

# Groupwise Maximin Fair Allocation of Indivisible Goods

**Siddharth Barman**  
Indian Institute of Science  
barman@iisc.ac.in

**Arpita Biswas**  
Indian Institute of Science  
arpitab@iisc.ac.in

**Sanath Kumar Krishnamurthy**  
Chennai Mathematical Institute  
sanathkumar9@cmi.ac.in

**Yadati Narahari**  
Indian Institute of Science  
narahari@iisc.ac.in

## Abstract

We study the problem of allocating *indivisible* goods among  $n$  agents in a *fair* manner. For this problem, maximin share (MMS) is a well-studied solution concept which provides a fairness threshold. Specifically, maximin share is defined as the minimum utility that an agent can guarantee for herself when asked to partition the set of goods into  $n$  bundles such that the remaining  $(n - 1)$  agents pick their bundles adversarially. An allocation is deemed to be fair if every agent gets a bundle whose valuation is at least her maximin share.

Even though maximin shares provide a natural benchmark for fairness, it has its own drawbacks and, in particular, it is not sufficient to rule out unsatisfactory allocations. Motivated by these considerations, in this work we define a stronger notion of fairness, called *groupwise maximin share guarantee* (GMMS). In GMMS, we require that the maximin share guarantee is achieved not just with respect to the grand bundle, but also among all the subgroups of agents. Hence, this solution concept strengthens MMS and provides an *ex-post* fairness guarantee. We show that in specific settings, GMMS allocations always exist. We also establish the existence of approximate GMMS allocations under additive valuations, and develop a polynomial-time algorithm to find such allocations. Moreover, we establish a scale of fairness wherein we show that GMMS implies approximate envy freeness.

Finally, we empirically demonstrate the existence of GMMS allocations in a large set of randomly generated instances. For the same set of instances, we additionally show that our algorithm achieves an approximation factor better than the established, worst-case bound.

## 1 Introduction

In recent years, the topic of fair division of *indivisible goods* has received significant attention in the computer science, mathematics, and economics communities, see, for instance, Chapter 12 in (Brandt et al. 2016). A central motivation behind this thread of research is the fact that classical notions of fairness—such as *envy freeness* (EF)<sup>1</sup> and *proportional-*

*ity*<sup>2</sup>—which were developed for divisible goods (that can be fractionally allocated), do not translate well to the indivisible case. For instance, if there is a single indivisible good and two agents, then no allocation can guarantee EF or proportionality. But, given that a number of real-world settings (such as budgeted course allocations (Budish 2011), division of inheritance, and partitioning resources in a cloud computing environment) entail allocation of discrete/indivisible goods, it is essential to define and study solution concepts which are applicable for a fair division of indivisible goods.

Classically, the applicability of solution concepts is studied via existence results. Understanding if and when a solution concept is guaranteed to exist is of fundamental importance in microeconomics and other related fields. Such existence results have been notably complemented by research—in algorithmic game theory and artificial intelligence—that has focused on computational issues surrounding the underlying solution concepts. Broadly, our results contribute to these key themes by establishing both existential and algorithmic results for a new notion of fairness.

In the context of fair division, the focus on developing efficient algorithms is motivated, in part, by websites such as *Spliddit*<sup>3</sup> (Goldman and Procaccia 2015) and *Adjusted Winner*<sup>4</sup> (Brams and King 2005), which offer fair solutions for dividing goods. Spliddit has attracted more than fifty thousand users and, among other services, it computes allocations which are fair with respect to the standard notions of fairness. One of the solution concepts considered by Spliddit is the *maximin share guarantee* (MMS).

The MMS solution concept was defined in the notable work of (Budish 2011), and it deems an allocation to be fair if each agent gets a bundle of value greater than or equal to an agent-specific threshold, called the *maximin share* of the agent. Specifically, the maximin share of an agent corresponds to the maximum value that the agent can attain for herself if she is hypothetically asked to partition the set of goods into  $n$  bundles and, then, the remaining  $(n - 1)$  agents pick their bundles adversarially. Hence, a risk-averse agent  $i$  would partition the goods (into  $n$  bundles) such that value of

Copyright © 2018, Association for the Advancement of Artificial Intelligence (www.aaai.org). All rights reserved.

<sup>1</sup>An allocation is called envy free if every agent values her bundle at least as much she values any other agent's bundle (Foley 1967; Varian 1974; Stromquist 1980).

<sup>2</sup>An allocation, among  $n$  agents, is called proportionally fair if every agent's value for her share is at least  $1/n$  times the total value of all the goods (Steinhaus 1948).

<sup>3</sup>[www.spliddit.org](http://www.spliddit.org)

<sup>4</sup><http://www.nyu.edu/projects/adjustedwinner/>

the least desirable bundle (according to her) in the partition is maximized. The value of the least desirable bundle in such a partition is called the *maximin share* of agent  $i$ . This definition can be interpreted as a natural generalization of the classical *cut-and-choose* protocol.

Although maximin shares provide a natural benchmark to define fairness, this solution concept has its own drawbacks. In particular, MMS is not sufficient to rule out unsatisfactory allocations; see Section 3 for an example. Moreover, different MMS allocations can be drastically different in terms of, say, the social welfare of the agents.

Motivated by these considerations, we define a strictly stronger notion of fairness, called *groupwise maximin share guarantee* (GMMS). Intuitively, GMMS provides an *ex-post* fairness guarantee: it ensures that, even after the allocation has been made, the maximin share guarantee is achieved not just with respect to the grand bundle of goods, but also among all the subgroups of agents  $J \subseteq [n]$ . Specifically, we say that an allocation is GMMS if, for all groups  $J \subseteq [n]$  and agents  $i \in J$ , the value of  $i$ 's bundle in the allocation is no less than the maximin share of  $i$  restricted to  $J$ . That is, if the agent  $i$  were to repeat the thought experiment (of dividing all the goods allocated to the agents in group  $J$ , so that the other  $|J| - 1$  agents pick their bundles adversarially) to calculate her maximin share restricted to  $J$ , then the value of her bundle is at least this threshold. This definition directly ensures that groupwise maximin share guarantee is a stronger solution concept: GMMS implies MMS. In Section 3, we show that GMMS can, in fact, be arbitrarily better than an allocation that just satisfies MMS.

GMMS also strictly generalizes *pairwise maximin share guarantee* (PMMS), a notion defined by (Caragiannis et al. 2016). In PMMS, the maximin share guarantee is required only for pairs of agents, but not necessarily for the grand bundle. Section 2.1, provides an example which establishes that GMMS is a strict generalization of PMMS and MMS.

The relevance of GMMS is also substantiated by the fact that it implies other complementary notions of fairness, which do not follow from MMS alone. In the context of indivisible goods, relaxations of envy freeness, such as EF1<sup>5</sup> (Budish 2011) and EFX<sup>6</sup> (Caragiannis et al. 2016) have also been studied. In Section 4 we show that (unlike MMS) GMMS fits into this scale of fairness and, in particular, a GMMS allocation is guaranteed to be EFX (and hence EF1). These implications essentially follow from the observation that, by definition, a GMMS allocation is PMMS as well. (Caragiannis et al. 2016) have shown that PMMS implies EFX, and hence we obtain the desired implications.

Throughout the paper, we focus on additive valuations, and a high-level contribution of our work is to show that under additive valuations, a number of useful (existence, algorithmic, and approximation) results which have been es-

<sup>5</sup>An allocation is said to be envy-free up to one good (EF1) if no agent envies any other after removing at most one good from the other agent's bundle; see Definition 4.

<sup>6</sup>An allocation is said to be envy-free up to the least valued good (EFX) if no agent envies any other agent after removing any positively valued good from the other agent's bundle; see Definition 5.

tablished for MMS continue to hold for GMMS as well.

## 1.1 Our Contributions

In addition to proving a scale of fairness for GMMS, we establish the following results:

1. Approximate groupwise maximin share allocations always exist under additive valuations. Prior work has shown that there are instances wherein no allocation is MMS (Procaccia and Wang 2014; Kurokawa, Procaccia, and Wang 2016). These non-existence results have motivated a detailed study of approximate MMS allocations, i.e., allocations in which each agent gets a bundle of value (multiplicatively) close to her maximin share. Along these lines, we consider approximate GMMS (see Definition 3), and show that, under additive valuations, a  $1/2$ -GMMS allocation always exists. In addition, such an allocation can be found in polynomial time.
2. GMMS allocations are guaranteed to exist when the valuations of the agents are either binary or identical.
3. Analogous to the experimental results for MMS (Bouveret and Lemaître 2014), GMMS allocations exist empirically. These simulation results indicate that we do not fall short on such generic existence results by strengthening the maximin solution concept.

## 1.2 Related Work

As mentioned earlier, (Budish 2011) introduced the notion of maximin share guarantee (MMS), and it has been extensively studied since then. In particular, (Bouveret and Lemaître 2014) showed that if the agents' valuations are additive, then an envy free (or proportional) allocation will be MMS as well. They also established that MMS exists under binary, additive valuations. Their experiments, using different distributions over the valuations, did not yield a single example wherein an MMS allocation did not exist. (Kurokawa, Procaccia, and Wang 2016) showed that MMS allocations exist with high probability when valuations are drawn randomly.

However, (Procaccia and Wang 2014) provided intricate counterexamples to refute the universal existence of MMS allocations, even under additive valuations. This motivated the study of approximate maximin share allocations,  $\alpha$ -MMS, where each agent obtains a bundle of value at least  $\alpha \in (0, 1)$  times her maximin share. (Procaccia and Wang 2014) established the existence of  $2/3$ -MMS, and developed a polynomial-time algorithm to obtain such an allocation when the number of agents is a fixed constant. Later, (Amnatidis et al. 2015) showed that a  $2/3$ -MMS can be computed in polynomial (in the number of players) time.

Approximate maximin share allocations have also been studied for general valuations. (Barman and Krishnamurthy 2017) have developed an efficient algorithm which finds a  $1/10$ -MMS allocation under submodular valuations. More recently, (Ghodsi et al. 2017) have improved the approximation guarantee for additive valuations to  $3/4$ . They have also developed constant-factor approximation guarantees for submodular and *XOS* valuations, along with a logarithmic approximation for subadditive valuations.

(Aziz et al. 2017) studied the fair division of indivisible chores (negatively valued goods) and have developed an efficient algorithm which finds a 2-MMS allocation.

(Caragiannis et al. 2016) defined another important fairness notion called *pairwise maximin share guarantee* (PMMS). As mentioned previously, under PMMS, the maximin share guarantee is required only for pairs of agents, and not even for the grand bundle. They also established that PMMS and MMS are incomparable: neither one of these solution concepts implies the other.

## 2 Preliminaries and Notation

We consider the problem of finding a *fair* allocation of a set of *indivisible* goods  $[m] = \{1, \dots, m\}$ , among a set of agents  $[n] = \{1, \dots, n\}$ . For a subset of goods  $S \subseteq [m]$  and integer  $t$ , let  $\Pi_t(S)$  denote the set of all  $t$  partitions of  $S$ . An *allocation* is defined as an  $n$ -partition  $(A_1, A_2, \dots, A_n) \in \Pi_n([m])$ , where  $A_i$  is the set of goods allocated to agent  $i$ .

The preference of the agents over the goods is specified via valuations. Specifically, we denote the valuation of an agent  $i \in [n]$  for a subset of goods  $S \subseteq [m]$  by  $v_i(S)$ . In this work, we assume the valuations to be *non-negative* and *additive*, i.e.,  $v_i(\{g\}) \geq 0$  for all  $g \in [m]$  and  $v_i(S) = \sum_{g \in S} v_i(\{g\})$ . For ease of presentation, we will use  $v_i(g)$  for agent  $i$ 's valuation of good  $g$ , i.e., for  $v_i(\{g\})$ .

As mentioned previously, the fairness notions considered in this work are defined using thresholds called maximin shares. Formally, given an agent  $i$ , parameter  $k \in \mathbb{Z}_+$ , and subset of goods  $S \subseteq [m]$ , the  $k$ -maximin share of  $i$  restricted to  $S$  is defined as  $\mu_i^k(S) := \max_{(P_1, \dots, P_k) \in \Pi_k(S)} \min_{j \in [k]} v_i(P_j)$ . Throughout,  $\text{MMS}_i$  will be used to denote the maximin share of an agent  $i$  with respect to the grand bundle,  $\text{MMS}_i := \mu_i^n([m])$ . We will now formally define *maximin share allocation* (MMS allocation).

**Definition 1** (Maximin Share Allocation). *An allocation  $(A_1, \dots, A_n)$  is said to be an MMS allocation iff for all agents  $i \in [n]$  we have  $v_i(A_i) \geq \text{MMS}_i$ .*

A different solution concept defined by (Caragiannis et al. 2016) requires the maximin share guarantee to hold only for every pair of agents, i.e., an allocation  $\mathcal{A} = (A_1, \dots, A_n)$  is said to achieve *pairwise maximin share guarantee* (PMMS) iff for all  $i, j \in [n]$ , we have  $v_i(A_i) \geq \mu_i^2(A_i \cup A_j)$  and  $v_j(A_j) \geq \mu_j^2(A_i \cup A_j)$ .

In this paper, we strengthen MMS and PMMS, and define a stronger threshold for each agent  $i \in [n]$ , namely *groupwise maximin share* (GMMS <sub>$i$</sub> ). Formally,

$$\text{GMMS}_i = \max_{J \subseteq [n], J \ni i} \mu_i^{|J|} \left( \bigcup_{j \in J} A_j \right).$$

Now we define *groupwise maximin share allocation* (GMMS allocation).

**Definition 2** (Groupwise Maximin Share Allocation). *An allocation  $(A_1, \dots, A_n)$  is said to be a GMMS allocation iff for all agents  $i \in [n]$  we have  $v_i(A_i) \geq \text{GMMS}_i$ .*

Note that an allocation  $(A_1, \dots, A_n)$  is GMMS iff  $v_i(A_i) \geq \mu_i^{|J|} \left( \bigcup_{j \in J} A_j \right)$  for all  $J \subseteq [n]$  such that  $i \in J$ .

The fact that there are fair division instances which do not admit an MMS allocation directly implies that GMMS allocation are not guaranteed to exist either. Therefore, we consider approximate GMMS allocations.

**Definition 3** (Approximate Groupwise Maximin Share Allocation). *An allocation  $(A_1, \dots, A_n)$  is said to be an  $\alpha$ -approximate groupwise maximin share allocation ( $\alpha$ -GMMS) iff for all agents  $i \in [n]$  we have  $v_i(A_i) \geq \alpha \text{GMMS}_i$ .*

A 1-approximate groupwise maximin share allocation is a GMMS allocation.

### 2.1 GMMS Strictly Generalizes MMS and PMMS

In this section, we provide an example to show that allocations which achieve both MMS and PMMS might not be GMMS. Let us consider an instance with 9 agents and 8 goods  $G = \{g_1, g_2, \dots, g_8\}$ . This ensures that  $\text{MMS}_i = 0$  for all  $i \in [n]$  and, hence, all the allocations are MMS. Let us fix an agent, say  $i = 1$ . Let her valuation for 8 goods be  $v_1(g_1) = v_1(g_2) = v_1(g_3) = 5$ ,  $v_1(g_4) = v_1(g_5) = 3$ , and  $v_1(g_6) = v_1(g_7) = v_1(g_8) = 1$ . For all other agents  $i \in \{2, \dots, 9\}$ , let  $v_i(g_z) = 0$  for all  $z \in \{1, \dots, 7\}$  and  $v_i(g_8) = 1$ . Now, let us consider an allocation  $A_1 = \{g_1\}$ ,  $A_2 = \{g_2, g_4\}$ ,  $A_3 = \{g_3, g_5\}$ ,  $A_4 = \{g_6, g_7, g_8\}$ , and  $A_5 = A_6 = A_7 = A_8 = \emptyset$ . This allocation satisfies both MMS and PMMS. However, the maximin share for the group of agents  $J = \{1, 2, 3, 4\}$  is  $\mu_1^4(A_1 \cup \dots \cup A_4) = 6 > 5 = v_1(A_1)$ , and hence the allocation is not GMMS. This illustrates the fact that GMMS is a distinct solution concept which strictly generalizes both PMMS and MMS. This example can be extended to construct instances that admit allocations which satisfy the maximin share guarantee for all subgroups of, say, size at most  $k$ , but are still not GMMS.

## 3 GMMS can be arbitrarily better than MMS

In this section, we provide a class of examples where an MMS allocation is not necessarily satisfactory in terms of agents' valuations. In particular, we show that imposing GMMS leads to allocations which Pareto dominate an allocation which only satisfies MMS.

Consider  $n$  agents and a set of  $n + 3$  goods (where  $[m] = \{g_1, \dots, g_{n+3}\}$ ), along with parameter  $V$  and a sufficiently small  $\varepsilon$ ;  $0 < \varepsilon \ll V$ . The valuations are assumed to be additive and are as follows:

For agents  $i \in \{1, \dots, n-1\}$ ,

$$v_i(\{g_z\}) = \begin{cases} V, & z \in \{1, \dots, n-3\} \\ V/2, & z \in \{n-2, n-1\} \\ V/2 - \varepsilon, & z \in \{n, n+2\} \\ V/2 + \varepsilon, & z \in \{n+1, n+3\} \end{cases}$$

For the agent  $n$ ,

$$v_n(\{g_z\}) = \begin{cases} V, & z \in \{1, \dots, n-1\} \\ 0, & z \in \{n, n+1\} \\ \varepsilon, & z \in \{n+2, n+3\} \end{cases}$$

Note that in this fair division instance the maximin share of the first  $(n-1)$  agents,  $\text{MMS}_i = V$  for all  $i \in \{1, \dots, n\}$ . In addition, the maximin share of the last agent,  $\text{MMS}_n$ , is  $2\varepsilon$ .

Now, consider the allocation  $\mathcal{A} = (A_1, \dots, A_n)$  wherein  $A_i = \{g_i\}$ ,  $i \in \{1, \dots, n-3\}$ ,  $A_{n-2} = \{g_{n-2}, g_{n-1}\}$ ,

$A_{n-1} = \{g_n, g_{n+1}\}$ , and  $A_n = \{g_{n+2}, g_{n+3}\}$ . Allocation  $\mathcal{A}$  is MMS since  $v_i(A_i) = V$  for each  $i \in \{1, \dots, n-1\}$  and  $v_n(A_n) = 2\varepsilon$ . In this allocation the valuation of agent  $n$  is unsatisfactorily low. Another relevant observation is that this allocation is not GMMS. Below, we show that in this instance *any* GMMS allocation allocates a bundle of value  $V$  to every agent, including agent  $n$ .

The fact that  $\mathcal{A}$  is not GMMS follows by considering the goods allocated to agents  $n-2$  and  $n$ , i.e., let  $S := A_{n-2} \cup A_n = \{g_{n-2}, g_{n-1}, g_{n+1}, g_{n+3}\}$ . Now,  $\mu_n^2(S) = V + \varepsilon$ , but  $v_n(A_n) = 2\varepsilon \ll V$ . Furthermore, it can be observed that it is necessary to allocate the agent  $n$  at least one of her high valued goods  $\{1, \dots, n-1\}$  to satisfy GMMS. Thus, GMMS allocation would always ensure that a bundle of at least  $V$  is allocated to all the agents. That is, in this instance, unsatisfactorily low valuations can be avoided by imposing the groupwise maximin share guarantee.

#### 4 Scale of Fairness

As mentioned earlier, *envy freeness* (EF) is a well-studied solution concept in the context of fair division of divisible items. However, for indivisible goods, a simple example with one positively valued good and two agents shows that envy-free allocations do not always exist. Hence, for indivisible goods, natural relaxations of envy freeness—in particular, EF1 and EFX—have been considered in the literature. We now provide formal definitions of these relaxations.

**Definition 4** (Envy-free up to one good (Budish 2011)). *An allocation  $\mathcal{A} = (A_1, A_2, \dots, A_n)$  is said to be envy-free up to one good (EF1) iff for every pair of agents  $i, j \in [n]$  there exists a good  $g \in A_j$  such that  $v_i(A_i) \geq v_i(A_j \setminus \{g\})$ .*

**Definition 5** (Envy-free up to the least positively valued good (Caragiannis et al. 2016)). *An allocation  $\mathcal{A} = (A_1, A_2, \dots, A_n)$  is said to be envy-free up to the least valued good (EFX) iff for every pair of agents  $i, j \in [n]$  and for all goods  $g \in A_j \cap \{g' \in [m] \mid v_i(g') > 0\}$  (i.e., for all goods  $g$  in  $A_j$  which are positively valued by agent  $i$ ) we have  $v_i(A_i) \geq v_i(A_j \setminus \{g\})$ .*

Note that the above mentioned definitions imply that, for additive valuations, an EFX allocation is necessarily EF1 as well. Next we show that, interestingly, these relaxed versions of envy freeness are implied by GMMS, but not by MMS.

**Proposition 1** (Scale of Fairness). *In any fair division instance with additive valuations*

1. *If an allocation  $\mathcal{A}$  is envy free (EF) then it achieves groupwise maximin share guarantee (GMMS) as well.*
2. *If an allocation  $\mathcal{B}$  is GMMS then it is EFX (and, hence, EF1) as well.*

*Proof.* First we will show that EF implies GMMS: Assume that the allocation  $\mathcal{A} = (A_1, \dots, A_n)$  is envy free, that is, for all  $i, j \in [n]$  we have  $v_i(A_i) \geq v_i(A_j)$ . Therefore, for any agent  $i$  and any group of agents  $J \subseteq [n]$  we have  $|J|v_i(A_i) \geq \sum_{j \in J} v_i(A_j)$ . Since the valuation  $v_i$  is additive, this inequality leads to the following bound  $v_i(A_i) \geq \frac{1}{|J|}v_i(S)$ ; here  $S = \cup_{j \in J} A_j$ . Now, an averaging argument establishes the following inequality: for

any  $|J|$ -partition of  $S$  if  $(P_1, P_2, \dots, P_{|J|}) \in \Pi_{|J|}(S)$  then  $v_i(A_i) \geq \min_{1 \leq k \leq |J|} v_i(P_k)$ . Hence,  $v_i(A_i) \geq \max_{(P_1, \dots, P_{|J|}) \in \Pi_{|J|}(S)} \min_k v_i(P_k) = \mu_i^{|J|}(S)$ , and we get that  $\mathcal{A}$  is GMMS.

Next, we argue that GMMS implies EFX: (Caragiannis et al. 2016) have shown that that PMMS implies EFX. By definition, GMMS implies PMMS. Hence, a GMMS allocation is guaranteed to be EFX.  $\square$

Note that (Caragiannis et al. 2016) also provided an example to show that maximin share guarantee by itself does not imply EF1. Hence, an MMS allocation is not necessarily EFX. Consider a fair division instance with three agents, five goods, and each agent values each good at 1. The maximin share of all the agents is 1. Thus, the allocation  $A_1 = \{g_1, g_2, g_3\}$ ,  $A_2 = \{g_4\}$  and  $A_3 = \{g_5\}$  satisfies MMS, but not EF1. This is because agents 2 and 3 continue to envy agent 1 even if a single good is removed from  $A_1$ .

This, overall, shows that while MMS is not enough to guarantee weaker notions of envy freeness, GMMS ensures fairness in terms of such notions, and secures a place in the scale of fairness.

Next we will consider the complementary direction of going from bounded envy to groupwise maximin fairness. In particular, we will establish existence and algorithmic results for approximate GMMS by considering a solution concept which is stronger than EF1, but weaker than EFX. Specifically, we will define allocations which are *envy-free up to one less-preferred good* (EFL)—see Definition 6 below—and show that such allocations are guaranteed to exist, when the valuations are additive. Note that, in contrast, the generic existence of EFX allocations remains an interesting open question. Furthermore, we will prove that EFL allocations can be computed in polynomial time and, under additive valuations, such allocations imply 1/2-GMMS.

**Definition 6.** *An allocation  $\mathcal{A} = (A_1, A_2, \dots, A_n)$  is said to be envy-free up to one less-preferred good (EFL) if for every pair of agents  $i, j \in [n]$  at least one of the following conditions hold:*

- *$A_j$  contains at most one good which is positively valued by  $i$ ;  $|A_j \cap \{g' \mid v_i(g') > 0\}| \leq 1$*
- *There exists a good  $g \in A_j$  such that  $v_i(A_i) \geq v_i(A_j \setminus \{g\})$  and  $v_i(A_i) \geq v_i(\{g\})$ .*

The fact that an EFL allocation is EF1 follows directly from the definitions of these solution concepts. Also, note that if an allocation  $(A_1, \dots, A_n)$  is EFX then for any pair of agents  $i, j \in [n]$  with  $|A_j \cap \{g' \mid v_i(g') > 0\}| > 1$  the second condition in the definition of EFL holds. In particular, write  $A_j^i := A_j \cap \{g' \mid v_i(g') > 0\}$  and consider two distinct goods  $\hat{g}, \tilde{g} \in A_j^i$ . Since the allocation is EFX, we have  $v_i(A_i) \geq v_i(A_j \setminus \{\hat{g}\})$  and  $v_i(A_i) \geq v_i(A_j \setminus \{\tilde{g}\})$ . Note that  $\hat{g} \in A_j \setminus \{\tilde{g}\}$ , and, hence,  $v_i(A_i) \geq v_i(\hat{g})$ . This implies that good  $\hat{g}$  satisfies the second condition in Definition 6. Hence, any EFX allocation is EFL as well.

With this new fairness notion, we have the following chain of implications:  $\text{EF} \Rightarrow \text{GMMS} \Rightarrow \text{EFX} \Rightarrow \text{EFL} \Rightarrow \text{EF1}$ .

## 5 An Approximation Algorithm for GMMS

Our main result in this section shows that 1/2-GMMS allocations always exist under additive valuation, and such allocations can be found efficiently.

**Theorem 1.** *Every fair division instance with additive valuations admits a 1/2-approximate groupwise maximin share allocation. Furthermore, such an allocation can be found in polynomial time.*

*Proof-Sketch* The proof proceeds in two steps. First, we provide a constructive proof for the existence of EFL allocations, under additive valuations (Section 5.1). Next, we complete the proof by showing that EFL implies 1/2-GMMS (in Section 5.2)  $\square$

### 5.1 Existence of EFL Allocations

This section shows that EFL allocations are guaranteed to exist when the valuations are additive. Specifically, we develop an algorithm that always finds such an allocation.

**Lemma 1.** *Given any fair division instance with additive valuations, Algorithm 1 finds an EFL allocation in polynomial time.*

Algorithm 1 iteratively allocates the goods and maintains a partial allocation,  $\mathcal{A} = (A_1, \dots, A_n)$ , of the goods assigned so far. In each iteration, the algorithm selects an agent  $i$  who is not currently envied by any other agent, and allocates  $i$  an unassigned good of highest value (under  $v_i$ ).

Throughout the execution of the algorithm, the existence of an unenvied agent is ensured by maintaining a directed graph,  $G(\mathcal{A})$ , that captures the envy between agents. The nodes in this envy graph represent the agents and it contains a directed edge from  $i$  to  $j$  iff  $i$  envies  $j$ , i.e.,  $v_i(A_i) < v_i(A_j)$ . The following lemma, established in (Lipton et al. 2004), shows that if any iteration leads to a cycle in the envy graph  $G(\mathcal{A})$ , then we can always *resolve* it to obtain an acyclic envy graph without decreasing the valuation of any agents. The fact that  $G(\mathcal{A})$  is acyclic for a partial allocation  $\mathcal{A}$  implies that it contains a source node, i.e., an agent  $i$  who is not envied by other agents.

Although, Algorithm 1 is similar to the one developed in (Lipton et al. 2004)—which efficiently finds EF1 allocations—here, instead of assigning goods in an arbitrary order, we always allocate to an unenvied agent the available good she values the most. This is crucial for obtaining an EFL allocation.

**Lemma 2.** (Lipton et al. 2004) *Given a partial allocation  $\mathcal{A} = (A_1, \dots, A_n)$  of a subset of goods  $S \subseteq [m]$ , we can find another partial allocation  $\mathcal{B} = (B_1, \dots, B_n)$  of  $S$  in polynomial time such that*

- (i) *The valuations of agents for their bundles do not decrease, that is,  $v_i(B_i) \geq v_i(A_i)$  for all  $i \in [n]$ .*
- (ii) *The envy graph  $G(\mathcal{B})$  is acyclic.*

*Proof of Lemma 1.* Write  $\mathcal{A} = (A_1, \dots, A_n)$  to denote the allocation returned by Algorithm 1. First, we note that an inductive argument proves that  $\mathcal{A}$  is EF1. In fact, we will show that the EF1 condition holds for  $\mathcal{A}$  with respect to the last (in terms of the algorithm’s allocation order) good  $g$  assigned to

---

### Algorithm 1 Finding an EFL Allocation

---

**Input :**  $n$  agents,  $m$  items, and valuations  $v_i\{g\}$  for each agent  $i \in [n]$  and for each good  $g \in [m]$ .

**Output:** An EFL allocation.

- 1: Initialize allocation  $\mathcal{A} = (A_1, A_2, \dots, A_n)$  with  $A_i = \emptyset$  for each agent  $i \in [n]$ , and  $M = [m]$ .
  - 2: Initialize directed envy graph  $G(\mathcal{A}) = ([n], E)$  where  $E = \emptyset$ .
  - 3: **while**  $M \neq \emptyset$  **do**
  - 4:   Pick a source vertex  $i$  of  $G(\mathcal{A})$ . {such a vertex always exists, since  $G(\mathcal{A})$  is acyclic.}
  - 5:   Pick  $g \in \arg \max_{j \in M} v_i\{g\}$ .
  - 6:   Update  $A_i \leftarrow A_i \cup \{g\}$  and  $M \leftarrow M \setminus \{g\}$ .
  - 7:   **while** the current envy graph  $G(\mathcal{A})$  contains a cycle **do**
  - 8:     Update  $\mathcal{A}$  (using Lemma 2) to remove the cycle.
  - 9:   **end while**
  - 10: **end while**
  - 11: Return  $\mathcal{A}$ .
- 

each bundle  $A_j$ . Write  $g_t$  to denote the good allocated in the  $t$ th iteration of Algorithm 1; hence, the goods are allocated in the following order  $g_1, g_2, \dots, g_m$ .

The initial partial allocation  $(\emptyset, \emptyset, \dots, \emptyset)$  is EF1 (in fact, it is envy free). Now, say that in the  $j$ th iteration the algorithm allocates good  $g_j$  to agent  $i$ . Write  $\mathcal{A}^j = (A_1^j, \dots, A_n^j)$  and  $\mathcal{A}^{j+1} = (A_1^{j+1}, \dots, A_n^{j+1})$ , respectively, to denote the partial allocations before and after the  $j$ th iteration. The induction hypothesis implies that  $\mathcal{A}^j$  is EF1 with respect to the last assigned good. Therefore, for every pair of agents  $r, s \in [n]$ , we have  $v_r(A_r^j) \geq v_r(A_s^j \setminus \{g_a\})$ , where  $g_a$  is the last good assigned to the bundle  $A_s^j$  (i.e., for any other good  $g_b \in A_s^j$ , we have  $b < a$ ).

Since the good  $g_j$  is allocated to agent  $i$ , it must be the case that  $i$  is a source vertex in  $G(\mathcal{A}^j)$ , i.e., no agent envies  $i$  under  $\mathcal{A}^j$ . This implies that,  $v_r(A_r^j) \geq v_r(A_i^j) = v_r((A_i^j \cup \{g_j\}) \setminus \{g_j\})$  for all  $r \in [n]$ . Note that at this point of time,  $g_j$  is the last good assigned to the bundle  $A_i^j$ . In addition, from the proof of Lemma 2, we know that  $\mathcal{A}^{j+1}$  is a permutation of the allocation  $(A_1^j, A_2^j, \dots, A_i^j \cup \{g_j\}, \dots, A_n^j)$ , and  $v_r(A_r^{j+1}) \geq v_r(A_r^j)$  for all  $r \in [n]$ . Hence, for every pair of agents  $r, s \in [n]$  there exists a good  $g_a \in A_s^{j+1}$  such that  $v_r(A_r^{j+1}) \geq v_r(A_s^{j+1} \setminus \{g_a\})$ . In addition,  $g_a$  is the last good assigned to the bundle  $A_s^{j+1}$ . That is, the stated property holds for  $\mathcal{A}^{j+1}$  as well.

Now, we will use this observation to prove that  $\mathcal{A}$  is EFL. Specifically, we show the EFL conditions hold for, say, agent 1 (analogous arguments establish the claim for the other agents). Suppose that, during the execution of the algorithm, agent 1 receives its first good,  $g_t$ , in the  $t$ th iteration. Note that the partial allocation before the  $t$ th iteration, say  $\mathcal{A}^t = (A_1^t, \dots, A_n^t)$ , satisfies  $|A_i^t \cap \{g' \in [m] : v_1(g') > 0\}| \leq 1$  for all  $i \in [n]$ . This bound follows from the observation that during any previous iteration  $s < t$ , the selected source vertex  $i$  (i.e., the agent that gets a new good during the  $s$ th iteration) does not contain any good which is positively valued by agent 1; otherwise,  $i$  would have been envied by 1, contradicting the fact that it is a source vertex. Hence, each bundle in  $\mathcal{A}^t$  contains at most one good from

the set  $\{g' \in [m] : v_1(g') > 0\}$ .

Let us now consider the final allocation  $\mathcal{A} = (A_1, \dots, A_n)$  and any agent  $j \in [n]$ . If  $|A_j \cap \{g' \in [m] : v_1(g') > 0\}| \leq 1$ , then the first condition in the definition of EFL holds and we are done, else if  $|A_j \cap \{g' \in [m] : v_1(g') > 0\}| > 1$ , then bundle  $A_j$  must have received a good after the  $t$ th iteration. This is consequence of the above-mentioned property of the partial allocation  $\mathcal{A}^t$ . Write  $g_\ell$  to denote the last good allocated to the bundle  $A_j$ . We have  $\ell > t$ , since  $g_\ell$  was assigned to  $A_j$  after the  $t$ th iteration. Also, in the  $t$ th iteration good  $g_t$  was selected instead of  $g_\ell$ , hence it must be the case that  $v_1(g_t) \geq v_1(g_\ell)$ .

Note that, as mentioned before, the EF1 condition holds for  $A_j$  with respect to  $g_\ell$ , i.e.,  $v_1(A_1) \geq v_1(A_j \setminus \{g_\ell\})$ . In addition, Lemma 2 implies that  $v_1(A_1) \geq v_1(\{g_t\})$ . Therefore, if  $|A_j \cap \{g' \in [m] : v_1(g') > 0\}| > 1$  for any  $j \in [n]$ , then there exists a good  $g_\ell$  such  $v_1(A_1) \geq v_1(A_j \setminus \{g_\ell\})$  and  $v_1(A_1) \geq v_1(g_\ell)$ . Hence,  $\mathcal{A}$  is an EFL allocation.  $\square$

## 5.2 EFL implies Approximate GMMS

**Lemma 3.** *In any fair division instance with additive valuations, if an allocation  $\mathcal{A} = (A_1, \dots, A_n)$  is EFL, then it is  $1/2$ -GMMS allocation as well.*

*Proof.* Fix an agent  $i$  and a set of  $k$  agents,  $J \subseteq [n]$  which contains  $i$ , i.e.,  $|J| = k$  and  $J \ni i$ . Also, let  $S = \cup_{j \in J} A_j$  denote the set of all the goods allocated to the agents in  $J$ . We will show that agent if  $(A_1, \dots, A_n)$  is EFL, then  $v_i(A_i)$  is at least  $\frac{1}{2}$  times the maximin share of  $i$  restricted to  $S$ , i.e.,  $v_i(A_i) \geq \frac{1}{2} \mu_i^k(S)$ . This establishes the stated claim.

Write  $S_i \subseteq [m]$  to denote the set of goods which are positively valued by  $i$ ,  $S_i := \{g' \in [m] \mid v_i(g') > 0\}$ . Now, among the set of agents  $J \setminus \{i\}$  consider the ones who are allocated at most one good from  $S_i$ ; specifically, let  $T := \{j \in J \setminus \{i\} \mid |A_j \cap S_i| \leq 1\}$ . Write  $J' := J \setminus T$ ,  $t' := |J'|$ , and  $S' = \cup_{j \in J'} A_j$ . Note that the agent  $i$  belongs to the group  $J'$ , and for all  $j \in J'$  we have  $|A_j \cap S_i| > 1$ . Therefore, the fact that  $\mathcal{A}$  is EFL implies that, for all  $j \in J'$ , there exists a good  $g_{(j)} \in A_j$  such that  $v_i(A_i) \geq v_i(A_j \setminus \{g_{(j)}\})$  and  $v_i(A_i) \geq v_i(g_{(j)})$ . In other words, for the additive valuation  $v_i$ , we have  $2v_i(A_i) \geq v_i(A_j)$ .

We will now establish the multiplicative bound with respect to  $\mu_i^{t'}(S')$  (the maximin share of  $i$  restricted to  $S'$ ) and prove that  $\mu_i^{t'}(S') \geq \mu_i^k(S)$ . This will complete the proof. Since  $v_i$  is additive, an averaging argument gives us

$$\mu_i^{t'}(S') \leq \frac{1}{t'} v_i(S') = \frac{1}{t'} \sum_{j \in J'} v_i(A_j) \leq \frac{1}{t'} 2t' v_i(A_i)$$

Here, the last inequality uses the bound  $2v_i(A_i) \geq v_i(A_j)$  for all  $j \in J'$ . Therefore,  $v_i(A_i) \geq \frac{1}{2} \mu_i^{t'}(S')$ . To complete the proof, we need to show that  $\mu_i^{t'}(S') \geq \mu_i^k(S)$ . Note that—while considering the maximin shares of agent  $i$ —we can restrict our attention to  $S_i$  (the set of goods which are positively valued by  $i$ ). In particular, the equalities  $\mu_i^{t'}(S') = \mu_i^{t'}(S' \cap S_i)$  and  $\mu_i^k(S) = \mu_i^k(S \cap S_i)$  imply that, without loss of generality, we can work under the assumptions that  $S \subseteq S_i$  and  $S' \subseteq S_i$ . Effectively, for all  $j \in J$ , we can slightly abuse notation and denote  $A_j \cap S_i$

by  $A_j$ . Now, consider the allocation  $\mathcal{M} = (M_1, \dots, M_t) \in \arg \max_{(B_1, \dots, B_t) \in \Pi_t(S)} \min_{j \in [t]} v_i(B_j)$ .

We have,  $\min_{j \in J} v_i(M_j) = \mu_i^t(S)$ . Also, note that  $|A_j| \leq 1$  for each agent  $j \in K = J \setminus J'$ . Therefore, there are at most  $|K| = t - t'$  bundles in  $\mathcal{M}$  with items from  $S \setminus S'$ . We choose  $t'$  bundles from  $\mathcal{M} = (M_1, \dots, M_t)$  which do not contain any item from  $S \setminus S'$ . Let us call these new bundles  $\mathcal{M}' = (M'_1, \dots, M'_{t'})$ . By the definition of  $\mathcal{M}$ , the value of each such bundle for agent  $i$  is greater than or equal to  $\mu_i^t(S)$ , that is,  $v_i(M'_j) \geq \mu_i^t(S)$  for all  $j \in [t']$ . Since we have assumed that all items have nonnegative values, adding more items from the remaining  $(t - t')$  bundles to any of the bundles in  $\mathcal{M}'$  can only increase the value of the partitions. Thus,  $\mu_i^{t'}(S') \geq \mu_i^t(S)$ , which implies that  $v_i(A_i) \geq \frac{1}{2} \mu_i^k(S)$ .  $\square$

Lemmas 1 and 3 directly establish Theorem 1.

We note that both in terms of the algorithm and EFL's implication the approximation guarantee established here is essentially tight; this observation can be proved by considering simple examples.

## 6 Guaranteed Existence of GMMS Allocations in Restricted Settings

In this section, we prove the existence of GMMS for relevant valuation classes.

**Theorem 2.** *Groupwise maximin share allocations always exist under additive, binary valuations, i.e., such allocations exist when the additive valuations satisfy  $v_i(g) \in \{0, 1\}$  for all agents  $i \in [n]$  and goods  $g \in [m]$ .*

*Proof.* To prove the theorem, it suffices to show that under additive, binary valuations, EF1 implies GMMS. Recall that EF1 allocations are guaranteed to exist. Let  $\mathcal{A} = (A_1, \dots, A_n)$  be an EF1 allocation. Fix an agent  $i$  and a group of agents  $J \ni i$ . Also, write  $S := \cup_{j \in J} A_j$ . Next we complete the proof by showing that  $v_i(A_i) \geq \mu_i^{|J|}(S)$ .

Since the valuations are binary, for any subset of goods  $B \subseteq [m]$ , the total value  $v_i(B)$  is a non-negative integer. Therefore,  $\mu_i^{|J|}(S)$  must be a non-negative integer. Moreover, since the valuations are additive, we have  $\frac{1}{|J|} v_i(S) \geq \mu_i^{|J|}(S)$ . In addition, the facts that (a)  $\mathcal{A}$  is an EF1 allocation and (b) the maximum value of any good is 1 imply  $v_i(A_i) \geq v_i(A_j) - 1$  for all  $j \in J$ . Therefore,

$$\begin{aligned} v_i(A_i) &= \frac{1}{|J|} |J| v_i(A_i) \geq \frac{1}{|J|} \left( v_i(A_i) + \sum_{j \in J \setminus \{i\}} (v_i(A_j) - 1) \right) \\ &\geq \frac{1}{|J|} (v_i(S) - |J| + 1) = \frac{1}{|J|} v_i(S) - \frac{|J| - 1}{|J|} \\ &\geq \mu_i^{|J|}(S) - \frac{|J| - 1}{|J|} > \mu_i^{|J|}(S) - 1. \end{aligned}$$

The last inequality  $v_i(A_i) - \mu_i^{|J|}(S) > -1$  implies  $v_i(A_i) \geq \mu_i^{|J|}(S)$  since  $v_i(A_i)$  and  $\mu_i^{|J|}(S)$  are integers. Overall, the inequality  $v_i(A_i) \geq \mu_i^{|J|}(S)$  shows that the EF1 allocation  $\mathcal{A}$  is GMMS as well. This completes the proof.  $\square$

Note that the following theorem holds for general combinatorial valuations and is not limited to the additive case.

**Theorem 3.** *If all the  $n$  agents in a fair division instance have the same valuation (i.e.,  $v_i = v_j$  for all  $i, j \in [n]$ ), then a groupwise fair allocation is guaranteed to exist.*

*Proof.* Given a fair division instance with  $m$  goods and  $n$  agents with the same valuation function  $v$ , we will define an order on  $\Pi_n([m])$ , the set of  $n$ -partitions of  $[m]$ . For vectors  $u, u' \in \mathbb{R}^n$ , we say that  $u$  *lexicographically dominates*  $u'$  if  $u = u'$ , or there exists an index  $a \in [n]$  such that  $u_{(a)} > u'_{(a)}$  and for all  $b < a$  we have  $u_{(b)} = u'_{(b)}$ ; here,  $u_{(k)}$  and  $u'_{(k)}$  denotes the  $k$ th smallest component of  $u$  and  $u'$ , respectively. Extending this definition, an allocation  $(A_1, \dots, A_n) \in \Pi_n([m])$  is said to lexicographically dominate another allocation  $(B_1, \dots, B_n) \in \Pi_n([m])$  iff the vector of valuations  $(v(A_1), \dots, v(A_n))$  lexicographically dominates  $(v(B_1), \dots, v(B_n))$ . Note that lexicographic domination defines a total order over the equivalence classes of  $n$ -partitions. In addition, up to permutations of the valuation vector, there exists a unique maximal allocation with respect to this order, i.e., there exists an allocation  $\mathcal{A}^* = (A_1^*, \dots, A_n^*)$  which lexicographically dominates all other allocations.

We will show that  $\mathcal{A}^*$  is GMMS. For contradiction, say that this is not the case, and the GMMS condition is violated for subset  $J \subseteq [n]$  and agent  $i \in J$ , i.e.,  $v_i(A_i^*) < \mu_i^{|J|}(S)$ ; here  $S := \cup_{j \in J} A_j^*$ . Since the valuations are identical, the maximin share of each agent in  $J$  restricted to  $S$  is the same, i.e.,  $\mu_i^{|J|}(S) = \mu_k^{|J|}(S)$ , for all  $k \in J$ . Hence, GMMS condition must also be violated for agent  $\arg \min_{j \in J} v(A_j^*)$ . This observation and the definition of maximin shares imply that there exists a  $|J|$ -partition of  $S$ , say  $(M_1, \dots, M_{|J|})$ , such that  $\min_{1 \leq a \leq |J|} v(M_a) > \min_{j \in J} v(A_j^*)$ .

Now, consider an allocation  $\mathcal{B} = (B_1, \dots, B_n)$  obtained from  $\mathcal{A}^*$  by (i) replacing the  $|J|$  bundles  $A_j^*$ s (for  $j \in J$ ) with  $M_a$ s (for  $1 \leq a \leq |J|$ ), and (ii) setting  $B_k = A_k^*$  for all  $k \notin J$ . Note that, even under any permutation of the bundles, the allocation  $\mathcal{B} \neq \mathcal{A}^*$ . Also, the construction of  $\mathcal{B}$  ensures that it lexicographically dominates  $\mathcal{A}^*$ . This contradicts the maximality (under the defined lexicographic order) of  $\mathcal{A}^*$  and, hence, the stated claim follows.  $\square$

## 7 Some Empirical Results

For an experimental investigation, we generated 1000 random instances, for several combinations of  $n$  agents and  $m$  goods ( $n$  ranging from 3 to 5, and  $m$  from 3 to 11), by randomly drawing the valuations from two different distributions (a) gaussian distribution with mean 0.5, standard deviation 0.1 (truncated at 0 to ensure nonnegative valuations) and (b) uniform distribution  $[0, 1]$ . These are the same set of experiments that (Bouveret and Lemaître 2014) carried out for studying the existence of various fairness notions under additive valuations.

In addition, we considered the modification of these instances wherein all the agents agree on the same ranking of the goods, but may have different valuations for the same

item. Such instances are said to have same-order preferences (SOP), and were studied by (Bouveret and Lemaître 2014). They showed that, when it comes to (empirically) finding an MMS allocation, SOP instances are the hardest. We observe similar results for GMMS: finding a GMMS allocation, done using brute-force search, requires noticeably more time in SOP instances, than in non-SOP instances. Our empirical results and observations are summarized below:

1. GMMS allocations exist empirically in all randomly generated instances (which is similar to the experimental results for MMS (Bouveret and Lemaître 2014)). These simulation results indicate that we may not fall short on such generic existence results by strengthening the maximin solution concept.
2. Allocations generated by the EFL algorithm (Algorithm 1) on the random instances achieve an approximation factor of 0.98 (on an average) which is better than our theoretically obtained worst-case bound of 0.5. The approximation factor (on average) is lower for SOP than for non-SOP instances. Moreover, we observe that, for randomly generated instances, the approximation factor is better when the number of items is a multiple of the number of agents. This may be because of the round robin nature of the algorithm which gives an agent her most desirable good at each round.
3. As expected, our proposed EFL algorithm ran much faster than the (exponential-time) algorithm for finding exact GMMS. Our algorithm was about  $10^7$  times faster than the exact GMMS computation on a machine with a quad Intel Core *i7* processor and 32 GB RAM.

## 8 Conclusion and Future Work

In this paper, we introduced the concept of GMMS to address fair allocation of indivisible goods, thereby strengthening the well-studied notions of MMS and PMMS. We established the existence of 1/2-GMMS under additive valuations, and developed an efficient algorithm for finding such allocations. We also proved that under specific settings exact GMMS allocations always exist. In addition, GMMS allocations were always found in our experiments (over a large number of randomly generated instances). This indicates why it seems nontrivial to obtain simpler<sup>7</sup> examples which refute the guaranteed existence of GMMS allocations. Finding an instance which admits an MMS allocation but not a GMMS allocation remains an interesting, open problem.

Our work addresses key questions about groupwise fairness when the valuations are additive, and it motivates the study of this notion in more general settings, e.g., under sub-modular valuations, or under additional constraints, such as the ones considered by (Bouveret et al. 2017). Another interesting direction of future work is to understand groupwise fair division of indivisible goods among strategic agents.

<sup>7</sup>The intricate examples showing the nonexistence of MMS under additive valuations (Procaccia and Wang 2014; Kurokawa, Procaccia, and Wang 2016) also serve as counterexamples for GMMS.

**Acknowledgments.** Siddharth Barman was supported by a Ramanujan Fellowship (SERB - SB/S2/RJN-128/2015). Arpita Biswas gratefully acknowledges the support of a Google PhD Fellowship Award.

## References

- Amanatidis, G.; Markakis, E.; Nikzad, A.; and Saberi, A. 2015. Approximation algorithms for computing maximin share allocations. In *International Colloquium on Automata, Languages, and Programming*, 39–51. Springer.
- Aziz, H.; Rauchecker, G.; Schryen, G.; and Walsh, T. 2017. Approximation algorithms for max-min share allocations of indivisible chores and goods. In *Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence*, 335–341.
- Barman, S., and Krishnamurthy, S. K. 2017. Approximation algorithms for maximin fair division. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, 647–664.
- Bouveret, S., and Lemaître, M. 2014. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. In *Proceedings of International Conference on Autonomous Agents and Multi-Agent Systems*, 1321–1328.
- Bouveret, S.; Cechlaova, K.; Elkind, E.; Igarashi, A.; and Peters, D. 2017. Fair division of a graph. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17*, 135–141.
- Brams, S. J., and King, D. L. 2005. Efficient fair division: Help the worst off or avoid envy? *Rationality and Society* 17(4):387–421.
- Brandt, F.; Conitzer, V.; Endriss, U.; Procaccia, A. D.; and Lang, J. 2016. *Handbook of Computational Social Choice*. Cambridge University Press.
- Budish, E. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy* 119(6):1061–1103.
- Caragiannis, I.; Kurokawa, D.; Moulin, H.; Procaccia, A. D.; Shah, N.; and Wang, J. 2016. The unreasonable fairness of maximum nash welfare. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, 305–322. ACM.
- Foley, D. 1967. Resource allocation in the public sector. *Yale Economic Essays* 7:73–76.
- Ghods, M.; HajiAghayi, M.; Seddighin, M.; Seddighin, S.; and Yami, H. 2017. Fair allocation of indivisible goods: Improvement and generalization. *arXiv preprint arXiv:1704.00222*.
- Goldman, J., and Procaccia, A. D. 2015. Spliddit: Unleashing fair division algorithms. *ACM SIGecom Exchanges* 13(2):41–46.
- Kurokawa, D.; Procaccia, A. D.; and Wang, J. 2016. When can the maximin share guarantee be guaranteed? In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence*, volume 16, 523–529.
- Lipton, R. J.; Markakis, E.; Mossel, E.; and Saberi, A. 2004. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM conference on Electronic commerce*, 125–131. ACM.
- Procaccia, A. D., and Wang, J. 2014. Fair enough: Guaranteeing approximate maximin shares. In *Proceedings of the Fifteenth ACM Conference on Economics and Computation*, 675–692. ACM.
- Steinhaus, H. 1948. The problem of fair division. *Econometrica* 16:101–104.
- Stromquist, W. 1980. How to cut a cake fairly. *The American Mathematical Monthly* 87(8):640–644.
- Varian, H. R. 1974. Equity, envy, and efficiency. *Journal of economic theory* 9(1):63–91.