmented with the auxiliary items), the respective welfare function  $W$  is submodular. Dutta and Ray (1989) prove that if  $W$  is submodular, then a Lorenz dominating allocation exists. Consequently, with our valuation functions  $f'$ , a Lorenz dominating allocation  $A'$  exists.

Uniqueness of the vector of utilities is a consequence of the fact that with the auxiliary items, the utility of a player uniquely identifies the player. Consequently, any two different vectors of utilities give two different sorted vectors. The sorted vector of a Lorenz dominating allocation is unique (by definition of the Lorenz domination partial order), and hence the (unsorted) vector is also unique.

Removing the auxiliary items from a Lorenz dominating allocation  $A'$  gives an allocation  $A$  that is Lorenz dominating with respect to the original MRF valuations  $f$ . For the sake of contradiction, suppose otherwise, that there is some allocation  $B$  such that  $A$  does not Lorenz dominate  $B$ . Then there is some  $k \leq n$  such that the sum of the first  $k$  terms of the sorted vector of  $B$  is larger than the sum of the first  $k$  terms of the sorted vector of  $A$ . As the values of both sums are integer, the difference between the two sums is at least 1. Consequently, even  $A'$  does not Lorenz dominate  $B$ , because the total contribution of auxiliary items is at most  $\sum_{i=1}^n \frac{i}{n^2} < 1$ , contradicting the assumption that  $A'$  was Lorenz dominating with respect to  $f'$ . ■

### 3.2 The Prioritized Egalitarian (PE) Mechanism

We can now present our allocation mechanism for MRF valuations, that we refer to as the *prioritized egalitarian (PE) mechanism*. We assume for this purpose that each MRF  $f_i$  has a succinct representation (of size polynomial in the number  $m$  of items) such that given this representation, for every  $S$  one may compute  $f_i(S)$  (answer *value queries*) in time polynomial in  $m$ .

1. The mechanism imposes an arbitrary priority order  $\sigma$  among the agents. For simplicity and without loss of gen-

erality, we assume that the order is from 1 to  $n$ , where player 1 has highest priority.

2. Every player is requested to report his MRF to the mechanism. A report that is not an MRF (or failure to provide a report at all) is considered illegal, and is replaced by the MRF that is identically 0 (and consequently, the non-redundant allocation will not give such a player any item)<sup>8</sup>.
3. Given the reported MRF valuation functions  $r_1, \dots, r_n$ , the mechanism computes a non-redundant Lorenz dominating allocation  $A = (A_1, \dots, A_n)$  with respect to these reports and  $\sigma$  (as implied by Theorem 4), and gives each player  $i$  the respective set  $A_i$ .

We now show that the PE mechanism is truthful. For this we introduce some notation. Given a valuation function  $f_i$  and a set  $D$  of items, we use  $f_{i|D}$  to denote the function  $f_i$  restricted to the items of  $D$ . Namely, for every set  $S$ ,  $f_{i|D}(S) = f_i(S \cap D)$ . We note that if  $f_i$  is an MRF, then so is  $f_{i|D}$ . Truthfulness will be a consequence of the following properties of the allocation mechanism.

We say that an allocation mechanism is *faithful* if the following holds for every collection  $f = (f_1, \dots, f_n)$  of valuation functions and for every player  $i$ . Let  $A_i$  denote the allocation of the mechanism to player  $i$  when the reported valuation functions are  $f$ . Then if instead player  $i$  reports valuation function  $f_{i|A_i}$  (and the reports of the other players remain unchanged), then the allocation to player  $i$  remains  $A_i$ . We say that an allocation mechanism is *strongly faithful* if it is faithful, and in addition, for every set  $A' \subset A_i$ , if player  $i$  reports valuation function  $f_{i|A'}$  (and the reports of the other players remain unchanged), then the allocation to player  $i$  becomes  $A'$ .

We say that an allocation mechanism is *monotone* if the following holds for every collection  $f = (f_1, \dots, f_n)$  of valuation functions, every player  $i$ , and every two sets of items  $S$  and  $T$  with  $S \subset T$ . Let  $A_{i|S}$  denote the allocation of the mechanism to player  $i$  when the reported valuation function for player  $i$  is  $f_{i|S}$  and the remaining reports are as in  $f$ . Then if instead player  $i$  reports a legal (see remark that follows) valuation function  $f_{i|T}$  (and the remaining reports remain unchanged), then the allocation  $A_{i|T}$  to player  $i$  satisfies  $f_i(A_{i|T}) \geq f_i(A_{i|S})$ . (*Remark.* It may happen that  $f_i$  is not an MRF,  $f_{i|T}$  is not an MRF, but  $f_{i|S}$  happens to be an MRF. In this case the PE mechanism might produce a nonempty  $A_{i|S}$  and an empty  $A_{i|T}$ , violating the inequality  $f_i(A_{i|T}) \geq f_i(A_{i|S})$ . For this reason we do not impose the monotonicity condition if the valuation function  $f_{i|T}$  is illegal with respect to the underlying allocation mechanism.)

We next prove that the PE mechanism is truthful.

**Theorem 5** *The PE mechanism is truthful for players with MRF valuations. Namely, for every player with an MRF valuation, reporting her true valuation function maximizes her utility, for any reports of the other players.*

**Proof.** Consider an arbitrary player  $v$ . Fix the reported valuation functions of all other players. All these reported valuation functions can be assumed to be MRFs, because the

<sup>8</sup>See full version for more details.

PE mechanism replaces every non-MRF reported function by the all 0 MRF. Let  $f_v$  be the MRF valuation of  $v$ . Let  $A_v$  denote the set of items that  $v$  receives when reporting  $f_v$ . Suppose now that instead  $v$  reports a different valuation function  $f'_v \neq f_v$ , and receives an allocation  $A'_v$ . We need to show that  $f_v(A_v) \geq f_v(A'_v)$ .

We may assume that  $f'_v$  is an MRF, as otherwise  $v$  gets no item and  $f_v(A_v) \geq 0 = f_v(\emptyset)$ . Change the report  $f'_v$  to  $f'_{v|A'_v}$ . By faithfulness (proven in the full version) the allocation to  $v$  remains  $A'_v$ . Let  $B \subseteq A'_v$  be a subset of smallest cardinality for which  $f_v(B) = f_v(A'_v)$ . Necessarily,  $|B| = f_v(A'_v) = f_v(B)$ , and  $B$  is a maximum size subset of  $A'_v$  that is independent with respect to the matroid underlying  $f_v$ . Change the report  $f'_{v|A'_v}$  to  $f'_{v|B}$ . By strong faithfulness (proven in full version) the allocation to  $v$  becomes  $B$ . Now change the report  $f'_{v|B}$  to  $f_{v|B}$ . This changes nothing because as functions  $f'_{v|B} = f_{v|B}$  (both  $f'_v$  and  $f_v$  give value 1 to items of  $B$ , value 0 to other items, and are additive over  $B$  – for  $f'_v$  additivity follows because the allocation is non-redundant, and for  $f_v$  because  $B$  was chosen to be an independent set of the matroid), and hence the allocation to  $v$  remains  $B$ . Finally, change the report  $f_{v|B}$  to  $f_v$ . By monotonicity (proven in full version), the resulting allocation to  $v$  (which is now simply  $A_v$ ) has value to  $v$  at least as high as  $B$  does. We conclude that  $f_v(A_v) \geq f_v(B) = f_v(A'_v)$ , as desired. ■

In the full version we prove that the PE mechanism can be computed efficiently, as claimed in Theorem 1.

### 3.3 Best-of-Both-Worlds via Random Priorities

We have presented the PE mechanism that is truthful, welfare maximizing and EFX (among other fairness properties). Yet, as this is a deterministic mechanism, it cannot be envy-free (deterministic envy freeness is clearly impossible: consider a single desired item and two agents). In this section we show that envy can be eliminated (ex-ante) by running the PE mechanism with uniformly random priorities, and that this holds even if agents are not risk neutral.

The *random priority egalitarian (RPE) mechanism* is the mechanism that first assigns the agents priorities uniformly at random, and then runs the PE allocation mechanism with the drawn priorities. This mechanism is universally truthful (truthful for any realization of the random priorities), welfare maximizing and obtains all the fairness properties of the PE mechanism ex-post. We also show that it is *stochastically envy free*. This establishes a best-of-both-worlds result: both stochastic envy freeness of the randomized allocation, and EFX (among other fairness properties) ex-post.

For given valuation functions  $(v_1, v_2, \dots, v_n)$ , a distribution over allocations is *stochastically envy free* if for every two agents  $i$  and  $j$ , and for every value  $t$ ,

$$Pr[v_i(A(i)) \geq t] \geq Pr[v_i(A(j)) \geq t],$$

where the probability is taken over the choice of random allocation according to the given distribution.

We note that this notion of stochastic envy-freeness implies *ex-ante envy-freeness*, that is, it implies  $E[v_i(A_i)] \geq$

$E[v_i(A_j)]$ , but it is stronger (see full version), and it implies that for any risk attitude, and not only when an agent is risk neutral, he prefers his own lottery over the lottery of any other agent (e.g., he can be risk seeking or risk averse).

We next present our result for the RPE mechanism:

**Theorem 6** *The random priority egalitarian (RPE) mechanism has the following properties when players have submodular dichotomous valuations:*

1. *Being truthful is a dominant strategy for any realization of the priorities (universally truthful).*
2. *When players are truthful the realized allocation is welfare maximizing.*
3. *When players are truthful, the realized allocation of the mechanism is a Lorenz dominating allocation, and consequently it enjoys additional fairness properties, including maximizing Nash social welfare, and being envy-free up to any item (EFX). If furthermore, the valuations are additive dichotomous, the allocation gives every player at least her maximin share.*
4. *The mechanism is stochastically envy-free, and thus is ex-ante envy free as well as ex-ante proportional.*

## 4 XOS Valuations

We next consider dichotomous valuations beyond the submodular case. A class of valuations that contains submodular valuations is the class of XOS valuations. An XOS valuation  $f$  is defined by a set of additive valuations  $\{f_1, \dots, f_k\}$  and for every  $S$ ,  $f(S) = \max_{i \in [k]} f_i(S)$ . An XOS dichotomous valuation, is a function that is both XOS and dichotomous. We use the following construction of an XOS dichotomous valuation. Given a family  $F$  of sets of items, we define  $f_F(S) = \max_{T \in F} |T \cap S|$ . Clearly  $f_F$  is XOS and dichotomous, since we can define for every  $T \in F$ , the additive function  $f_T(S) = |T \cap S|$ , and  $f_F$  is the max over the  $\{f_T\}_{T \in F}$ .

**Proposition 7** *For the setting with two dichotomous XOS players and four items, there is no randomized truthful-in-expectation mechanism that always maximizes welfare.*

**Proof.** Let the set of items  $M = \{1, 2, 3, 4\}$ . Given a family  $F$  of feasible subsets of  $M$ , let  $f_F$  be the XOS dichotomous function

$$f_F(S) = \max_{T \in F} |T \cap S|$$

Consider any mechanism that always picks an allocation that maximizes the welfare. If both players have the same family  $F_1$  with only one feasible set  $T = \{2, 3, 4\}$ , then there is a player that gets more than one item in expectation, as welfare maximization implies that all three items in  $\{2, 3, 4\}$  must be allocated. W.l.o.g., we assume that player 1 is that player. Suppose now that player 1 has the family  $F_2$ , that contains  $T$  and the set  $\{1\}$ . If player 1 reports  $F_2$  (and player 2 reports  $F_1$ ), a welfare maximizing mechanism must allocate item 1 to player 1 and the remaining items to player 2. Yet player 1 can get higher expected value by reporting  $F_1$ , and thus the mechanism is not truthful in expectation. ■

For additional impossibility results, see the full version.

## References

- Aleksandrov, M. D.; Aziz, H.; Gaspers, S.; and Walsh, T. 2015. Online fair division: Analysing a food bank problem. In *Twenty-Fourth International Joint Conference on Artificial Intelligence*.
- Aziz, H. 2020a. Simultaneously Achieving Ex-ante and Ex-post Fairness. In *Web and Internet Economics - 16th International Conference, WINE 2020, Beijing, China, December 7-11, 2020, Proceedings*.
- Aziz, H. 2020b. Strategyproof multi-item exchange under single-minded dichotomous preferences. *Autonomous Agents and Multi-Agent Systems* 34.
- Babaioff, M.; Lavi, R.; and Pavlov, E. 2009. Single-Value Combinatorial Auctions and Algorithmic Implementation in Undominated Strategies. *J. ACM* 56(1).
- Benabbou, N.; Chakraborty, M.; Igarashi, A.; and Zick, Y. 2020. Finding Fair and Efficient Allocations When Valuations Don't Add Up. In *Algorithmic Game Theory - 13th International Symposium, SAGT 2020, Augsburg, Germany, September 16-18, 2020, Proceedings*.
- Bogomolnaia, A.; and Moulin, H. 2004. Random Matching Under Dichotomous Preferences. *Econometrica* 72: 257 – 279.
- Bogomolnaia, A.; Moulin, H.; and Stong, R. 2005. Collective choice under dichotomous preferences. *Journal of Economic Theory* 122(2): 165 – 184. ISSN 0022-0531.
- Bouveret, S.; and Lang, J. 2008. Efficiency and Envy-freeness in Fair Division of Indivisible Goods: Logical Representation and Complexity. *J. Artif. Intell. Res.* 32: 525–564.
- Brams, S. J.; and Taylor, A. D. 1996. *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press.
- Brandt, F.; Conitzer, V.; Endriss, U.; Lang, J.; and Procaccia, A. D. 2016. *Handbook of Computational Social Choice*. USA: Cambridge University Press, 1st edition. ISBN 1107060435.
- Budish, E. 2011. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. *Journal of Political Economy* 119(6): 1061–1103.
- Caragiannis, I.; Kurokawa, D.; Moulin, H.; Procaccia, A. D.; Shah, N.; and Wang, J. 2019. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation (TEAC)* 7(3): 1–32.
- Dutta, B.; and Ray, D. 1989. A Concept of Egalitarianism Under Participation Constraints. *Econometrica* 57(3): 615–635.
- Freeman, R.; Shah, N.; and Vaish, R. 2020. Best of Both Worlds: Ex-Ante and Ex-Post Fairness in Resource Allocation. In *EC '20: The 21st ACM Conference on Economics and Computation, Virtual Event, Hungary, July 13-17, 2020*.
- Freitas, G. 2010. Combinatorial assignment under dichotomous preferences. Manuscript.
- Halpern, D.; Procaccia, A. D.; Psomas, A.; and Shah, N. 2020. Fair Division with Binary Valuations: One Rule to Rule Them All. In *Web and Internet Economics - 16th International Conference*.
- Kurokawa, D.; Procaccia, A. D.; and Shah, N. 2018. Leximin Allocations in the Real World. *ACM Trans. Econ. Comput.* 6(3–4).
- Lehmann, B.; Lehmann, D. J.; and Nisan, N. 2006. Combinatorial auctions with decreasing marginal utilities. *Games and Economic Behavior* 55(2): 270–296.
- Mishra, D.; and Roy, S. 2013. Implementation in multidimensional dichotomous domains. *Theoretical Economics* 8(2): 431–466.
- Moulin, H. 2004. *Fair division and collective welfare*. MIT press.
- Ortega, J. 2020. Multi-unit assignment under dichotomous preferences. *Mathematical Social Sciences* 103: 15 –24.
- Roth, A. E.; Sonmez, T.; and Utku Unver, M. 2005. Pairwise kidney exchange. *Journal of Economic Theory* 125(2): 151–188.