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Robust Stable Marriage

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Abstract

Stable Marriage (SM) is a well-known matching problem, where the aim is to match a set of men and women. The resulting matching must satisfy two properties: there is no unassigned person and there are no other assignments where two people of opposite gender prefer each other to their current assignments. We propose a new version of SM called as Robust Stable Marriage (RSM) by combining stability and robustness. We define robustness by introducing *(a,b)-supermatches*, which has been inspired by (a, b)-supermodels (Ginsberg, Parkes, and Roy 1998). An (a, b)-supermatch is a stable matching, where if at most a pairs want to break up, it is possible to find another stable matching by breaking at most b other pairs.

Introduction & Background

Many matching problems involve the assignment of agents to agents while respecting an optimality criterion. Such problems include the assignment of junior doctors to hospitals, workers to firms, and training programs (Gusfield and Irving 1989). Stable marriage, introduced by Gale and Shapley (Gale and Shapley 1962), is a classic approach to solving such problems. An instance of the stable marriage problem involves two disjoint sets of men and women. Each person is associated with a strictly ordered preference list containing members of the opposite sex. A *stable matching* M is one between men and women in which each person is matched to at most one person in his/her preference list such that no two persons prefer each other to their current match in M. A stable matching M_i dominates a stable matching M_j, denoted by $M_i \preceq M_j$, if every man prefers his match in M_i to M_j or is indifferent between them. The structure that represents all stable matchings according to the dominance order is a lattice. We denote by M_0 (M_z) the man-optimal (respectively, woman-optimal or man-pessimal) stable matching.

A rotation ρ corresponds to an ordered list of pairs taking part in a stable matching M such that *eliminating* these pairs from M and matching each man to the woman of the next man (with respect to ordering of ρ) corresponds to a stable matching. In this case ρ is said to be *exposed* in M. An important property of rotations is that every pair is a part of at most one rotation (Gusfield and Irving 1989). A rotation ρ dominates another rotation ρ' if ρ' cannot be exposed until exposing ρ. This dominance relation defines a partial order on the set of rotations and the associated structure is called a *rotation poset*. Although the number of all stable matchings can be exponential in the number of men/women, the size of rotation poset remains polynomial. A closed subset S is a set of rotations such that for any rotation ρ in S, if there exists a rotation ρ' that dominates ρ then ρ' is also in S. There is a one-to-one correspondence between such closed subsets and the set of stable matchings (Gusfield and Irving 1989).

Motivation

We introduce the notion of (a, b)-supermatch which we define as a stable matching M such that if at most a men (or women) decide to break their matches in M, it is possible to "repair" the matching (i.e., find another stable matching) by changing the partners of those a men/women and at most b others. This concept is inspired by the notion of (a, b)-supermodels in boolean satisfiability (Ginsberg, Parkes, and Roy 1998). We study the problem of finding a stable matching M_i that is closest to a given stable matching M if a pair (man,woman) decides to break their match in M. The distance between matchings is measured as the total number of men that have different partners between M and M_i.

The intuition behind using this notion of robustness is to provide solutions that can be repaired as a result of unforeseen cases with bounded cost. The most robust matching is the one that requires the minimum number of repairs (i.e. minimum b) amongst all stable matchings. The concept of (a, b)-supermatches is also meaningful in the context of other matching problems such as carpooling, stable roommate problems, etc.

Methodology

Let M be a stable matching and S be the closed subset that corresponds to M. Suppose that $\langle m, w\rangle \in M$ is a pair to remove. Let ρ_p be the unique rotation that produces $\langle m, w\rangle$ and ρ_e be the unique rotation that eliminates $\langle m, w\rangle$.

The closest matching that does not include $\langle m, w\rangle$ is a matching in which either $\langle m, w\rangle$ was not produced at all or was eliminated. Hence, if ρ_p exists, there exists a set of stable matchings S_u that dominate M and do not include $\langle m, w\rangle$. Similarly, if ρ_e exists, there exists another set of stable matchings S_d that are dominated by M and do not include $\langle m, w\rangle$.

Predecessors of a rotation ρ in rotation poset are denoted by $N^{-}(\rho)$ and successors are denoted by $N^{+}(\rho)$. Note that, due to the transitivity property of rotation posets, these two sets correspond to all predecessors and successors of the rotation ρ. For all $\langle m, w\rangle \in M$ such that $\langle m, w\rangle \notin M_0$ there exists ρ_p that produces $\langle m, w\rangle$. If there exists such a ρ_p, we can define a specific closed subset, S_1^*, as follows:

$$S_1^* = S \setminus \Big\{\{\rho_p\} \cup \{N^{+}(\rho_p) \cap S\}\Big\}. \quad (1)$$

Similarly, if $\langle m, w\rangle \notin M_z$ there exists ρ_e that eliminates $\langle m, w\rangle$. If there exists such a ρ_e we define S_2^* as follows:

$$S_2^* = S \cup \Big\{\{\rho_e\} \cup \{N^{-}(\rho_e) \setminus S\}\Big\}. \quad (2)$$

Let M_1^* (respectively M_2^*) be the stable matching corresponding to S_1^* (respectively S_2^*). By construction, we have $M_1^* \in S_u$ and $M_2^* \in S_d$. We can show that any stable matching $M_i \notin \{M_1^*, M_2^*\}$ that does not include the pair $\langle m, w\rangle$ cannot be closer to M than M_1^* or M_2^*. Two rotations are incomparable if one does not precede the other. Let $X(S)$ denote the set of men that are exposed in at least one of the rotations in S. Let $d(M_i, M_j)$ be the distance calculated by the number of men that have different partners. We introduce the following two lemmas but omit their proof for space limitation.

Lemma 1. *Given two incomparable rotations ρ and ρ', $X(\{\rho\}) \cap X(\{\rho'\}) = \emptyset$.*

Lemma 2. *Given three stable matchings M_i, M_j and M_k where $M_i \preceq M_j \preceq M_k$, then $d(M_j, M_k) \leq d(M_i, M_k)$ and $d(M_i, M_j) \leq d(M_i, M_k)$.*

Lemma 3. *If there exists an M_x that does not contain $\langle m, w\rangle$, dominates M and different from M_1^*, then M_x dominates M_1^*.*

Proof. $M_1^* \preceq M$ by definition. Let $\langle m, w\rangle$ be the unwanted pair. Suppose by contradiction that there exists an M_x such that $\langle m, w\rangle \notin M_x$ and $M_1^* \preceq M_x \preceq M$. It implies that $S_1^* \subset S_x \subset S$. In this case, $(S_x \setminus S_1^*) \subset \Big\{\{\rho_p\} \cup \{N^{+}(\rho_p) \cap S\}\Big\}$. However, this set contains ρ_p and the rotations preceded by ρ_p. Adding any rotation from this set to S_x results in a contradiction by either adding $\langle m, w\rangle$ to the matching, thereby not breaking that couple, or because the resulting set is not a closed subset. □

Lemma 4. *If there exists an M_x that does not contain $\langle m, w\rangle$ dominated by M but different from M_2^*, then M_2^* dominates M_x.*

Proof. Similar to the proof above, suppose that there exists an M_x such that $\langle m, w\rangle \notin M_x$ and $M \preceq M_x \preceq M_2^*$. Therefore $S \subset S_x \subset S_2^*$. It implies $(S_x \setminus S) \subset \Big\{\{\rho_e\} \cup \{N^{-}(\rho_e) \setminus S\}\Big\}$. This set contains the rotation ρ_e that eliminates the pair and the rotations preceding ρ_e. In order to add ρ_e all other rotations must be added to form a closed subset. If all rotations are added, $S = S_2^*$ which results in a contradiction. □

Lemma 5. *For any stable matching M_i incomparable with M such that M_i does not contain the pair $\langle m, w\rangle$, M_1^* is closer to M than M_i.*

Proof. Let S_i be the closed subset corresponding to M_i, and S be that corresponding to M.

First, we consider the case in which $S_i \cap S = \emptyset$. If the closed subsets have no rotations in common the rotations in these sets are incomparable. Using Lemma 1 $X(S_i) \cap X(S) = \emptyset$. Therefore, $d(M_i, M) = |X(S_i)| + |X(S)|$, whereas $d(M_1^*, M) \leq |X(S)|$.

Second, we consider the case in which $S_i \cap S \neq \emptyset$. Let M_c be the closest dominating stable matching of both M_i and M_1^*, along with S_c as its corresponding closed subset. Using Lemma 2 we know that $d(M_1^*, M_s) \leq d(M_c, M_s)$, where $d(M_c, M_s) = |X(S \setminus S_c)|$.

Using Lemma 1 we know that $X(S_i \setminus S_c) \cap X(S \setminus S_c) = \emptyset$. Therefore, $d(M_i, M) = |X(S_i \setminus S_c)| + |X(S \setminus S_c)|$. By substituting the formula above, $d(M_i, M) \geq |X(S_i \setminus S_c)| + d(M_1^*, M)$. Using the fact that $|X(S_i \setminus S_c)| > 0$ from the definition of M_i, we can conclude that $d(M_i, M) > d(M_1^*, M)$. □

The following theorem is immediate from Lemmas 3, 4, 5.

Theorem 6. *The closest stable matching of a stable matching M is either M_1^* or M_2^*.*

Enumerating all closed subsets requires exponential time since the number of stable matchings is exponential. However, using the properties of rotations and their partial order provides a way to compute M_1^* and M_2^* in polynomial time. A direct consequence of this work is that checking if a stable matching is a $(1, b)$-supermatch can be performed in polynomial time.

Conclusion & Future Research

We have introduced the novel concept of stable matching involving robustness. We show how to "repair" a solution in polynomial time if one pair is to be eliminated. There exist a number of future research directions for this new topic. For instance, the complexity of finding a $(1, b)$-supermatch is still open. It is not clear how one can find the most robust stable matching. Moreover, it would be interesting to study the more general problem of finding (a, b)-supermatches.

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