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New Lower Bound for the Minimum Sum Coloring Problem

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Abstract

The Minimum Sum Coloring Problem (MSCP) is an NP-Hard problem derived from the graph coloring problem (GCP) and has practical applications in different domains such as VLSI design, distributed resource allocation, and scheduling. There exist few exact solutions for MSCP, probably due to its search space much more elusive than that of GCP. On the contrary, much effort is spent in the literature to develop upper and lower bounds for MSCP. In this paper, we borrow a notion called motif, that was used in a recent work for upper bounding the minimum number of colors in an optimal solution of MSCP, to develop a new algebraic lower bound called $LBM\Sigma$ for MSCP. Experiments on standard benchmarks for MSCP and GCP show that $LBM\Sigma$ is substantially better than the existing lower bounds for several families of graphs.

Introduction

The Minimum Sum Coloring Problem (MSCP) is a NP-hard problem introduced in 1989 by Kubicka and Schwenk (Kubicka and Schwenk 1989). As the widely studied Graph Coloring Problem (GCP), MSCP also consists in assigning one color to each vertex of a given graph G, respecting the neighborhood constraints. But, MSCP considers in addition a weight associated with each color. While an optimal solution of GCP is a coloring of G that uses the minimum number of colors, called the chromatic number of G and denoted by $\chi(G)$, an optimal solution of MSCP is a coloring of G such that the sum of weights associated with the used colors is minimum. This minimum sum is called the *chromatic sum* of G and is denoted by $\Sigma(G)$. The minimum number of colors required in an optimal solution of MSCP is called the *strength* (or *chromatic strength*) of G, and is denoted by $s(G)$. It holds that $\chi(G) \leq s(G)$.

Apart from the theoretical interest, MSCP has practical applications in different domains such as VLSI design, distributed resource allocation, scheduling (Bar-Noy et al. 1998; Malafiejski 2004), and more particularly in the field of the management of networks. Indeed, the quality of service (QoS) through a network of customers can easily be reduced to an instance of MSCP of the corresponding graph. To illustrate this, we consider an Allocation Resource Problem (ARP) (Chandy and Misra 1984; Bar-Noy et al. 1998) with a set of jobs $\{J_1,J_2,...,J_n\}$ that have to run on some computer servers $\{P_1,P_2,...,P_m\}$. Each job needs a unit of time to be completed and two jobs requiring the same server cannot be executed simultaneously. The set of constraints can be represented by a conflict graph, where a vertex represents a job and an edge represents a conflict between two jobs. The aim is to compute a job scheduling that minimizes the average response time for jobs. Such a solution is given by an optimal sum coloring of the conflict graph where colors are time units. Figure 1 shows an ARP instance of 8 jobs and 7 computer servers. Figure 2 shows the associated conflict graph, from which GCP resolution provides the following scheduling : $\{J_1, J_2, J_3, J_5\}$ at time 1 and $\{J_4, J_6, J_7, J_8\}$ at time 2 with the average response time $(4 \times 1 + 4 \times 2)/8 = 1.5$. MSCP resolution provides the following scheduling : $\{J_1, J_2, J_3, J_6, J_7, J_8\}$ at time 1, $\{J_4\}$ at time 2 and $\{J_5\}$ at time 3 with the average response time $(6 \times 1 + 2 + 3)/8 = 1.3$. So, even if only 2 units of time are sufficient to schedule the 8 jobs, it is better to use a third unit of time to reduce the average response time and to increase the global QoS.

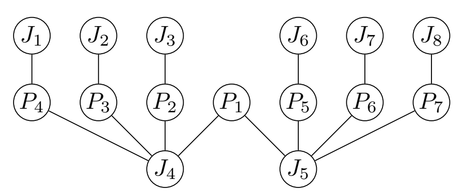
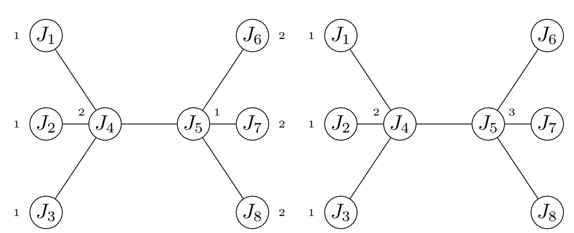


Figure 1: An allocation resource problem



(a) $\Sigma(X) = 12$, $\mathcal{X}(G) = 2$ (b) $\Sigma(G) = 11$, $s(G) = 3$

Figure 2: GCP (a) and MSCP (b) on the conflict graph

Recently, there is a growing interest on MSCP because of its practical applications. Unfortunately, solving MSCP is generally much harder than solving GCP, because it is substantially harder to reduce the number of colors to be considered when solving MSCP than when solving GCP. In fact, when a GCP algorithm finds a valid coloring solution with k colors, it immediately prunes the sub-space with k or more colors. However, a MSCP algorithm cannot do this, because an optimal solution of MSCP can involve more than k colors. For this reason, most MCSP algorithms in the literature are heuristic/meta-heuristic (Li et al. 2009; Sghiouer et al. 2010; Jin, Hao, and Hamiez 2014; Benlic and Hao 2012; Moukrim et al. 2010), providing approximate solutions to the problem. The few exact MSCP algorithms are based on the branch-and-bound schema (Lecat et al. 2015), linear programming (Wang, Hao, and Glover 2013), CSP solving (Lecat et al. 2015; Minot, Ndiaye, and Solnon 2016), or MaxSAT/MinSAT solving (Lecat et al. 2015), and can only solve small graphs.

In this paper, we borrow a notion called *motif*, that was used in (Lecat, Lucet, and Li 2016) for upper bounding the *chromatic strength* $s(G)$ of a graph G, to develop a new algebraic lower bound called $LBM\Sigma$ for $\Sigma(G)$. A tight lower bound lb of $\Sigma(G)$ is useful, as illustrated by the following two examples.

- A heuristic MSCP algorithm generally gives a valid coloring X of G and the corresponding sum $\Sigma(X)$ of weights of the used colors. The quality of this solution can be estimated using lb. In particular, if $lb = \Sigma(X)$, $\Sigma(X)$ is proved to be an optimal solution of MSCP.
- A branch-and-bound MSCP algorithm needs to compute lb at every search tree node. Let X_{best} be the best valid coloring found so far. If $lb \geq \Sigma(X_{best})$, then search below the tree node can be pruned.

This paper is organized as follows. Section 2 presents the necessary definitions. Section 3 presents the notion of motifs and summarizes properties introduced in (Lecat, Lucet, and Li 2016). Section 4 describes our approach based on motifs to compute the lower bound $LBM\Sigma$ of the chromatic sum $\Sigma(G)$. Section 5 analyses the empirical results. Finally, Section 6 concludes the paper.

Basic Definitions

Let $G = (V, E)$ be an undirected graph, where V is the set of n vertices ($|V| = n$) and $E \subseteq V^2$ is the set of edges. A coloring of G is a function $c : V \mapsto \{1, 2, ..., k\}$ that assigns a color $c(v)$, represented by an integer, to each vertex $v \in V$. A coloring is valid iff $c(v) \neq c(v') \; \forall (v, v') \in E$. The color class X_i ($1 \leq i \leq k$), is the set of vertices v such that $c(v) = i$. A valid coloring is denoted by $X = \{X_1, X_2, ..., X_k\}$. For MSCP, a weight w_i is associated with each color i. In this paper, we consider that $w_i = i$. We denote by $\Sigma(X)$ the sum of color weights of the valid coloring X, calculated by Equation 1.

$$\Sigma(X) = 1 \times |X_1| + 2 \times |X_2| + ... + k \times |X_k| \quad (1)$$

An optimal solution of MSCP for a graph G, denoted by $\Sigma(G)$, is defined as follows :

$$\Sigma(G) = min\{\Sigma(X) | X \; is \; a \; valid \; coloring \; of \; G\} \quad (2)$$

Refering to the graph in Figure 3, $X = \{\{c, d, f\}_1, \{a, e\}_2, \{b\}_3\}$ is a valid coloring where the vertices c, d and f are colored with the color 1, a and e with the color 2 and b with the color 3. X is an optimal solution for MSCP, and the chromatic sum is $\Sigma(G) = \Sigma(X) = 10$.

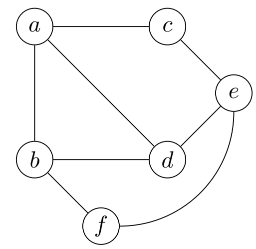


Figure 3: A simple graph

A stable set of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that $\forall (v, v') \in S^2$, $(v, v') \notin E$. A clique of G is a subset $C \subseteq V$ such that $\forall (v, v') \in C^2$, $(v, v') \in E$. We denote by $\alpha(G)$ the cardinality of a maximum stable set of graph G, called the *independence number* of G.

$$\alpha(G) = max(|S| \mid \forall (v, v') \in S^2, \; (v, v') \notin E) \quad (3)$$

A graph $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{(v, v') \in V^2 | (v, v') \notin E\}$, is called the *complement graph* of G. A clique of G is clearly a stable set of $\overline{G}$, and vice versa. Since the vertices in a clique of G need different colors to be colored, $\alpha(\overline{G})$ is a lower bound of the chromatic number of G, which itself is a lower bound of the strength of G :

$$\alpha(\overline{G}) \leq \chi(G) \leq s(G) \quad (4)$$

The permutation of color classes of a valid coloring X has no impact on its validity. Thus, the permutation set of color classes of X forms an equivalent class, denoted by $\Psi(X)$. The colorings of $\Psi(X)$ are symmetric, and use the same number of colors. However, the sum of color weights of the symmetric colorings is not necessarily the same. We can notice it on Figure 3, where the sum associated to $\{\{c, d, f\}_1, \{a, e\}_2, \{b\}_3\}$ is 10, while the sum associated to the symmetric coloring $\{\{a, e\}_1, \{b\}_2, \{c, d, f\}_3\}$ is 13. Because MSCP consists in minimizing the sum of weights, it could be interesting to focus on colorings having the smallest sum among those in $\Psi(X)$. We call such a coloring a *major coloring*, that is defined as follows :

Definition 1 *A major coloring, denoted by X^m, is a coloring $X^m = \{X_1, X_2, \ldots, X_k\}$ such that $|X_1| \geq |X_2| \geq \ldots \geq |X_k|$.*

Property 1 *Let $\Psi(X)$ be the set of symmetric colorings of X, and X^m be a major coloring of $\Psi(X)$, then $\forall X' \in \Psi(X)$, $\Sigma(X^m) \leq \Sigma(X')$.*

Property 1 is a consequence of the specific structure of major colorings that can be summarized by the intuitive idea that the smallest weight should be assigned to the vertices in the largest color class. Referring to Figure 3, the coloring $\{\{c, d, f\}_1, \{a, e\}_2, \{b\}_3\}$ is a major coloring with sum equal to 10. Consequently, we only consider major colorings in this paper, which allow to define the notion of *motifs* in the next section. Such motifs are used to construct our lower bound of the chromatic sum $LBM\Sigma$.

Motifs

The notion of motifs was used in (Bonomo and Valencia-Pabon 2009; 2014) to solve MSCP for P4-sparse graphs and in (Lecat, Lucet, and Li 2016) to define an algebraic upper bound and an algorithmic upper bound for the chromatic strength of the graph. In this section, we briefly recall this notion and its properties useful in the construction of $LBM\Sigma$.

A motif is a representation of a major coloring by a non-increasing sequence of integers as described in Definition 2.

Definition 2 *The motif associated with a major coloring $X = \{X_1, X_2, \ldots, X_k\}$ is a non-increasing sequence of positive integers, denoted by $p = (|X_1|, |X_2|, \ldots, |X_k|)$. The i^{th} integer is $p[i] = |X_i|$. The sum of color weights corresponding to p can be computed using Equation 5. The length of p is the number of colors used by X, denoted by $|p|$ ($|p| = k$).*

$$\Sigma(X) = \Sigma(p) = 1 \times p[1] + 2 \times p[2] + ... + k \times p[k] \quad (5)$$

According to this definition, the motif associated with $X = \{\{c, d, f\}_1, \{a, e\}_2, \{b\}_3\}$ is $p = (3, 2, 1)$, where $p[1] = 3$, $p[2] = 2$ and $p[3] = 1$. $\Sigma(X) = \Sigma(p) = 10$. We note that different major colorings can share the same associated motif. Indeed, the colorings $X^a = \{\{c, d, f\}_1, \{a, e\}_2, \{b\}_3\}$ and $X^b = \{\{c, d, f\}_1, \{b, e\}_2, \{a\}_3\}$ both correspond to $p = (3, 2, 1)$, their sum of color weights are equal to $\Sigma(p)$. Thus, if we consider motifs instead of major colorings, we reduce further the representation of the search space.

For a graph $G = (V, E)$ with $|V| = n$, we denote by $\phi(n)$ the set of all possible motifs associated with major colorings of G, and $\phi(n, k)$ the set of motifs such that $|p| = k$. The sets $\phi(n, k)$ in $\phi(n)$ are sorted by k in increasing order and the motifs in $\phi(n, k)$ are sorted in the decreasing lexicographical order defined below.

Definition 3 *Let p_k and q_k be two motifs in $\phi(n, k)$. We say that p_k lexicographically precedes (or is greater than) q_k iff $p_k[1] > q_k[1]$ or $\exists t > 0$ such that $p_k[x] = q_k[x]$ for each $x < t$ and $p_k[t] > q_k[t]$.*

The decreasing lexicographical order allows to assign an index i to each motif in $\phi(n, k)$: p^i_k is the i^{th} motif in $\phi(n, k)$ in the decreasing lexicographical order. Table 1 lists the motifs of $\phi(8)$ and the corresponding sums of color weights.

We observe that the motif $p^1_3 = (6, 1, 1)$ is lexicographically greater than the motif $p^4_3(4, 2, 2)$, so p^1_3 precedes p^4_3 in $\phi(8, 3)$. All motifs in $\phi(8, 3)$ precede the motifs in $\phi(8, 4)$. This order is of great importance as shown in (Lecat, Lucet, and Li 2016) to compute the upper bound of the chromatic strength $s(G)$. We take the same way to construct the lower bound of MSCP, $LBM\Sigma$, based on the dominance relation between motifs defined below.

ϕ(n,k)	p$\in \phi$(n,k)	$\Sigma(p)$
ϕ(8,1)	(8)	8
ϕ(8,2)	(7,1)	9
	(6,2)	10
	(5,3)	11
	(4,4)	12
ϕ(8,3)	(6,1,1)	11
	(5,2,1)	12
	(4,3,1)	13
	(4,2,2)	14
	(3,3,2)	15
ϕ(8,4)	(5,1,1,1)	14
	(4,2,1,1)	15
	(3,3,1,1)	16
	(3,2,2,1)	17
	(2,2,2,2)	20
ϕ(8,5)	(4,1,1,1,1)	18
	(3,2,1,1,1)	19
	(2,2,2,1,1)	21
ϕ(8,6)	(3,1,1,1,1,1)	23
	(2,2,1,1,1,1)	24
ϕ(8,7)	(2,1,1,1,1,1,1)	29
ϕ(8,8)	(1,1,1,1,1,1,1,1)	36

Table 1: $\phi(8)$: the set of all possible motifs for $n = 8$ vertices

Definition 4 *Let p and q be two motifs in $\phi(n)$. We say that p dominates q, denoted by $p \succeq q$, if and only if $\forall t$ such that $1 \leq t \leq min\{|p|, |q|\}$, $\sum_{x=1}^{t} p[x] \geq \sum_{x=1}^{t} q[x]$.*

Example 1 *Let p and q be two motifs of a graph G with 8 vertices, $p = (5, 2, 1)$ and $q = (4, 3, 1)$.*
For $t = 1 : 5 > 4$;
For $t = 2 : 5 + 2 \geq 4 + 3$;
For $t = 3 : 5 + 2 + 1 \geq 4 + 3 + 1$.
Thus $p \succeq q$.

Property 2 *Let G be a graph, p and q two motifs, respectively associated with two valid colorings X and X' of G. If $p \succeq q$, then $\Sigma(X) \leq \Sigma(X')$.*

We note that the dominance relation is a partial order, contrary to the decreasing lexicographical order which is total. Referring to Table 1, $(4, 3, 1) \succeq (3, 2, 1, 1, 1)$, but $(3, 1, 1, 1, 1)$ and $(2, 2, 2, 2)$ are incomparable, because $(3, 1, 1, 1, 1) \not\succeq (2, 2, 2, 2)$ and $(2, 2, 2, 2) \not\succeq (3, 1, 1, 1, 1)$.

The first motif in $\phi(n, k)$ in the decreasing lexicographical order is p^1_k. It has the following specific structure:

$$(n - k + 1, \overbrace{1, 1, \ldots, 1}^{k-1 \ times})$$

In $\phi(8)$, $p_2^1 = (7,1)$, $p_3^1 = (6,1,1)$, $p_4^1 = (5,1,1,1)$, etc. The Properties 3 and 4 are consequences of this structural property.

Property 3 *Let $G(V,E)$ be a graph, and $k \leq n$ an integer. The motif $p_k^1 = (n-k+1, 1, 1, \dots, 1)$ dominates all other motifs in $\phi(n,k)$.*

Proof 1 *Let q be a motif in $\phi(n,k)$. Then $\forall t$ such that $1 \leq t \leq k$, $\sum_{x=t+1}^{k} p_k^1[x] \leq \sum_{x=t+1}^{k} q[x]$, because $p_k^1[x] = 1 \leq q[x]$ when $x > 1$. So, $\sum_{x=1}^{t} p_k^1[x] = n - \sum_{x=t+1}^{k} p_k^1[x] \geq n - \sum_{x=t+1}^{k} q[x] = \sum_{x=1}^{t} q[x]$.*

Property 4 *Let $G(V,E)$ be a graph, and k and k' be two integers such that $1 \leq k < k' \leq n$. $p_k^1 = (n-k+1, 1, 1, \dots, 1)$ dominates $p_{k'}^1 = (n-k'+1, 1, 1, \dots, 1)$.*

Proof 2 *According to the specific structure of a first motif, $\forall t \leq k$, we have $\sum_{x=1}^{t} p_k^1[x] = n-k+1+t-1 \geq n-k'+1+t-1 = \sum_{x=1}^{t} p_{k'}^1[x]$.*

Consequently, we conclude that p_k^1 dominates all its successors in $\phi(n)$ in the decreasing lexicographical order, and then, its associated sum of color weights is the smallest one over all the colorings that use at least k colors, in the search space.

Algebraic Lower Bound for Chromatic Sum

An algebraic lower bound LB_{Kok} for the chromatic sum $\Sigma(G)$ was proposed in (Kokosiński and Kwarciany 2007). In this Section, we firstly re-state LB_{Kok} in terms of motifs, and then present the new lower bound $LBM\Sigma$ of $\Sigma(G)$.

As mentionned in the previous section, Properties 3 and 4 mean that the motif p_k^1 dominates all the motifs $p_{k'}^i$ such that $i \geq 1$ and $k' \geq k$. That is to say, we cannot find an MSCP solution X that uses k or more colors, such that $\Sigma(X) < \Sigma(p_k^1)$. More formally:

$$\forall q \in \bigcup_{x=k}^{n} \{\phi(n,x)\}, p_k^1 \succeq q \text{ and } \Sigma(p_k^1) \leq \Sigma(q).$$

The sum of color weights associated with p_k^1 is easily computed using Equation 6. Moreover, we know that the minimum number of colors to color a given graph G is its chromatic number $\chi(G)$. Then $\Sigma(p_{\chi(G)}^1)$ is the lowest sum of color weights that can be reached. We deduce the lower bound LB_{Kok} given by Equation 7, equal to $\Sigma(p_{\chi(G)}^1)$. Of course, the chromatic number is not necessarily known for G. In that case, any lower bound of $\chi(G)$ (e.g., the cardinality of a maximum clique of G) can replace $\chi(G)$ in Equation 7 to give a lower bound of $\Sigma(G)$.

$$\Sigma(p_k^1) = (n-k) + \frac{k \times (k+1)}{2} \tag{6}$$

$$LB_{Kok} = (n - \chi(G)) + \frac{\chi(G) \times (\chi(G)+1)}{2} \tag{7}$$

The main drawback of LB_{Kok} is the lack of consideration of the structural properties of the graph G. Hence, we propose to consider one of the characteristics of G, the independence number $\alpha(G)$ combined with the motif representation to improve the lower bound. Indeed, since each integer in a motif represents the cardinality of one color class and each color class is a stable set, any motif containing an integer greater than $\alpha(G)$ cannot match a valid coloring of G. In particular, if $p_k^i[1] > \alpha(G)$, p_k^i can be excluded from the search space.

The principal idea of the construction of $LBM\Sigma$ consists in finding a motif p_k^i ($i \geq 1$ and $1 \leq k \leq n$) such that $p_k^i[1] = \alpha(G)$ and p_k^i dominates all its successors q in $\phi(n)$ with $q[1] \leq \alpha(G)$. If such a motif exists, the corresponding associated sum $\Sigma(p_k^i)$ is a lower bound for $\Sigma(G)$. For this purpose, we borrow the notion of *major motif* that has been introduced in (Lecat, Lucet, and Li 2016) based on the following observation : Let X be a valid major coloring, it could be possible to find two color classes X_i and X_j, with $|X_i| \geq |X_j|$ and $i < j$, such that shifting a vertex of X_j to X_i induces a valid coloring X' with $\Sigma(X') < \Sigma(X)$. Thus, we define for a motif p the *left-shifting operation* as follows.

Definition 5 *Let p be a motif. The left-shifting operation is the transformation of p to p' such that $p'[i] = p[i]+1$, $p'[j] = p[j]-1$ for two positive integers $i < j$, and $p'[x] = p[x]$ for $x \neq i, x \neq j$, and p' remains a non-increasing sequence of positive integers.*

Because p' should remain a non-increasing sequence of positive integers, the left-shifting operation is not always applicable. Thus, a motif in $\phi(n,k)$ that cannot be transformed into another motif in $\phi(n,k)$ by a left-shifting operation without incrementing the first integer is called *major motif* in (Lecat, Lucet, and Li 2016). A left-shifting operation incrementing the first integer of a motif is excluded in the notion of major motif, because incrementing the first integer may make it bigger than $\alpha(G)$, so that the obtained motif does not correspond to any valid coloring of G.

Roughly speaking, a motif is major if it gives the smallest sum of color weights among all motifs that use the same number of colors and in which the first integer is the same. Referring to Table 1, there are two motifs in $\Phi(8,3)$ in which the first integer is 4: (4, 3, 1) and (4, 2, 2), the motif (4, 3, 1) is major; and there are two motifs in $\Phi(8,4)$ in which the first integer is 3: (3, 3, 1, 1) and (3, 2, 2, 1), the motif (3, 3, 1, 1) is major. Intuitively, a major motif in $\phi(n,k)$ contains the maximum number (β) of integers equal to its first integer, denoted by λ ($\lceil \frac{n}{k} \rceil \leq \lambda \leq n-k+1$). The remaining value of n (i.e. $n - \beta \times \lambda$) should be partitioned into $k-\beta$ positive integers. So β is the maximum integer satisfying $n - \beta \times \lambda \geq k - \beta$, or $\beta \leq \frac{n-k}{\lambda-1}$ after excluding the trivial motif (1, 1, …, 1) and assuming $\lambda > 1$. So, $\beta = \left\lfloor \frac{n-k}{\lambda-1} \right\rfloor$.

A major motif is formally defined in (Lecat, Lucet, and Li 2016) as follows.

Definition 6 *Let λ and β be two integers such that $\lceil \frac{n}{k} \rceil \leq$*

$\lambda \leq n-k+1$ *and* $\beta = \left\lfloor \frac{n-k}{\lambda-1} \right\rfloor$. *A motif* p_k^i *in* $\phi(n,k)$ *is major if :*

1. $p_k^i[x] = \lambda,\ if\ 1 \leq x \leq \beta;$
2. $p_k^i[x] = n - \beta \times \lambda - (k - \beta - 1),\ if\ x = \beta + 1;$
3. $p_k^i[x] = 1,\ if\ \beta + 1 < x \leq k.$

It is easy to see that the left-shifting operation is not applicable to a major motif without incrementing its first integer. The following property gives an insight to the notion of motifs.

Property 5 *(1) Let* p *be a motif in* $\phi(n,k)$*, then* $\left\lceil \frac{n}{k} \right\rceil \leq p[1] \leq n-k+1$. *(2) Let* O *denote the decreasing lexicographical order. For any number* λ *such that* $\left\lceil \frac{n}{k} \right\rceil \leq \lambda \leq n-k+1$*, consider the subset of motifs* $\{p \mid p[1] = \lambda\}$ *of* $\phi(n,k)$*, the left-shifting operation is not applicable to the greatest motif* m *w.r.t.* O *in this subset without incrementing* $m[1]$*; furthermore, if a motif* q *in this subset is not the greatest motif w.r.t.* O*, the left-shifting operation is always applicable to* q *to transform it into a greater one w.r.t.* O*.*

Proof 3 *(1) If* $p[1] > n-k+1$*, then* $p[1]+p[2]+\cdots+p[k] > n$*, because* $p[x] \geq 1$ *for* $x > 1$*. If* $p[1] < \left\lceil \frac{n}{k} \right\rceil$*, then* $p[1]+p[2]+\cdots+p[k] \leq k \times \left\lceil \frac{n}{k} \right\rceil - k < n$*.*

(2) It is easy to see that a left-shifting operation applied to a motif p *results in a motif* p' *such that* p' *precedes* p *w.r.t.* O*. So, the left-shifting operation cannot be applied to the greatest motif* m *w.r.t.* O *in* $\{p \mid p[1] = \lambda\}$ *of* $\phi(n,k)$ *without incrementing* $m[1]$*. Let* q *be a motif in the subset such that* m *precedes* q *w.r.t.* O*. By definition, there is a number* $t > 0$ *such that* $m[x] = q[x]$ *for* $x < t$ *and* $m[t] > q[t]$*. Let* y *be the greatest number such that* $m[y] < q[y]$ *(*y *exists because* $\sum_{x=1}^{k} q[x] = \sum_{x=1}^{k} m[x] = n$*). We can apply the left-shifting operation by decrementing* $q[y]$ *and incrementing* $q[t]$ *to obtain a motif preceding* q *w.r.t.* O*.*

Property 5 says that only the greatest motif w.r.t. O in the subset $\{p \mid p[1] = \lambda\}$ of $\phi(n,k)$ is major, meaning that, although the total number of motifs in $\phi(n)$ is exponential, the number of major motifs is only quadratic.

The following properties are the consequences of the structure of major motifs.

Property 6 *Let* p *and* q *be two motifs in* $\phi(n,k)$ *such that* p *is major and* $p[1] = q[1]$*, then* $p \succeq q$*.*

Proof 4 *Let* $\beta = \left\lfloor \frac{n-k}{p[1]-1} \right\rfloor$*. Since* p *is major, we have* $p[1] = p[2] = \ldots = p[\beta] = q[1] \geq q[2] \geq \ldots \geq q[\beta]$*. So,* $\forall t$ *such that* $1 \leq t \leq \beta$*,* $\sum_{x=1}^{t} p[x] \geq \sum_{x=1}^{t} q[x]$*.*

Moreover, $\forall t$ *such that* $\beta+1 \leq t < k$*, we have* $p[t+1] = p[t+2] = \ldots = p[k] = 1 \leq q[k] \leq q[k-1] \leq \ldots \leq q[t+1]$*. So,* $\sum_{x=1}^{t} p[x] = n - \sum_{x=t+1}^{k} p[x] \geq n - \sum_{x=t+1}^{k} q[x] = \sum_{x=1}^{t} q[x]$*. Therefore,* $p \succeq q$*.*

Property 7 *Let* p *and* q *be two major motifs in* $\phi(n,k)$*. If* $p[1] > q[1]$ *then* $p \succeq q$*.*

Proof 5 *Let* $\beta_p = \left\lfloor \frac{n-k}{p[1]-1} \right\rfloor$ *and* $\beta_q = \left\lfloor \frac{n-k}{q[1]-1} \right\rfloor$*. Clearly* $\beta_p \leq \beta_q$*. We have* $p[1] = p[2] = \ldots = p[\beta_p] > q[1] = q[2] = \ldots = q[\beta_p]$*. So,* $\forall t$ *such that* $1 \leq t \leq \beta_p$*,* $\sum_{x=1}^{t} p[x] > \sum_{x=1}^{t} q[x]$*.*

Moreover, $\forall t$ *such that* $\beta_p+1 \leq t < k$*, we have* $p[t+1] = p[t+2] = \ldots = p[k] = 1 \leq q[k] \leq q[k-1] \ldots, \leq q[t+1]$*, implying* $\sum_{x=1}^{t} p[x] = n - \sum_{x=t+1}^{k} p[x] \geq n - \sum_{x=t+1}^{k} q[x] = \sum_{x=1}^{t} q[x]$*. Therefore,* $p \succeq q$*.*

Property 8 *Let* $p_k^i \in \phi(n,k)$ *be a major motif and* $p_k^j \in \phi(n,k)$ *such that* $j > i$*. Then* $p_k^i \succeq p_k^j$*.*

Proof 6 *This result is a consequence of the Property 6 and the Property 7.*

Properties 6, 7 and 8 show that major motifs dominate their successors in $\phi(n,k)$. So, the sum coloring associated with major motif p_k^i such that $p_k^i[1] = \alpha(G)$ is minimal in $\phi(n,k)$ for G.

Property 9 *Let* k *and* k' *be two integers such that* $k < k'$*. Let* p_k^i *be a major motif. Then* $\forall q \in \bigcup_{y=k'}^{n} \phi(n,y)$ *such that* $q[1] \leq p_k^i[1]$*, we have* $p_k^i \succeq q$*.*

Proof 7 *Let* $\beta = \left\lfloor \frac{n-k}{p_k^i[1]-1} \right\rfloor$*. Since* p_k^i *is major, we have* $p_k^i[1] = p_k^i[2] = \ldots = p_k^i[\beta] \geq q[1] \geq q[2] \geq \ldots \geq q[\beta]$*. So,* $\forall t$ *such that* $1 \leq t \leq \beta$*,* $\sum_{x=1}^{t} p_k^i[x] \geq \sum_{x=1}^{t} q[x]$*.*

Moreover, $\forall t$ *such that* $\beta+1 \leq t < k$*, we have* $p_k^i[t+1] = p_k^i[t+2] = \ldots = p_k^i[k] = 1 \leq q[k] \leq q[k-1] \leq \ldots \leq q[t+1]$ *(*q *is a sequence of non-increasing positive integers), implying* $\sum_{x=1}^{t} p_k^i[x] = n - \sum_{x=t+1}^{k} p_k^i[x] \geq n - \sum_{x=t+1}^{k} q[x] = \sum_{x=1}^{t} q[x]$*. Therefore,* $p_k^i \succeq q$*.*

Property 9 shows that a major motif p_k^i dominates its successors q in $\phi(n)$ such that $q[1] \leq p_k^i[1]$. Then, the sum coloring associated with major motif p_k^i such that $p_k^i[1] = \alpha(G)$ is minimal in $\phi(n)$ for G. $\Sigma(p_k^i)$ is a lower bound for $\Sigma(G)$, named $LBM\Sigma$. Based on Definition 6, $LBM\Sigma$ can be computed by Equation 8.

$$\begin{aligned} LBM\Sigma = {} & \alpha(G) \times \frac{\beta \times (\beta+1)}{2} & + \\ & (n - \beta \times \alpha(G) - (k - \beta - 1)) \times (\beta + 1) & + \\ & \sum_{j=\beta+2}^{k} j \end{aligned} \tag{8}$$

In our results, we used $\chi(G)$ as the value for k if it is known and the best known lower bound of $\chi(G)$ otherwise. If $\alpha(G)$ is unknown then we used an upper bound of $\alpha(G)$.

Graph	$\|V\|$	α^*	k^*	LB_{Kok}	$LBM\Sigma$	$T_{LBM\Sigma}$	LB_{Best}	$T_{LB_{Best}}$	UB_{Best}
DSJC1000.1	1000	(175)	20	1190	**3480**	1540	2762	5193	8991
DSJC1000.5	1000	15	87	1105	**33835**	408	6708	155	37594
DSJC1000.9	1000	6	215	24005	**85235**	223	26557	2741	103464
DSJC125.1	125	34	5	135	**297**	0	247	24	973
DSJC125.5	125	10	16	245	**851**	0	549	2040	1012
DSJC125.9	125	4	43	1028	**2108**	0	1691	1128	2503
DSJC250.1	250	44	4	256	**840**	117	570	2940	970
DSJC250.5	250	12	26	575	**2745**	0	1287	3936	3210
DSJC250.9	250	5	71	2735	**6651**	24	4312	N/A	8277
DSJC500.1	500	(85)	12	566	**1746**	1039	1250	1269	2841
DSJC500.5	500	13	13	578	**9867**	16	2923	3936	10897
DSJC500.9	500	5	123	8025	**25581**	84	11053	4116	29869
flat1000-50-0	1000	20	50	2225	**25500**	263	6601	118	25500
flat1000-60-0	1000	17	60	2770	**29914**	296	6640	414	30100
flat1000-76-0	1000	15	76	3850	**33880**	466	6632	98	37167
flat300-20-0	300	15	20	490	**3150**	0	1531	4506	3150
flat300-26-0	300	12	26	625	**3901**	0	1548	4212	3966
flat300-28-0	300	12	28	678	**3906**	0	1547	3750	4238
le450-15c	450	41	15	555	**2705**	1602	2610	3438	3487
le450-15d	450	41	15	555	**2705**	2266	2628	3294	3504
le450-5a	450	(94)	5	460	**1310**	2430	1193	4044	1350
le450-5b	450	(96)	5	460	**1290**	1083	1189	4020	1350
le450-5c	450	90	5	460	**1350**	2	1278	4008	1350
le450-5d	450	90	5	460	**1350**	2	1282	4296	1350
myciel3	11	5	4	17	**20**	0	17	0	21
myciel4	23	11	5	33	**41**	0	34	0	45
myciel5	47	23	6	62	**81**	0	70	0	93
myciel6	95	47	7	116	**158**	0	142	18	189
myciel7	191	95	8	219	**308**	0	286	144	381

Table 2: Our new lower bound $LBM\Sigma$ of the chromatic sum for some instances of the $DSJC$, $flat$, $le450$ and $myciel$ families of graphs and the runtime $T_{LBM\Sigma}$ to compute $LBM\Sigma$, compared with the results LB_{Kok} of (Kokosiński and Kwarciany 2007), the best known lower bound LB_{Best} in the literature and the runtime $T_{LB_{Best}}$ to compute LB_{Best}. When the independence number α^* of a graph cannot be obtained in reasonable time, an upper bound between parentheses was used to compute $LBM\Sigma$. UB_{Best} is the best known upper bound of the chromatic sum in the literature and k^* is the best known lower bound of $\chi(G)$.

Empirical Results and Analysis

In this section, we present our lower bound $LBM\Sigma$ on the instances of the literature. To compute a maximum stable set of a graph G, we work on the complement graph $\overline{G}$ and compute a maximum clique of $\overline{G}$ by running the state-of-the-art exact MaxClique algorithm IncMaxCLQ (Li, Fang, and Xu 2013). The test set is composed of DIMACS (Dimacs) and COLOR (Color02) graphs. The experimental results are obtained on a processor Intel Westmere Xeon E7-8837 (2.66GHz) under Linux.

Table 2 shows the results of $LBM\Sigma$. For each graph, we denote by $Graph$ its name, $|V|$ the number of vertices, α^* the cardinality of a maximum stable set of G when this one is known, an upper bound between brackets otherwise , k^* the best known lower bound for the chromatic number $\chi(G)$, LB_{Kok} the results according to (Kokosiński and Kwarciany 2007), $LBM\Sigma$ the result of our new algebraic bound, $T_{LBM\Sigma}$ the time expressed in seconds to compute $LBM\Sigma$ (including the time of the maximum stable set searching). Column LB_{BEST} gives the best known lower bound of $\Sigma(G)$ in the literature (Jin and Hao 2016; Moukrim et al. 2010; Qinghua and Jin-Kao 2013), all are based on a partition of the graph into cliques and $T_{LB_{Best}}$ is time in seconds to compute LB_{BEST}. UB_{BEST} is the best known upper bound of $\Sigma(G)$ (Jin and Hao 2016; Sghiouer 2011).

The results in Table 2 focus on instances of the DIMACS and COLOR benchmarks where $LBM\Sigma$ improves results of the literature. Among the 70 tested instances, $LBM\Sigma$ gives an improved value of the lower bound of MSCP for 29 instances and reaches the best known lower bound $\Sigma(G)$ in the literature for 13 instances. In particular, $LBM\Sigma$ proves the optimal solution for $MSCP$ for the instances *flat1000-50-0*, *flat300-20-0*, *le450-5c* and *le450-5d* for the first time in the literature, because the lower bound $LBM\Sigma$ is equal to the best known upper bound for these graphs. However, for 28 instances our algebraic bound $LBM\Sigma$ is less effective.

The following observations can be made from Table 2:

- The heuristic methods of the literature are generally based on a partition of G into cliques (Moukrim et al. 2010; Qinghua and Jin-Kao 2013) : if G can be partitioned into r cliques $C_1, C_2, \dots, C_r$, then $\Sigma(G) \geq \sum_{i=1}^{r} \frac{|C_i| \times (|C_i| + 1)}{2}$. These methods can give a lower bound of $\Sigma(G)$ near the optimum for some graphs. In this case, it is hard for any other method to give a better lower bound.
- Our approach is based on searching a maximum stable set of G, which can be very hard for large sparse graphs. In this case, an upper bound of the independence number can be used to compute $LBM\Sigma$.
- For other graphs G, such as random graphs $DSJC$ and $flat$, whose independence number $\alpha(G)$ is known or can be efficiently computed using IncMaxCLQ, our lower bound $LBM\Sigma$ gives good results. Indeed, although the maximum stable set problem is a NP-hard, we observe that the time needed to find a maximum stable and to compute $LBM\Sigma$ is comparable with the time of the methods based on partition of graph into cliques.

Conclusion

We present in this paper a new lower bound $LBM\Sigma$ for the Minimum Sum Coloring Problem (MSCP). We first remind the notion of motifs, that is an abstraction for the solutions of MSCP. Next, we re-state a previous algebraic lower bound for MSCP in terms of motifs. Then, we extend and improve such a lower bound by focusing on the major motifs and using a structural property of a graph G, i.e., the independence number $\alpha(G)$. This one is computed using the efficient IncMaxCLQ solver on the complement graph $\overline{G}$. Finally, we evaluate this approach by comparing our results with those of the literature, showing that $LBM\Sigma$ gives substantially better lower bound of the chromatic sum for some families of graphs in the famous benchmarks $DIMACS$ and $COLOR$. In particular, $LBM\Sigma$ allows to reach optimal MSCP solutions for four graphs in these benchmarks for the first time in the literature.

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