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Resource Allocation Polytope Games: Uniqueness of Equilibrium, Price of Stability, and Price of Anarchy

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Abstract

We consider a two-player resource allocation polytope game, in which the strategy of a player is restricted by the strategy of the other player, with common coupled constraints. With respect to such a game, we formally introduce the notions of independent optimal strategy profile, which is the profile when players play optimally in the absence of the other player; and common contiguous set, which is the set of top nodes in the preference orderings of both the players that are exhaustively invested on in the independent optimal strategy profile. We show that for the game to have a unique PSNE, it is a necessary and sufficient condition that the independent optimal strategies of the players do not conflict, and either the common contiguous set consists of at most one node or all the nodes in the common contiguous set are invested on by only one player in the independent optimal strategy profile. We further derive a socially optimal strategy profile, and show that the price of anarchy cannot be bound by a common universal constant. We hence present an efficient algorithm to compute the price of anarchy and the price of stability, given an instance of the game. Under reasonable conditions, we show that the price of stability is 1. We encounter a paradox in this game that higher budgets may lead to worse outcomes.

Introduction

The problem of resource allocation is relevant to a large number and wide variety of applications, from small household applications to citywide, marketwide, and worldwide applications (Johari and Tsitsiklis 2004; Wei et al. 2010; Clearwater 1996; Thomas 1990). A primary goal of an agent is to allocate its resources or budget in such a way that its utility is maximized. In most scenarios, there exist competing agents who also aim to allocate their resources with the aim of maximizing their own utilities. Furthermore, there could be correlation among the agents' utilities, for instance, an investment by an agent on a node may benefit or harm the utility of another agent (Borodin, Filmus, and Oren 2010).

A node or machine for which the resources are to be allocated (or on which investments are to be made) may have a bound or capacity beyond which it cannot be invested on. So the set of feasible investment profiles would be restricted. This would result in strategic investment by the agents, not only because they have to invest on multiple nodes, but also because there would be competition among the agents for investing on the nodes. This results in a game whose players are the agents and a player's strategy is how to allocate its resources among the nodes while respecting node capacities.

We now describe the setting in detail, and see how it belongs to the class of games called *polytope games* (Bhattacharjee, Thuijsman, and Vrieze 2000).

Setting

We label the two players as A and B, and the set of nodes as N. Let $n = |N|$. Let w_{Ai} be the benefit that A gets by investing a unit amount on node i. Similarly, let w_{Bi} be the benefit that B gets by investing a unit amount on i. Consistent with most applications, we assume $w_{Ai}, w_{Bi} > 0, \forall i \in N$. Let x_i and y_i be the respective investments made by A and B on i. Since the benefit that A gets by investing on i would be an increasing function of x_i and w_{Ai}, we assume the benefit to be $w_{Ai}x_i$ for analytical tractability. Similarly, $w_{Bi}y_i$ is the benefit that B gets by investing on i. Let $\mathbf{w}_A = (w_{Ai})_{i\in N}$, $\mathbf{w}_B = (w_{Bi})_{i\in N}$, $\mathbf{x} = (x_i)_{i\in N}$, $\mathbf{y} = (y_i)_{i\in N}$.

There may be correlation between the players' utilities by investing on a node, for example, A's investment of x_i on node i could result in an added amount of $\alpha w_{Ai}x_i$ in B's utility. This could be a benefit if $\alpha > 0$, a loss if $\alpha < 0$, or an uninfluential term if $\alpha = 0$. So the marginal utility that B gets from i is $(w_{Bi}y_i + \alpha w_{Ai}x_i)$. Similarly, if B's investment of y_i results in an added amount of $\beta w_{Bi}y_i$ in A's utility, the marginal utility that A gets from i is $(w_{Ai}x_i + \beta w_{Bi}y_i)$. Let $u_A(\mathbf{x},\mathbf{y})$ and $u_B(\mathbf{x},\mathbf{y})$ denote their respective utilities. So the net total utility of A summed over all nodes is $u_A(\mathbf{x},\mathbf{y}) = \sum_{i\in N}(w_{Ai}x_i + \beta w_{Bi}y_i)$ and that of B is $u_B(\mathbf{x},\mathbf{y}) = \sum_{i\in N}(w_{Bi}y_i + \alpha w_{Ai}x_i)$.

The players have budget constraints stating that A can invest a total of, say k_A, across all nodes, and B can invest a total of k_B. That is, $\sum_{i\in N} x_i \le k_A$, $\sum_{i\in N} y_i \le k_B$. Also, the total amount that can be invested on a node is bounded. We assume that all nodes have a common bound or capacity. We assume this bound to be 1 without loss of generality. So we have another set of constraints: $x_i + y_i \le 1, \forall i$, which are common coupled constraints (a player's constraints are satisfied if and only if constraints of the other player are satisfied for every strategy profile). We assume that $k_A + k_B \le |N|$, that is, there are enough nodes to be able to invest on.

So players A and B aim to maximize their own utilities:

$$u_A(\mathbf{x},\mathbf{y}) = \sum_{i\in N}(w_{Ai}x_i + \beta w_{Bi}y_i),$$
$$u_B(\mathbf{x},\mathbf{y}) = \sum_{i\in N}(w_{Bi}y_i + \alpha w_{Ai}x_i) \qquad (1)$$

$$\text{subject to} \begin{cases} x_i, y_i \geq 0, \forall i \in N \\ \sum_{i\in N} x_i \leq k_A, \sum_{i\in N} y_i \leq k_B \\ x_i + y_i \leq 1, \forall i \in N \end{cases}$$

Since the common coupled constraints and the utility functions are linear, it can be classified as a polytope game.

Motivation

There are several scenarios where there would be bound on allocation on each node by the players combined. Such a bound could account for critical scenarios where exceeding a certain limit is infeasible or highly undesirable. For instance, players may want to allocate jobs to machines (nodes), where each machine cannot accept more than a certain total load, beyond which it would overheat and crash. In scenarios where investing on a node means providing information and convincing arguments (such as during elections), the bounding constraint may arise owing to the attention capacity of a node, beyond which any information may be ignored. In routing, the links usually have capacities, which are responsible for the cost or time expended; in scenarios where there is a time limit before which the data transfers should be completed, the amount of data that can be transferred over a link would be bounded. Such resource allocation examples with linear bounding constraints form our motivation to study resource allocation polytope games.

The study of existence and uniqueness of equilibrium, and price of stability and price of anarchy, is often important for games inspired by practical applications. There have been extensive studies on these topics in resource allocation setting (such as routing) and other games such as congestion games, where there is an underlying cost function for allocating resources (or assigning job) to a node. The fundamental assumption in such studies is that the cost function is continuous, while most studies also assume smoothness for deriving the price of stability and the price of anarchy. An additional assumption of strict concavity is made to prove uniqueness of equilibrium. Our setting can be transformed so as to have a cost function instead of a bound on nodes, however such a cost function would have to be discontinuous, since the cost would shoot to infinity beyond the bound. So though the fundamental base is common, replacing continuous cost functions with bounding constraints demands a very different treatment, which this paper aims to study.

Preliminaries

Definition 1 (Feasible strategy)**.** *We say that $\mathbf{x}$ is a feasible strategy, given the strategy $\mathbf{y}$, if and only if $\forall i, 0 \leq x_i \leq 1 - y_i$ and $\sum_i x_i \leq k_A$. Similarly, $\mathbf{y}$ is a feasible strategy, given $\mathbf{x}$, if and only if $\forall i, 0 \leq y_i \leq 1 - x_i$ and $\sum_i y_i \leq k_B$.*

So a strategy profile $(\mathbf{x},\mathbf{y})$ is feasible if and only if $\sum_i x_i \leq k_A, \sum_i y_i \leq k_B$ and $\forall i, 0 \leq x_i + y_i \leq 1$. Given a strategy $\mathbf{y}$ of player B, we represent the set of feasible strategies of player A by $F(\mathbf{y})$. And given a strategy $\mathbf{x}$ of A, let the set of feasible strategies of B be $F(\mathbf{x})$.

Definition 2 (Pure strategy Nash equilibrium (PSNE))**.** *A feasible strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ is a PSNE if and only if*

$$\forall \mathbf{x}' \in F(\mathbf{y}^*), u_A(\mathbf{x}^*, \mathbf{y}^*) \geq u_A(\mathbf{x}', \mathbf{y}^*)$$
$$\textit{and } \forall \mathbf{y}' \in F(\mathbf{x}^*), u_B(\mathbf{x}^*, \mathbf{y}^*) \geq u_B(\mathbf{x}^*, \mathbf{y}')$$

Since the feasible strategy set of a player depends on the strategy of the other player, this equilibrium is termed *generalized Nash equilibrium* (Facchinei and Kanzow 2007).

The linear utility function and a bound on investment per node, result in a preference ordering on nodes by the players. It can be seen that $u_A(\mathbf{x}', \mathbf{y}) \geq u_A(\mathbf{x}'', \mathbf{y}) \iff \sum_i w_{Ai}x'_i \geq \sum_i w_{Ai}x''_i$. So if $w_{Ai} > w_{Aj}$, then A would invest on node j only if it is not possible to further invest on node i (owing to constraint $x_i \leq 1 - y_i$). Hence, $w_{Ai} > w_{Aj}$ implies that A prefers i over j; let us denote this by $i \succ_A j$.

One of the primary goals of this paper is to study conditions under which the game has a unique PSNE. However, if multiple nodes hold the same benefit for a player, investing an amount in one node would be as good as investing this amount in another node holding the same benefit, which also would be as good as distributing this amount over multiple nodes holding the same benefit. So, in order to avoid trivial non-uniqueness of PSNE due to ties, we assume that w_{Ai}'s are distinct, that is, $w_{Ai} \neq w_{Aj}$ for $i \neq j$. Similarly, $w_{Bi} \neq w_{Bj}$ for $i \neq j$. So each player has a strict ordering over nodes. Hence $\mathbf{w}_A$ induces a strict preference ordering on nodes with respect to player A, say π_A, such that

$$r_1 > r_2 \iff \pi_A(r_1) \succ_A \pi_A(r_2) \iff w_{A\pi_A(r_1)} > w_{A\pi_A(r_2)}$$

where $\pi_A(r)$ is the r^{th} node in the preference ordering of player A. Similarly, $\mathbf{w}_B$ induces ordering π_B for player B.

Related Work

As explained earlier, the game we consider falls in the class of polytope games (Bhattacharjee, Thuijsman, and Vrieze 2000), and the notion of equilibrium we study is generalized Nash equilibrium (Facchinei and Kanzow 2007). A notable study (Rosen 1965) shows existence of equilibrium in a constrained game, and its uniqueness in a strictly concave game. Another study (Altman and Solan 2009) focuses on equilibrium behavior in games with common coupled constraints.

There have been studies on existence and uniqueness of Nash equilibrium with respect to a variety of applications transformed into games (Szidarovszky and Okuguchi 1997; Yamazaki 2008). It is known that PSNE is guaranteed to exist in a class of games having an underlying potential function, popularly known as potential games (Monderer and Shapley 1996). There have also been studies on convergence to Nash equilibrium with respect to a number of applications (Arslan and Shamma 2004; Brandt, Fischer, and Harrenstein 2013). A two-node multiple links system has been shown to have a unique equilibrium under certain convexity conditions (Orda, Rom, and Shimkin 1993).

The quality or goodness of Nash equilibria has been a topic of study in several application, and has been of particular interest in network games with regard to the price of stability (Fiat et al. 2006; Anshelevich et al. 2008) and the price of anarchy (Roughgarden 2005; Christodoulou and Koutsoupias 2005; Bharathi, Kempe, and Salek 2007).

Our Contributions

Though there have been studies on generalized Nash equilibria and the existence of equilibrium in polytope games is known, it is not clear if it is unique and what the price of stability and the price of anarchy are. Most studies on uniqueness leverage the strict concavity (or convexity) of the underlying game. Since our game is neither strictly convex nor strictly concave, it requires a different treatment to determine the conditions under which the game would have a unique equilibrium. Also, though price of stability and price of anarchy have been studied with respect to congestion and other resource allocation games, such studies assume the cost functions to be continuous and do not consider common coupled constraints. Hence, this is the first game theoretic study on resource allocation polytope games, with respect to determining the conditions for uniqueness of equilibrium and deriving the price of stability and the price of anarchy.

Conditions for Uniqueness of PSNE

We first provide a simple existential proof.

Lemma 1. *PSNE exists in the considered game.*

Proof. Suppose that player A plays a strategy $\mathbf{x}'$ that maximizes $\sum_i w_{Ai}x_i$, that is, $\sum_i w_{Ai}x'_i \geq \sum_i w_{Ai}x_i, \forall \mathbf{x} \in [0,1]^n$. Let player B play a strategy $\mathbf{y}' \in F(\mathbf{x}')$ such that $\sum_i w_{Bi}y'_i \geq \sum_i w_{Bi}y_i, \forall \mathbf{y} \in F(\mathbf{x}')$. Adding $\sum_i \alpha w_{Ai}x'_i$ on both sides, we get $u_B(\mathbf{x}',\mathbf{y}') \geq u_B(\mathbf{x}',\mathbf{y}), \forall \mathbf{y} \in F(\mathbf{x}')$.

Since $\mathbf{y}' \in F(\mathbf{x}')$, we have $x'_i + y'_i \leq 1, \forall i$, and hence $\mathbf{x}' \in F(\mathbf{y}')$. As $\mathbf{x}'$ is such that $\sum_i w_{Ai}x'_i \geq \sum_i w_{Ai}x_i, \forall \mathbf{x} \in [0,1]^n$, we would have $\sum_i w_{Ai}x'_i \geq \sum_i w_{Ai}x_i, \forall \mathbf{x} \in F(\mathbf{y}')$. Adding $\sum_i \beta w_{Bi}y'_i$ on both sides, we get $u_A(\mathbf{x}',\mathbf{y}') \geq u_A(\mathbf{x},\mathbf{y}'), \forall \mathbf{x} \in F(\mathbf{y}')$. So strategy profile $(\mathbf{x}',\mathbf{y}')$ is a PSNE. Since we can always find such a strategy profile with this procedure, there exists a PSNE. □

Also, $u_A(\mathbf{x}',\mathbf{y}) - u_A(\mathbf{x}'',\mathbf{y}) = \sum_i(w_{Ai}x'_i + w_{Bi}y_i) - \sum_i(w_{Ai}x''_i + w_{Bi}y_i), \forall \mathbf{x}',\mathbf{x}'' \in F(\mathbf{y})$ (by adding $\sum_i(1-\beta)w_{Bi}y_i$ to both $u_A(\mathbf{x}',\mathbf{y})$ and $u_A(\mathbf{x}'',\mathbf{y})$). Similarly, we have $u_B(\mathbf{x},\mathbf{y}') - u_B(\mathbf{x},\mathbf{y}'') = \sum_i(w_{Ai}x_i + w_{Bi}y'_i) - \sum_i(w_{Ai}x_i + w_{Bi}y''_i), \forall \mathbf{y}',\mathbf{y}'' \in F(\mathbf{x})$. So the game can be classified as an *exact restricted potential game* (Schöbel and Schwarze 2006), with potential function $\Phi(\mathbf{x},\mathbf{y}) = \sum_i(w_{Ai}x_i + w_{Bi}y_i)$ and the restrictions on the strategies of A and B being $\mathbf{x} \in F(\mathbf{y})$ and $\mathbf{y} \in F(\mathbf{x})$, respectively. Since there exists a PSNE in an exact restricted potential game, this gives an alternative proof of Lemma 1. The lemma could also be viewed as a special case of a more general existential result (Arrow and Debreu 1954).

We now introduce some important terminologies.

Definition 3 (Independent optimal strategy)**.** *An independent optimal strategy of a player is the strategy that it would play in the absence of the other player.*

Let $\hat{\mathbf{x}} = (\hat{x}_i)_{i\in N}$, $\hat{\mathbf{y}} = (\hat{y}_i)_{i\in N}$ be the independent optimal strategies of A and B, respectively. The independent optimal strategy of A is to invest on nodes, one at a time, according to its ordering π_A, with a maximum of 1 unit per node, until its budget k_A is exhausted. That is, $\hat{x}_{\pi_A(r)} = 1, \forall r \leq \lfloor k_A \rfloor$ and $\hat{x}_{\pi_A(\lfloor k_A \rfloor+1)} = \text{frac}(k_A) = k_A - \lfloor k_A \rfloor$ and $\hat{x}_{\pi_A(r)} = 0, \forall r \geq \lfloor k_A \rfloor + 2$. The independent optimal strategy of B is analogous. Let $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ be the independent optimal strategy profile (IOS). As we assume orderings π_A, π_B to be strict (hence unique), we have that the IOS is unique.

Definition 4 (Non-conflicting IOS)**.** *The IOS $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ is non-conflicting if and only if $\hat{x}_i + \hat{y}_i \leq 1, \forall i \in N$.*

Lemma 2. *For the game to have a unique PSNE, it is necessary that the IOS is non-conflicting.*

Proof. If the IOS is conflicting, there exists a node i such that $\hat{x}_i + \hat{y}_i > 1$. On similar lines as the proof of Lemma 1, if player A plays first, it would invest $\hat{x}_i$ on node i, and B would then be able to invest $1 - \hat{x}_i < \hat{y}_i$ on node i. On the other hand, if player B plays first, it would invest $\hat{y}_i$ on node i, and A would then be able to invest $1 - \hat{y}_i < \hat{x}_i$ on node i. These result in two different PSNE's since $\hat{x}_i + \hat{y}_i \neq 1$.

In general, for every $x_i \in [1-\hat{y}_i, \hat{x}_i]$ and $y_i = 1 - x_i$, the nodes in $N \setminus \{i\}$ can be invested upon by A and B with respective budgets $k_A - x_i$ and $k_B - y_i$, according to the procedure in the proof of Lemma 1. It can be seen that such an allocation would be a PSNE. Since $[1-\hat{y}_i, \hat{x}_i]$ is an uncountable set, we have uncountable number of PSNE's. □

Lemma 3. *The IOS being non-conflicting is not sufficient for the uniqueness of PSNE.*

Proof. We provide a counterexample. Let $k_A = k_B = 2$, $\mathbf{w}_A = (5\ 4\ 3\ 2\ 1)$, $\mathbf{w}_B = (3\ 1\ 5\ 2\ 4)$. These result in non-conflicting IOS: $\hat{\mathbf{x}} = (1\ 1\ 0\ 0\ 0), \hat{\mathbf{y}} = (0\ 0\ 1\ 0\ 1)$. But it has multiple PSNE's, for instance, $\mathbf{x} = (1\ 1\ 0\ 0\ 0), \mathbf{y} = (0\ 0\ 1\ 0\ 1)$ and also $\mathbf{x} = (0\ 1\ 1\ 0\ 0), \mathbf{y} = (1\ 0\ 0\ 0\ 1)$. □

We introduce some notation to facilitate our proofs. The notation can be understood with the illustration in Figure 1.

Let T_A be the set of nodes on which player A would prefer to invest 1 unit each, that is, it is the set of top $\lfloor k_A \rfloor$ nodes in the ordering π_A. Let T_B be defined analogously. That is,

$$T_A = \{i : \hat{x}_i = 1, \hat{y}_i = 0\} = \{\pi_A(r)\}_{r=1}^{\lfloor k_A \rfloor}$$
$$T_B = \{i : \hat{y}_i = 1, \hat{x}_i = 0\} = \{\pi_B(r)\}_{r=1}^{\lfloor k_B \rfloor}$$

If there is a residual budget of player A ($\text{frac}(k_A) = k_A - \lfloor k_A \rfloor$) after investing in T_A, let l_A be the node on which it would prefer to invest this residual budget. Note that l_A does not exist when k_A is an integer, and if it exists, it is $\pi_A(\lfloor k_A \rfloor + 1)$. Let l_B be defined analogously. That is,

$$l_A = i \text{ s.t. } \hat{x}_i = \text{frac}(k_A) \in (0,1)$$
$$\text{and } \exists l_A \implies l_A = \pi_A(\lfloor k_A \rfloor + 1)$$

$$l_B = i \text{ s.t. } \hat{y}_i = \text{frac}(k_B) \in (0,1)$$
$$\text{and } \exists l_B \implies l_B = \pi_B(\lfloor k_B \rfloor + 1)$$

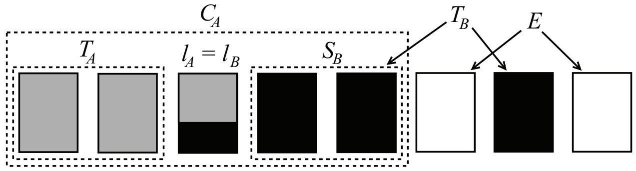

Figure 1: An example illustration of terminologies with respect to player A's ordering π_A, where $k_A = 2.7, k_B = 3.3$ (grey corresponds to $\hat{\mathbf{x}}$, black corresponds to $\hat{\mathbf{y}}$)

If l_A and l_B is the same node, we denote it by l_{AB}. Note that with respect to non-conflicting IOS, l_{AB} exists only if $\text{frac}(k_A) + \text{frac}(k_B) \leq 1$. Finally, let E be the set of nodes on which neither player opts to invest in the IOS. That is,

$$E = \{i : \hat{x}_i = \hat{y}_i = 0\}$$

Definition 5 (Contiguous set)**.** *We define the contiguous set in a player's preference ordering to be the set of top nodes in its ordering until we encounter a node which has partial or zero combined investment in the IOS.*

Let C_A and C_B denote the contiguous set in the preference orderings of players A and B, respectively. So,

$$C_A = \{\pi_A(r)\}_{r=1}^{q_A-1}$$
$$\text{s.t. } \hat{x}_i + \hat{y}_i = 1, \forall i \in C_A \text{ and } \hat{x}_{\pi_A(q_A)} + \hat{y}_{\pi_A(q_A)} < 1$$

$$C_B = \{\pi_B(r)\}_{r=1}^{q_B-1}$$
$$\text{s.t. } \hat{x}_i + \hat{y}_i = 1, \forall i \in C_B \text{ and } \hat{x}_{\pi_B(q_B)} + \hat{y}_{\pi_B(q_B)} < 1$$

Definition 6 (Common contiguous set)**.** *We define common contiguous set to be the set of nodes belonging to the contiguous sets in the preference orderings of both the players.*

Let C_{AB} denote the common contiguous set. So,

$$C_{AB} = C_A \cap C_B$$

Let S_A denote the set of nodes in T_A, which also belong to the common contiguous set in the ordering of player B. Let S_B be defined analogously. So we have

$$S_A = C_B \cap T_A \text{ and } S_B = C_A \cap T_B$$

Lemma 4. *If IOS* $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *is non-conflicting, then a strategy profile* $(\mathbf{x}, \mathbf{y})$ *is a PSNE only if* $x_i + y_i = \hat{x}_i + \hat{y}_i, \forall i \in N$.

Proof. Since the IOS is non-conflicting, we have that $\hat{x}_i + \hat{y}_i \leq 1, \forall i \in N$. If $\exists i \in T_A$ s.t. $x_i + y_i < 1$, since player A's budget $k_A \geq |T_A|$, it must have invested in some node $j \notin T_A$, that is, $x_j > 0$. Now since $i \succ_A j, \forall i \in T_A, \forall j \notin T_A$, player A can gain by transferring an amount $\epsilon > 0$ from node j to node i, that is, by investing $x_i + \epsilon$ in node i and $x_j - \epsilon$ in node j. So a strategy profile in which $\exists i \in T_A$ s.t. $x_i + y_i < 1$, cannot be a PSNE. So $(\mathbf{x}, \mathbf{y})$ is a PSNE only if $x_i + y_i = 1, \forall i \in T_A$. Similarly, $(\mathbf{x}, \mathbf{y})$ is a PSNE only if $x_i + y_i = 1, \forall i \in T_B$. So we have proved that $(\mathbf{x}, \mathbf{y})$ is a PSNE only if $x_i + y_i = \hat{x}_i + \hat{y}_i \,(= 1), \forall i \in T_A \cup T_B$.

The total budget to be invested over all nodes by both the players combined is $k_A + k_B$. Now we consider different cases depending on the existence of l_A, l_B (or l_{AB}) and show that $x_i + y_i = \hat{x}_i + \hat{y}_i$ for these nodes in PSNE.

If $\nexists l_A, \nexists l_B$, there is nothing to prove.

If $\exists l_A, \exists l_B$ ($l_A \neq l_B$), we have $x_i + y_i = 1, \forall i \in T_A \cup T_B$, so the total amount invested in $T_A \cup T_B$ is $\lfloor k_A \rfloor + \lfloor k_B \rfloor$. Since the budget invested by both the players combined is $k_A + k_B$, the residual amount of $(k_A + k_B) - (\lfloor k_A \rfloor + \lfloor k_B \rfloor) = \text{frac}(k_A) + \text{frac}(k_B) = \hat{x}_{l_A} + \hat{y}_{l_B}$ has to be distributed over nodes not in $T_A \cup T_B$, namely, $E \cup \{l_A\} \cup \{l_B\}$. That is,

$$\sum_{i \in E}(x_i + y_i) + (x_{l_A} + y_{l_A}) + (x_{l_B} + y_{l_B}) = \hat{x}_{l_A} + \hat{y}_{l_B} \quad (2)$$

If $x_{l_A} + y_{l_A} < \hat{x}_{l_A} = \text{frac}(k_A)$, A's investment in $T_A \cup \{l_A\}$ is less than $|T_A| + \hat{x}_{l_A} = \lfloor k_A \rfloor + \text{frac}(k_A) = k_A$, which is its budget. So it must have invested in some node $j \notin T_A \cup \{l_A\}$, that is, $x_j > 0$. Now since $l_A \succ_A j, \forall j \notin T_A \cup \{l_A\}$, A can gain by transferring an amount $\epsilon > 0$ from j to l_A, that is, by investing $x_{l_A} + \epsilon$ in l_A and $x_j - \epsilon$ in j. So a profile in which $x_{l_A} + y_{l_A} < \hat{x}_{l_A}$, cannot be a PSNE. So $(\mathbf{x}, \mathbf{y})$ is a PSNE only if $x_{l_A} + y_{l_A} \geq \hat{x}_{l_A} = \text{frac}(k_A)$. Similarly, $(\mathbf{x}, \mathbf{y})$ is a PSNE only if $x_{l_B} + y_{l_B} \geq \hat{y}_{l_B} = \text{frac}(k_B)$. These, along with Equation (2), give our desired condition:

$$\sum_{i \in E}(x_i + y_i) = 0 \;,\; (x_{l_A} + y_{l_A}) = \hat{x}_{l_A} \;,\; (x_{l_B} + y_{l_B}) = \hat{y}_{l_B}$$

The cases $\nexists l_A, \exists l_B$ and $\exists l_A, \nexists l_B$ can be proved on similar lines as the above case.

We now consider the case $\exists l_{AB}$ ($l_A = l_B$). If $\exists l_{AB}$, the residual amount of $(k_A + k_B) - (\lfloor k_A \rfloor + \lfloor k_B \rfloor) = \text{frac}(k_A) + \text{frac}(k_B) = \hat{x}_{l_{AB}} + \hat{y}_{l_{AB}}$ has to be distributed over nodes not belonging to $T_A \cup T_B$, namely, $E \cup \{l_{AB}\}$. That is,

$$\sum_{i \in E}(x_i + y_i) + (x_{l_{AB}} + y_{l_{AB}}) = \hat{x}_{l_{AB}} + \hat{y}_{l_{AB}} \quad (3)$$

If $x_{l_{AB}} + y_{l_{AB}} < \hat{x}_{l_{AB}} + \hat{y}_{l_{AB}} = \text{frac}(k_A) + \text{frac}(k_B)$, the combined investment of A and B in $T_A \cup T_B \cup \{l_{AB}\}$ is less than $|T_A| + |T_B| + \hat{x}_{l_{AB}} + \hat{y}_{l_{AB}} = \lfloor k_A \rfloor + \lfloor k_B \rfloor + \text{frac}(k_A) + \text{frac}(k_B) = k_A + k_B$. So A or B must have invested in some node $j \in E$, that is, $x_j > 0$. Now since $l_{AB} \succ_A j, l_{AB} \succ_B j, \forall j \in E$, any player which has invested in node j can gain by transferring an amount $\epsilon > 0$ from node j to node l_{AB}, that is, by investing $x_{l_{AB}} + \epsilon$ in node l_{AB} and $x_j - \epsilon$ in node j. So a strategy profile in which $x_{l_{AB}} + y_{l_{AB}} < \hat{x}_{l_{AB}} + \hat{y}_{l_{AB}}$, cannot be a PSNE. So $(\mathbf{x}, \mathbf{y})$ is a PSNE only if $x_{l_{AB}} + y_{l_{AB}} \geq \hat{x}_{l_{AB}} + \hat{y}_{l_{AB}} = \text{frac}(k_A) + \text{frac}(k_B)$. This, along with Equation (3), gives our desired condition:

$$\sum_{i \in E}(x_i + y_i) = 0 \text{ and } (x_{l_{AB}} + y_{l_{AB}}) = \hat{x}_{l_{AB}} + \hat{y}_{l_{AB}}$$

So in all the cases, we have shown that, if IOS $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is non-conflicting and strategy profile $(\mathbf{x}, \mathbf{y})$ is a PSNE, then $x_i + y_i = \hat{x}_i + \hat{y}_i, \forall i \in N$. $\square$

Corollary 1. *If IOS* $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *is non-conflicting and a strategy profile* $(\mathbf{x}, \mathbf{y})$ *is a PSNE, then* $\mathbf{x} = \hat{\mathbf{x}} \iff \mathbf{y} = \hat{\mathbf{y}}$.

We now present the necessary and sufficient conditions for the uniqueness of PSNE. The reader may refer to Table 1 for better understanding the different cases in the proof.

π_A	π_B	k_A	k_B	T_A	$\{m\}$	S_B	T_B	C_A	C_B	C_{AB}	C_{AB} invested on by one player?	Case #
$(a\ b\ c\ d\ e)$	$(c\ b\ a\ d\ e)$	1.5	1.4	$\{a\}$	$\{\}$	$\{\}$	$\{c\}$	$\{a\}$	$\{c\}$	$\{\}$	–	1
$(a\ b\ c\ d\ e)$	$(c\ e\ a\ d\ b)$	2	1.5	$\{a,b\}$	$\{\}$	$\{c\}$	$\{c\}$	$\{a,b,c\}$	$\{c\}$	$\{c\}$	Yes	1
$(a\ b\ c\ d\ e)$	$(c\ d\ e\ a\ b)$	2	2	$\{a,b\}$	$\{\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b,c,d\}$	$\{c,d\}$	$\{c,d\}$	Yes	1
$(a\ b\ c\ d\ e)$	$(d\ b\ c\ a\ e)$	1.5	1.5	$\{a\}$	$\{b\}$	$\{\}$	$\{d\}$	$\{a,b\}$	$\{d,b\}$	$\{b\}$	No	2
$(a\ b\ c\ d\ e)$	$(c\ e\ a\ d\ b)$	2	2	$\{a,b\}$	$\{\}$	$\{c\}$	$\{c,e\}$	$\{a,b,c\}$	$\{c,e,a\}$	$\{a,c\}$	No	3
$(a\ b\ c\ d\ e)$	$(c\ b\ a\ d\ e)$	1.5	1.5	$\{a\}$	$\{b\}$	$\{c\}$	$\{c\}$	$\{a,b,c\}$	$\{c,b,a\}$	$\{a,b,c\}$	No	3

Table 1: Examples for different cases in the proof of Proposition 1

Proposition 1. *Assuming that nodes can be strictly ordered by both players, the game has a unique PSNE if and only if the IOS is non-conflicting and either (a) the common contiguous set consists of at most one node, or (b) all the nodes in the common contiguous set are invested on by only one player in the IOS. Also, if the game has a unique PSNE, it is same as the IOS, else the number of PSNE's is uncountable.*

Proof. Since it is necessary that the IOS is non-conflicting, we look at all possibilities of non-conflicting IOS.

Recall that a contiguous set consists of nodes which are exhaustively invested on in the IOS. Such nodes can be invested on by player A or B or both. If such a node i is completely invested on by player A in IOS, then $i \in T_A$, while if it is invested on by player B, then $i \in T_B$. If it is invested on by both the players combined, then $i = l_{AB}$. In what follows, if node l_{AB} is such that $\hat{x}_{l_{AB}} + \hat{y}_{l_{AB}} = 1$, we label the node as m. Now, the contiguous set of player A (C_A) would typically consist of all the nodes on which it would want to invest 1 unit each (T_A), followed perhaps by a node on which it would want to invest the residual fractional part of its budget (m), followed perhaps by some nodes on which player B would want to invest 1 unit each (S_B). Similar would be the contiguous set of player B (C_B). We now consider all possible cases to prove the result.

Case 1 ($C_A = T_A$) or ($C_B = T_B$):

We prove for $C_A = T_A$ (proof for $C_B = T_B$ is similar).

If $C_A = T_A$, we have that the node $i = \pi_A(\lfloor k_A \rfloor + 1)$ following the last node of T_A (and hence C_A) in the ordering of A, is such that $\hat{x}_i + \hat{y}_i < 1$. Note also that $i \notin T_B$, since $\hat{x}_j + \hat{y}_j = 1, \forall j \in T_B$. So we have $i \succ_A j, \forall j \in T_B$. Since $\hat{x}_i + \hat{y}_i < 1$, any PSNE would follow $x_i + y_i < 1$ (from Lemma 4). So a strategy profile in which player A invests $x_j > 0$ in some node $j \in T_B$ cannot be a PSNE, since it can gain by transferring an amount $\epsilon > 0$ from node j to node i. So player A does not invest in any $j \in T_B$ in a PSNE.

So if $\nexists l_B$, we have $\mathbf{y} = \hat{\mathbf{y}}$ and so $\mathbf{x} = \hat{\mathbf{x}}$ (Corollary 1).

If $\exists l_{AB}$, it has to be $\pi_A(\lfloor k_A \rfloor + 1)$, in which case, $\hat{x}_{l_{AB}} + \hat{y}_{l_{AB}} < 1$ (it cannot be exhausted since $C_A = T_A$). Since l_{AB} is shared node, it is partially invested on by B, and so it has to also be $\pi_B(\lfloor k_B \rfloor + 1)$. Since l_{AB} follows the last node of T_B in the ordering of B and is not exhausted in IOS, we have $C_B = T_B$. Since assuming $C_A = T_A$, we showed A does not invest in T_B, in this case where $C_B = T_B$, we can similarly show B does not invest on T_A. So in any PSNE, A invests $\lfloor k_A \rfloor$ in T_A and B invests $\lfloor k_B \rfloor$ in T_B. So the residual budget of A (frac$(k_A) = \hat{x}_{l_{AB}}$) would be invested in l_{AB} since $l_{AB} \succ_A j, \forall j \notin T_A \cup \{l_{AB}\}$; that is, in a PSNE, $x_{l_{AB}} = \hat{x}_{l_{AB}}$. Since we now have $\mathbf{x} = \hat{\mathbf{x}}$, it implies $\mathbf{y} = \hat{\mathbf{y}}$.

Now if $\exists l_B$ (not shared in IOS), l_B is not invested on by A in IOS, and since by definition, $l_B \notin T_B$, we have $\hat{x}_{l_B} + \hat{y}_{l_B} < 1$. Since l_B follows the last node of T_B in the ordering of B and is not exhausted in IOS, we have $C_B = T_B$. So with the same argument as the above case of $\exists l_{AB}$, in any PSNE, A invests $\lfloor k_A \rfloor$ in T_A and B invests $\lfloor k_B \rfloor$ in T_B. If B does not invest the residual amount of frac$(k_B) = \hat{y}_{l_B}$ on l_B, it would have invested some amount in node $j \notin T_B \cup \{l_B\}$; and B can gain by transferring some amount from j to l_B. So in any PSNE, we would have B investing frac$(k_B) = \hat{y}_{l_B}$ in l_B. And since we now have $\mathbf{y} = \hat{\mathbf{y}}$, it implies $\mathbf{x} = \hat{\mathbf{x}}$. This follows regardless of whether or not T_A, T_B are empty.

Since $C_{AB} \subseteq C_A, C_{AB} \subseteq C_B$, this case had $C_{AB} \subseteq T_A$ (if $C_A = T_A$) or $C_{AB} \subseteq T_B$ (if $C_B = T_B$), and so all nodes of C_{AB} were invested on by only one player in the IOS, or $C_{AB} = \{\}$. We showed for this case, IOS is the only PSNE.

Since we have considered the case where $C_A = T_A$ or $C_B = T_B$, the remaining cases have $\{m\}$ or S_B or both in C_A, and $\{m\}$ or S_A or both in C_B.

Case 2 ($C_A = T_A \cup \{m\}$ and $C_B = T_B \cup \{m\}$):

Here, we have that $i = \pi_A(\lfloor k_A \rfloor + 2)$ following node m in the ordering of A, is such that $\hat{x}_i + \hat{y}_i < 1$. Since there is shared node m, it can be the only shared node, and since $i \notin T_A \cup T_B$ because $\hat{x}_j + \hat{y}_j = 1, \forall j \in T_A \cup T_B$, we have $i \in E$ and hence $\hat{x}_i + \hat{y}_i = 0$. So we have $i \succ_A j, \forall j \in T_B$ and $i \succ_B j, \forall j \in T_A$. Since $\hat{x}_i + \hat{y}_i = 0$, any PSNE would follow $x_i + y_i = 0$ (from Lemma 4). So a strategy profile in which A invests $x_j > 0$ in some $j \in T_B$ cannot be a PSNE, since it can gain by transferring an amount $\epsilon > 0$ from node j to node i. So A does not invest in any $j \in T_B$ in a PSNE. Similarly, B does not invest in any $j \in T_A$ in a PSNE.

So in any PSNE, A invests $\lfloor k_A \rfloor$ in T_A and B invests $\lfloor k_B \rfloor$ in T_B. So the residual budget of A is frac(k_A) and that of B is frac(k_B). Since there is a node m, by its definition, we have frac$(k_A)+$frac$(k_B) = 1$. Since $m \succ_A j, \forall j \in T_B \cup E$ and $m \succ_B j, \forall j \in T_A \cup E$, the residual budget of both the players would be invested in node m in any PSNE. So we have a unique PSNE $(\mathbf{x}, \mathbf{y})$ which follows $\mathbf{x} = \hat{\mathbf{x}}, \mathbf{y} = \hat{\mathbf{y}}$. This follows regardless of whether or not T_A, T_B are empty.

In this case, C_{AB} consisted of only one node (m), and for this case, we showed that the IOS is the only PSNE.

Case 3 ($C_A = T_A \cup \{m\} \cup S_B$ and $C_B = T_B \cup \{m\}$) or ($C_A = T_A \cup \{m\}$ and $C_B = T_B \cup \{m\} \cup S_A$) or ($C_A = T_A \cup \{m\} \cup S_B$ and $C_B = T_B \cup \{m\} \cup S_A$) or ($C_A = T_A \cup S_B$ and $C_B = T_B \cup S_A$) :

In this case, we have $|C_{AB}| \geq 2$, $\sum_{i \in C_{AB}} \hat{x}_i > 0$, $\sum_{i \in C_{AB}} \hat{y}_i > 0$. If we have an allocation $\sum_{i \in C_{AB}} x_i = \sum_{i \in C_{AB}} \hat{x}_i, \sum_{i \in C_{AB}} y_i = \sum_{i \in C_{AB}} \hat{y}_i$ and $x_j = \hat{x}_j, y_j = \hat{y}_j, \forall j \notin C_{AB}$, player A cannot improve by removing any amount from C_{AB}, since $\forall i \in C_{AB}$, any node $t \succ_A i$ would be exhaustively invested on (because $C_{AB} \subseteq C_A$ and from Lemma 4). Similarly, B cannot improve since $C_{AB} \subseteq C_B$. So any allocation satisfying the following conditions would be a PSNE: $\sum_{i \in C_{AB}} x_i = \sum_{i \in C_{AB}} \hat{x}_i > 0, \sum_{i \in C_{AB}} y_i = \sum_{i \in C_{AB}} \hat{y}_i > 0$ and $x_j = \hat{x}_j, y_j = \hat{y}_j, \forall j \notin C_{AB}$. This results in uncountable number of possible allocations, and hence uncountable number of PSNE's.

In this case, the common contiguous set consisted of at least two elements and all of these elements were not invested on by only one player in the IOS. For this case, we showed the existence of uncountable number of PSNE's.

So we have proved that, if condition (a) or (b) of the proposition is satisfied (Cases 1 and 2), we have that the game has a unique PSNE and it is same as the IOS. Conversely, if neither of the conditions is satisfied (Case 3), we have that the game has uncountable number of PSNE's. $\square$

Price of Stability and Price of Anarchy

A socially optimal strategy profile is a profile that maximizes the sum of players' utilities. In our game, it is a profile $(\mathbf{x}, \mathbf{y})$ that maximizes $\sum_i (w_{Ai} x_i + \beta w_{Bi} y_i) + \sum_i (w_{Bi} y_i + \alpha w_{Ai} x_i) = \sum_i ((1+\alpha) w_{Ai} x_i + (1+\beta) w_{Bi} y_i)$. Let 'best PSNE' be a PSNE that maximizes the sum of players' utilities, and 'worst PSNE' be a PSNE that minimizes it.

The *price of stability* is defined as the ratio between the sum of players' utilities in a socially optimal strategy profile and that in the best PSNE. Similarly, the *price of anarchy* is the ratio between the sum of players' utilities in a socially optimal strategy profile and that in the worst PSNE.

Socially Optimal Strategy Profile

Let $z_{Ai} = (1+\alpha) w_{Ai}$ and $z_{Bi} = (1+\beta) w_{Bi}$. So a socially optimal strategy profile, and hence the maximum sum of players' utilities, can be obtained by maximizing $\sum_i (z_{Ai} x_i + z_{Bi} y_i)$ over the set of feasible strategy profiles.

If $\alpha \leq -1$ and $\beta > -1$, it can be seen that the socially optimal strategy profile would have player A not investing at all and player B investing $\hat{\mathbf{y}}$, so the sum of players' utilities would be $\sum_i (1+\beta) w_{Bi} \hat{y}_i$. Similarly, if $\beta \leq -1$ and $\alpha > -1$, the socially optimal profile would have player B not investing at all and player A investing $\hat{\mathbf{x}}$, thus resulting in the sum of players' utilities as $\sum_i (1+\alpha) w_{Ai} \hat{x}_i$. If $\alpha, \beta \leq -1$, neither player would invest in the socially optimal profile, and so the sum of players' utilities would be zero.

We now analyze the more involved case when $\alpha, \beta > -1$. In this case, $\mathbf{z}_A = (1+\alpha)\mathbf{w}_A$ and $\mathbf{z}_B = (1+\beta)\mathbf{w}_B$ are constant positive scaling. So the orderings of A and B (π_A and π_B) remain unchanged if they respectively order the nodes according to $\mathbf{z}_A$ and $\mathbf{z}_B$, instead of $\mathbf{w}_A$ and $\mathbf{w}_B$. In an optimal profile, let j_B be the last node in the preference ordering of player B on which B invests, that is, B does not invest beyond this node. Let j_A be defined analogously. Let

$$I_A = \{\pi_A(r)\}_{r=1}^{j_A - 1} \text{ and } I_B = \{\pi_B(r)\}_{r=1}^{j_B - 1}$$
$$J_A = \{\pi_A(r)\}_{r=j_A+1}^{n} \text{ and } J_B = \{\pi_B(r)\}_{r=j_B+1}^{n}$$

If $\exists i \in I_B : x_i + y_i < 1$, value of $\sum_i z_{Bi} y_i$ can be increased (without altering $\sum_i z_{Ai} x_i$), by transferring some of B's investment from $\pi_B(j_B)$ to i. So in a social optimal profile, it should be that $\forall i \in I_B : x_i + y_i = 1$, that is, $y_i = 1 - x_i$. Also, $\forall i \in J_B : y_i = 0$ (by definition). So we have

$$\begin{aligned}
&\max_{\mathbf{x}} \max_{\mathbf{y} \leq \mathbf{1} - \mathbf{x}} \sum_{i \in N} (z_{Ai} x_i + z_{Bi} y_i) \\
&= \max_{\mathbf{x}} \max_{j_B} \sum_{i \in N} z_{Ai} x_i + \sum_{i \in I_B} (1 - x_i) z_{Bi} \\
&\qquad\qquad + \Big(k_B - \sum_{i \in I_B} (1 - x_i)\Big) z_{B\pi_B(j_B)} \\
&= \max_{\mathbf{x}} \max_{j_B} \sum_{i \in N} z_{Ai} x_i - \sum_{i \in I_B} x_i \left(z_{Bi} - z_{B\pi_B(j_B)}\right) \\
&\qquad\qquad + \sum_{i \in I_B} \left(z_{Bi} - z_{B\pi_B(j_B)}\right) + k_B z_{B\pi_B(j_B)} \\
&= \max_{j_B} \max_{\mathbf{x}} \sum_{i \in N} z_{Ai} x_i - \sum_{i \in N} x_i \left(\max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\}\right) \\
&\qquad\qquad + \sum_{i \in N} \max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\} + k_B z_{B\pi_B(j_B)} \\
&= \max_{j_B} \Big[\max_{\mathbf{x}} \Big(\sum_{i \in N} x_i \big(z_{Ai} - \max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\}\big) \Big) \\
&\qquad + \sum_{i \in N} \max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\} + k_B z_{B\pi_B(j_B)} \Big] \quad (4)
\end{aligned}$$

For checking the consistency of j_B, we check if the amount left for B after investing in I_B, that is, the amount allocated for $\pi_B(j_B)$, is between 0 and $1 - x_{\pi_B(j_B)}$. That is,

$$0 \leq k_B - \sum_{i \in I_B} (1 - x_i) \leq 1 - x_{\pi_B(j_B)}$$

$$\text{Lower bound} \iff \sum_{i \in I_B} x_i \geq (j_B - 1) - k_B \quad (5)$$

$$\text{Upper bound} \iff \sum_{i \in I_B} x_i + x_{\pi_B(j_B)} \leq j_B - k_B \quad (6)$$

Since at least $\lceil k_B \rceil$ nodes are required for B to spend its budget, we have $y_{\pi_B(r)} = 1 - x_{\pi_B(r)}, \forall r < \lceil k_B \rceil$. So from the definition of j_B, we have $j_B \geq \lceil k_B \rceil$. Also, if A invests in the most preferred nodes of B (amounting to a maximum of k_A), B would invest its available amount k_B in nodes so as to be a feasible investment strategy, given A's investment. So B would not invest in any node which is beyond $\pi_B(\lceil k_A + k_B \rceil)$ in its ordering. That is, $j_B \leq \lceil k_A + k_B \rceil$.

Player A's strategy in socially optimal profile can be obtained by maximizing (4) subject to Constraints (5) and (6),

Algorithm 1: Socially optimal strategy profile

Input: $\mathbf{w}_A, \mathbf{w}_B, k_A, k_B, \alpha, \beta$
Output: Strategy profile $(\mathbf{x}, \mathbf{y})$ that maximizes $v = u_A(\mathbf{x}, \mathbf{y}) + u_B(\mathbf{x}, \mathbf{y})$

```
v* ← −∞
for j_B ← ⌈k_B⌉ to ⌈k_A + k_B⌉ do
    for i ← 1 to n do
        ν_i^(j_B) = (1 + α)w_Ai
                 − max{(1 + β)(w_Bi − w_Bπ_B(j_B)), 0}
    χ^(j_B) = max_x Σ_i x_i ν_i^(j_B) s.t.
        Σ_{i=1}^{j_B−1} x_π(i) ≥ (j_B − k_B) − 1 and
        Σ_{i=1}^{j_B} x_π(i) ≤ j_B − k_B   (using greedy method)
    v^(j_B) = χ^(j_B) + k_B(1 + β)w_Bπ_B(j_B)
        + Σ_i max{(1 + β)(w_Bi − w_Bπ_B(j_B)), 0}
    if v^(j_B) > v* then
        v* ← v^(j_B)
        x* ← x
y* = arg max_y Σ_i y_i z_Bi s.t. y ≤ 1 − x*
```

and $x_i \in [0, 1], \forall i$, over values of $j_B \in [\lceil k_B \rceil, \lceil k_A + k_B \rceil]$. With j_B fixed, we use a greedy algorithm (instead of solving the linear program), where A invests in nodes i one at a time (up to 1 unit per node) in ascending order of the value $(z_{Ai} - \max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\})$, until k_A is exhausted. If this investment, say $\mathbf{x}^{(o)}$, is consistent with (5) and (6), it is our solution. If it is inconsistent, we make Constraint (5) tight (then (6) is automatically satisfied), and invest greedily on nodes in I_B, a total of $(j_B - 1) - k_B$ (such an investment is possible since $|I_B| > (j_B - 1) - k_B$). The residual amount is invested greedily on nodes in $N \setminus I_B$. Suppose this results in investment $\mathbf{x}^{(l)}$. We similarly check by making Constraint (6) tight, and obtain the corresponding greedy investment $\mathbf{x}^{(u)}$. Owing to linearity of the system, if $\mathbf{x}^{(o)}$ is inconsistent with the constraints, either $\mathbf{x}^{(l)}$ or $\mathbf{x}^{(u)}$ has to be optimal. So our solution is $\mathbf{x}^{(l)}$ or $\mathbf{x}^{(u)}$, whichever gives a higher value of $\sum_i x_i(z_{Ai} - \max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\})$.

To maximize (4), we iterate over $j_B \in [\lceil k_B \rceil, \lceil k_A + k_B \rceil]$ to obtain socially optimal strategy of A, say $\mathbf{x}^*$. The socially optimal strategy of B, say $\mathbf{y}^*$, can be obtained by investing greedily subject to a maximum of $1 - x_i^*$ in node i, until k_B is exhausted. The social optimal profile is thus, $(\mathbf{x}^*, \mathbf{y}^*)$. This method is presented as algorithm in Algorithm 1.

The Price of Anarchy

We first show that we cannot have a universal constant bound for the price of anarchy for the entire class of such games.

Example 1. *Say $N = \{i, j\}$, $k_A = k_B = 1$. Consider $w_{Ai} = w_{Bj} = M > 1$ and $w_{Aj} = w_{Bi} = 1$. Let $\alpha = \beta = 0$. A socially optimal profile has $x_i = 1, y_i = 0$ and $x_j = 0, y_j = 1$. Now there is a PSNE with $x_i = 0, y_i = 1$ and $x_j = 1, y_j = 0$. The ratio between the sum of players' utilities in socially optimal profile and that in this PSNE, is $\frac{M+M}{1+1} = M$. So the price of anarchy can be arbitrarily large for arbitrarily large M.*

In order to compute the price of anarchy for an instance of the game, we first provide a characterization of PSNE.

Lemma 5. *A strategy profile $(\mathbf{x}, \mathbf{y})$ is a PSNE if and only if there exist integers j_A, j_B such that*

$$\forall i \in I_A \cup I_B : x_i + y_i = 1$$
$$\textit{For } i = \pi_A(j_A), \pi_B(j_B) : x_i + y_i \leq 1$$
$$\forall i \in J_A : x_i = 0,\ y_i \leq 1, \quad \forall i \in J_B : y_i = 0,\ x_i \leq 1$$
$$\sum_{i \in N} x_i = k_A, \quad \sum_{i \in N} y_i = k_B, \quad \forall i \in N : x_i, y_i \geq 0$$

Proof. Since the number of nodes and budgets are finite, for any feasible strategy profile, there would always exist nodes $\pi_A(j_A)$ and $\pi_B(j_B)$ in the preference orderings of players A and B respectively, beyond which A and B would not invest; so there would exist integers j_A, j_B corresponding to any PSNE. Note however that, given integers j_A, j_B, we can have several strategy profiles which may or may not be feasible, and so may not correspond to any PSNE. We need to show that we would obtain a PSNE if and only if we are able to find integers j_A, j_B which satisfy the above conditions.

Note that the last three conditions are generic with respect to the studied problem (the budget constraints are tight since it is suboptimal for players to not exhaust their entire budgets). Moreover, the conditions $\forall i \in J_A :\ x_i = 0,\ y_i \leq 1$ and $\forall i \in J_B :\ y_i = 0,\ x_i \leq 1$ always hold due to the definitions of j_A, j_B and hence J_A, J_B. So we need to only prove that the first condition is necessary and sufficient, given the generic conditions and definitions result in feasible j_A, j_B.

If $x_i + y_i = 1, \forall i \in I_A \cup I_B$, we have $x_i + y_i = 1, \forall i \in I_A$, and so player A cannot deviate to a better strategy since all the top nodes in π_A are invested on to their limits. Similarly, we have $x_i + y_i = 1, \forall i \in I_B$, and so B cannot deviate to a better strategy. So strategy profile $(\mathbf{x}, \mathbf{y})$ is a PSNE.

Suppose $\exists i \in I_A$ such that $x_i + y_i < 1$, and A has invested in node $\pi_A(j_A)$, it can gain by transferring an amount $\epsilon > 0$ from $\pi_A(j_A)$ to i since $i \succ_A \pi_A(j_A)$. So the strategy profile $(\mathbf{x}, \mathbf{y})$ is not a PSNE. Similar is the case for player B. So if $\exists i \in I_A \cup I_B$ such that $x_i + y_i \neq 1$, $(\mathbf{x}, \mathbf{y})$ is not a PSNE if A has invested in $\pi_A(j_A)$ or B has invested in $\pi_B(j_B)$.

Note that if $\exists i \in I_A$ such that $x_i + y_i \neq 1$, and A has not invested in node $\pi_A(j_A)$ or any node $t \succ_A i$, we redefine j_A to be j'_A so that $\pi_A(j'_A) = i$ and redefine I_A to be I'_A accordingly. Similarly, we can redefine j_B and I_B to be j'_B and I'_B if required. If for a given strategy profile $(\mathbf{x}, \mathbf{y})$, any j'_A, j'_B result in $x_i + y_i \neq 1$ for some $i \in I_A \cup I_B$, $(\mathbf{x}, \mathbf{y})$ is not a PSNE because of the above argument. $\square$

The following proposition follows immediately.

Proposition 2. *A worst PSNE can be obtained by minimizing the value of $\sum_i((1 + \alpha)w_{Ai}x_i + (1 + \beta)w_{Bi}y_i)$ over all integers j_A, j_B that satisfy the conditions in Lemma 5.*

A solution can be obtained efficiently without solving the linear program, by using a greedy allocation. The idea is to partition the set of nodes in which A would invest $(I_A \cup \{\pi_A(j_A)\})$ into different subsets, and each subset is allotted a part of the total budget based on the requirements enforced by the conditions in Lemma 5; the nodes in each partition are then greedily invested on, one at a time, until the partition's share of the budget is exhausted.

Greedy Algorithm for Finding Worst PSNE

From Lemma 5, we have $\forall i \in I_A \cup I_B : x_i + y_i = 1$ and $\forall i \in J_B : y_i = 0, x_i \leq 1$, which give $\forall i \in I_A \cap J_B : x_i = 1$. Further, since player A exhausts its budget k_A by allocating among nodes only belonging to $I_A \cup \pi_A(j_A)$, we have that $\sum_{i \in I_A} x_i + x_{\pi_A(j_A)} = k_A$. As earlier, we check the consistency of j_B by enforcing Inequalities (5) and (6). Also, if $\pi_B(j_B) \in I_A$, the amount allocated by player B for node $\pi_B(j_B)$ would be $1 - x_{\pi_B(j_B)}$ (since the allocations by both players should sum to 1). This would mean that upper bound in Inequality (6) would be tight, thus leading to $\sum_{i \in I_B} x_i + x_{\pi_B(j_B)} = j_B - k_B$. Hence our optimization problem is:

$$\min_{\mathbf{x}} \sum_{i \in N} x_i \left(z_{Ai} - \max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\} \right)$$

subject to

$$\forall i \in N : \; x_i \in [0, 1]$$

$$\forall i \in I_A \cap J_B : \; x_i = 1$$

$$\sum_{i \in I_A} x_i + x_{\pi_A(j_A)} = k_A$$

$$\text{if } \pi_B(j_B) \in I_A : \left\{ \; \sum_{i \in I_B} x_i + x_{\pi_B(j_B)} = j_B - k_B \right.$$

$$\text{if } \pi_B(j_B) \notin I_A : \left\{ \begin{array}{l} \displaystyle\sum_{i \in I_B} x_i \geq (j_B - 1) - k_B \\ \displaystyle\sum_{i \in I_B} x_i + x_{\pi_B(j_B)} \leq j_B - k_B \end{array} \right.$$

Case 1 ($\pi_B(j_B) \in I_A$):

Case 1(a) ($\pi_A(j_A) \in J_B$):

Since we should have $\forall i \in I_A \cap J_B, \forall i \in I_A \cap J_B$, the total budget allocated by player A for the set $I_A \cap J_B$ should be $|I_A \cap J_B|$. Also we should have $\sum_{i \in I_B} x_i + x_{\pi_B(j_B)} = j_B - k_B$, that is, the total budget allocated by player A for the set $I_B \cup \pi_B(j_B)$ should be $j_B - k_B$. Since player A invests only in nodes belonging to $I_A \cup \pi_A(j_A)$ and $\pi_A(j_A) \in J_B$ (that is, $\pi_A(j_A) \notin I_B \cup \pi_B(j_B)$), we have that the budget allocated by player A for the set $(I_A \cup \pi_A(j_A)) \cap (I_B \cup \pi_B(j_B)) = I_A \cap (I_B \cup \pi_B(j_B))$ should be $j_B - k_B$. The residual budget can then be allocated to $\{\pi_A(j_A)\}$.

So the set $I_A \cup \{\pi_A(j_A)\}$ can be partitioned into three subsets, with the allocation for each partition as follows:

Partition (Z)	Allocated budget $\left(\sum_{i \in Z} x_i\right)$
$I_A \cap J_B$	$\lvert I_A \cap J_B \rvert$
$I_A \cap (I_B \cup \{\pi_B(j_B)\})$	$j_B - k_B$
$\{\pi_A(j_A)\}$	$k_A - \lvert I_A \cap J_B \rvert - (j_B - k_B)$

The nodes in each partition are filled one at a time, in ascending order of the value $(1 + \alpha)w_{Ai} - \max\{(1 + \beta)(w_{Bi} - w_{B\pi_B(j_B)}), 0\}$, until the allocation for that partition is reached. The budget allocation is valid if and only if the allocated budget for each partition is non-negative and not larger than the size of the partition, and the allocated budgets for the partitions sum to k_A.

The other cases follow on similar lines; we now present the allocations for the partitions in these cases.

Case 1(b) ($\pi_A(j_A) \in I_B \cup \{\pi_B(j_B)\}$):

Set $I_A \cup \{\pi_A(j_A)\}$ can be partitioned into subsets,

Partition (Z)	Allocation by A $\left(\sum_{i \in Z} x_i\right)$
$I_A \cap J_B$	$\lvert I_A \cap J_B \rvert$
$I_A \cap (I_B \cup \{\pi_B(j_B)\})$	$j_B - k_B$

Case 2 ($\pi_B(j_B) \notin I_A$):

Partition (Z)	Allocation by A $\left(\sum_{i \in Z} x_i\right)$
$I_A \cap J_B$	$\lvert I_A \cap J_B \rvert$
$(I_A \cup \{\pi_A(j_A)\}) \setminus (I_A \cap J_B)$	$k_A - \lvert I_A \cap J_B \rvert$

Here, the allocation is valid if two conditions are satisfied:

$$\sum_{i \in I_B} x_i \geq (j_B - 1) - k_B$$

$$\sum_{i \in I_B} x_i + x_{\pi_B(j_B)} \leq j_B - k_B$$

If any of the above two conditions is violated, we need to restructure the allocation budgets to forcibly satisfy one of the two extreme possibilities:

Possibility 2(a) ($\sum_{i \in I_B} x_i = (j_B - 1) - k_B$):

Case 2(a)[i] ($\pi_A(j_A) \notin I_B$):

Partition (Z)	Allocation by A $\left(\sum_{i \in Z} x_i\right)$
$I_A \cap J_B$	$\lvert I_A \cap J_B \rvert$
$I_A \cap I_B$	$(j_B - 1) - k_B$
$\pi_A(j_A)$	$k_A - \lvert I_A \cap J_B \rvert - (j_B - 1 - k_B)$

Case 2(a)[ii] ($\pi_A(j_A) \in I_B$):

Partition (Z)	Allocation by A $\left(\sum_{i \in Z} x_i\right)$
$I_A \cap J_B$	$\lvert I_A \cap J_B \rvert$
$(I_A \cup \pi_A(j_A)) \cap I_B$	$(j_B - 1) - k_B$

Possibility 2(b) ($\sum_{i \in I_B} x_i + x_{\pi_B(j_B)} = j_B - k_B$):

Note that this reduces to Case 1, since it also requires that $\sum_{i \in I_B} x_i + x_{\pi_B(j_B)} = j_B - k_B$.

We then take the minimum of the values obtained in Possibilities 2(a) and 2(b).

We obtain a worst PSNE by taking the minimum of the following expression over possible values of j_A, j_B:

$$\sum_{i \in N} x_i \left(z_{Ai} - \max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\} \right)$$
$$+ \sum_{i \in N} \max\{z_{Bi} - z_{B\pi_B(j_B)}, 0\} + k_B z_{B\pi_B(j_B)}$$

Algorithm 2 presents the concise algorithm for finding worst PSNE. The time complexity of determining the preference orderings is $O(n \log n)$, following which, the time complexity for finding worst PSNE is $O(n k_A k_B)$.

Remark 1. *The greedy algorithm outputs a strategy profile in which, there could be at most two nodes with non-integral allocation by player A (similarly by player B). Also, if both k_A and k_B are integers, all the nodes would have integral allocation by both the players.*

Since we know the socially optimal strategy profile and worst PSNE, the price of anarchy can hence be computed.

Algorithm 2: Worst PSNE

Input: $\mathbf{w}_A, \mathbf{w}_B, k_A, k_B, \alpha, \beta$
Output: PSNE $(\mathbf{x}, \mathbf{y})$ that minimizes
$v = u_A(\mathbf{x}, \mathbf{y}) + u_B(\mathbf{x}, \mathbf{y})$

```
v* ← +∞
for j_A ← ⌈k_A⌉ to ⌈k_A + k_B⌉ do
    for j_B ← ⌈k_B⌉ to ⌈k_A + k_B⌉ do
        for i ← 1 to n do
            ν_i^(j_B) = (1 + α) w_Ai
                    − max{(1+β)(w_Bi − w_Bπ_B(j_B)), 0}
        χ^(j_B) = min_x Σ_i x_i ν_i^(j_B)
            where x is obtained using greedy method
        v^(j_B) = χ^(j_B) + k_B (1 + β) w_Bπ_B(j_B)
            + Σ_i max{(1+β)(w_Bi − w_Bπ_B(j_B)), 0}
        if v^(j_B) < v* then
            v* ← v^(j_B)
            x* ← x
y* = arg max_y Σ_i y_i w_Bi s.t. y ≤ 1 − x*
```

The Price of Stability

Similar to Proposition 2, the following result can be proved.

Proposition 3. *A best PSNE can be obtained by maximizing the value of* $\sum_i ((1+\alpha)w_{Ai}x_i + (1+\beta)w_{Bi}y_i)$ *over all integers* j_A, j_B *that satisfy the conditions in Lemma 5.*

Algorithm 2 can be modified to find a best PSNE by initializing $v^* \leftarrow -\infty$ (instead of $+\infty$), and assigning $\chi^{(j_B)} = \max_{\mathbf{x}} \sum_i x_i \nu_i^{(j_B)}$ (instead of $\min_{\mathbf{x}} \sum_i x_i \nu_i^{(j_B)}$). That is, in the greedy algorithm, the nodes in each partition should be filled in descending order (instead of ascending order) of the value $(1+\alpha)w_{Ai} - \max\{(1+\beta)(w_{Bi} - w_{B\pi_B(j_B)}), 0\}$, until the allocation for that partition is reached. Since we know the socially optimal strategy profile and best PSNE, the price of stability can be computed.

We now present a specific result for the price of stability when $\alpha, \beta > -1$. The condition $\alpha, \beta > -1$ can be viewed as a practically reasonable one, since in usual scenarios, if a player's action (such as allocating job to a machine or sending data through a link) fetches it a certain benefit, it is the direct effect of its action; the indirect effect of this action on the other player's utility would usually not be negatively amplified, unless the setting is excessively antagonistic.

Proposition 4. *If* $\alpha, \beta > -1$*, the price of stability is 1.*

Proof. Consider a strategy profile $(\mathbf{x}', \mathbf{y}')$ that maximizes $\sum_i ((1+\alpha)w_{Ai}x'_i + (1+\beta)w_{Bi}y'_i)$. Suppose there exists a strategy $\mathbf{x}''$ to which A can deviate so that $u_A(\mathbf{x}'', \mathbf{y}') > u_A(\mathbf{x}', \mathbf{y}')$, that is, $\sum_i (w_{Ai}x''_i + \beta w_{Bi}y'_i) > \sum_i (w_{Ai}x'_i + \beta w_{Bi}y'_i)$ or equivalently, $\sum_i w_{Ai}x''_i > \sum_i w_{Ai}x'_i$. Since $\alpha > -1$, this would result in $\sum_i (1+\alpha)w_{Ai}x''_i > \sum_i (1+\alpha)w_{Ai}x'_i$, hence $\sum_i ((1+\alpha)w_{Ai}x''_i + (1+\beta)w_{Bi}y'_i) > \sum_i ((1+\alpha)w_{Ai}x'_i + (1+\beta)w_{Bi}y'_i)$. This implies $(\mathbf{x}', \mathbf{y}')$ is not socially optimal, a contradiction. So there is no strategy to which A can unilaterally deviate to improve its utility. Similarly, since $\beta > -1$, there is no strategy to which B can unilaterally deviate to improve its utility. So the socially optimal strategy profile $(\mathbf{x}', \mathbf{y}')$ is a PSNE. □

A Note on Non-Strict Preference Orderings

Under the assumption that players have strict preference orderings over nodes, we had the following condition in Lemma 5: $x_i + y_i = 1, \forall i \in I_A \cup I_B$. However, if the orderings are not strict, this condition would no longer be valid. Recall that non-strict ordering would mean that we have $w_{Ai} = w_{Aj}$ or $w_{Bi} = w_{Bj}$ for some $i \neq j$. We now discuss how this condition can be modified, and hence how the price of anarchy and the price of stability can be computed, when the ordering is not strict for at least one player.

Consider an ordering obtained by breaking ties using any tie breaking rule. Since player A invested in $\pi_A(j_A)$, all nodes strictly more beneficial for A than $\pi_A(j_A)$, must be exhausted; else A could transfer some investment from $\pi_A(j_A)$ to such nodes. Let P_A denote the set of such nodes. However, nodes in I_A which are as beneficial for A as $\pi_A(j_A)$ may not be exhausted. This would still be a PSNE since player A transferring some investment from $\pi_A(j_A)$ to these nodes would not change its utility. Let Q_A be the set of these nodes. The argument for player B is analogous (with P_B and Q_B defined accordingly). So the condition: $x_i + y_i = 1, \forall i \in (I_A \cup I_B)$ changes to the two conditions:

$$x_i + y_i = 1, \forall i \in (P_A \cup P_B),$$
$$x_i + y_i \leq 1, \forall i \in (Q_A \cup Q_B).$$

With these modified conditions in Lemma 5, Proposition 2 can be used to determine the worst PSNE by solving the linear program (our greedy algorithm cannot be used). Similarly, Proposition 3 would give the best PSNE. For $\alpha, \beta > -1$, Proposition 4 still holds (the price of stability is 1), since it does not require the unique ordering assumption; so this best PSNE is the socially optimal profile. For the cases when $\alpha \leq -1$ or $\beta \leq -1$ or both, the corresponding socially optimal profiles are on same lines as those under strict preference orderings. Since we know the socially optimal profile, the worst PSNE, and the best PSNE, we can compute the price of anarchy and the price of stability.

A Paradox

In Example 1, we found a PSNE which results in the sum of players' utilities to be 2. However, if we reduce the budget of one of the players (say B) by an infinitesimal amount $\epsilon > 0$, the common contiguous set would be empty, thus leading to a unique PSNE, which would be same as the IOS (from Proposition 1). The sum of players' utilities in this new PSNE is $(M + M - M\epsilon)$, which would be significantly higher than 2, for large values of M. So reducing the budget may lead to a better 'worst PSNE'. In fact, with $k_A = k_B = 1$, the set of PSNE's can be characterized by allocation $x_i = y_j = \rho, x_j = y_i = 1 - \rho$, where $\rho \in [0, 1]$. The sum of players' utilities would thus be $2M\rho + 2(1-\rho)$, which for almost all values of ρ, would be lesser than $2M - M\epsilon$ (which is the sum of utilities in the unique PSNE when B's budget is reduced). Further, both players would individually gain with this reduced budget with respect to almost all values of ρ.

Though we used a particular example to show that lowering the budget may lead to a better outcome, the underlying reasoning is general. If the IOS is such that reducing players' budgets by relatively small amounts, leads to a break in the contiguity and hence contraction of the common contiguous set, the resulting IOS may satisfy the conditions in Proposition 1. This would lead to a unique PSNE, the IOS itself, which is desirable both individually and socially. On the other hand, the original higher budgets would have been such that they led to either the violation of the uniqueness conditions owing to excessive contiguity, or a conflicting IOS. This would result in uncountable number of PSNE's, of which a significant fraction may be starkly undesirable.

Conclusion

We considered a resource allocation game with linear utility function and a bound on resources that can be allocated to any node by the two players combined; these resulted in linear common coupled constraints and hence a resource allocation polytope game. We showed that, assuming players have strict preference orderings over nodes, the game has a unique PSNE if and only if the independent optimal strategy profile (IOS) is non-conflicting and either (a) the common contiguous set consists of at most one node, or (b) all the nodes in the common contiguous set are invested on by only one player in the IOS. Also, if the game has a unique PSNE, it is same as the IOS, else the number of PSNE's is uncountable. We also derived a socially optimal strategy profile. For obtaining the price of anarchy and the price of stability, we provided a characterization of PSNE, developed a linear program, and proposed an efficient greedy algorithm. Under reasonable conditions, we showed that the price of stability is 1. We concluded by presenting an interesting paradox in this game, that higher budgets may lead to worse outcomes.

A possible future direction is to consider more general utility functions and complex common coupled constraints. It would be interesting to study this game with more than two players to see if the results are fundamentally different. The paradox encountered in this game, has a potential of a detailed study. It may be interesting to measure contiguity or conflict in IOS that would lead to such a paradox.

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