Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence (AAAI-17)

An Ambiguity Aversion Model for Decision Making under Ambiguity

Wenjun Ma,[1] Xudong Luo,[2*] Yuncheng Jiang[1†]
[1] School of Computer Science, South China Normal University, Guangzhou, China.
[2] Institute of Logic and Cognition, Department of Philosophy, Sun Yat-sen University, Guangzhou, China.

Abstract

In real life, decisions are often made under ambiguity, where it is difficult to estimate accurately the probability of each single possible consequence of a choice. However, this problem has not been solved well in existing work for the following two reasons. (i) Some of them cannot cover the Ellsberg paradox and the Machina Paradox. Thus, the choices that they predict could be inconsistent with empirical observations. (ii) Some of them rely on parameter tuning without offering explanations for the reasonability of setting such bounds of parameters. Thus, the prediction of such a model in new decision making problems is doubtful. To the end, this paper proposes a new decision making model based on D-S theory and the emotion of ambiguity aversion. Some insightful properties of our model and the validating on two famous paradoxes show that our model indeed is a better alternative for decision making under ambiguity.

.

Introduction

In real life, due to some factors such as time pressure, lack of data, noise disturbance, and random outcome of some attributes, decisions often have to be made in the case that no precise probability distributions over the single possible outcomes are available. In order to distinguish these cases from the risk cases (in which the possible outcomes of choice are already formulated in terms of a unique probability distribution), we called such cases *decision making under ambiguity*. Since decision making under ambiguity is inevitable in real-world applications, this topic is a central concern in decision science (Tversky and Kahneman 1992) as well as artificial intelligence (Dubois, Fargier, and Perny 2003; Liu 2006; Yager and Alajlan 2015).

In the literature, various theoretic models have been proposed with the aim of capturing how ambiguity can affect decision making and some of them are based on Dempster-Shafer theory (Shafer 1976). However, the existing models still have drawbacks: (i) Some of them still violate a series of experimental observation on the human's choices in real world, for example, the Ellsberg paradox (Ellsberg 1961) and the Machina Paradox (Machina 2009). (ii) Some of them can solve the paradoxes but conditionally by applying some additional parameters setting. In other words, it is unclear why a decision maker will select different bounds of parameter variations in various situation. Thus, the prediction of such models in new decision problems is doubtful. More details discussion can be found in related work section.

To tackle the above problems, this paper will identify a set of basic principles that should be followed by a preference ordering for decision making under ambiguity. Then we will construct a decision model to set a determinate preference ordering based on expected utility intervals and the emotion of ambiguity aversion by using evidence theory of Dempster and Shafer (D-S theory) (Shafer 1976) and the generalised Hartley measure (Dubois and Prade 1985). Accordingly, we will reveal some insightful properties of our model. Finally, we will valid our model by solving two famous paradoxes.

This paper advances the state of art in the field of decision-making under uncertainty in the following aspects: (i) identify a set of basic principles for comparing interval-valued expected utilities; (ii) give a normative decision model for decision making under ambiguity; (iii) disclose a set of properties, which are consistent with human's intuitions; and (iv) use our model to solve both well-known Ellsberg paradox and Machina paradox without any extra parameters setting, which most of the existing models cannot.

The remainder of this paper is organised as follows. First, we recap some basic concepts of D-S theory. Second, we give the formal definition of decision problems under ambiguity. Third, we discuss some principles for setting a proper preference ordering in decision making under ambiguity. Fourth, we present the our decision making model that employs the ambiguity aversion to find an optimal choice based on a set of basic principles. Fifth, we also reveal some properties of our model. Sixth, we validate our model by showing that it can solve well both Ellsberg paradox and Machina paradox. Seventh, we discuss the related work. Finally, we conclude the paper with the further work.

Preliminaries

This section recaps some basic concepts of D-S theory.

*The corresponding author: luoxd3@mail.sysu.edu.cn
†The corresponding author: ycjiang@scnu.edu.cn

Definition 1 (Shafer 1976) *Let Θ be a set of exhaustive and mutually exclusive elements, called a frame of discernment (or simple a frame). Function $m : 2^{\Theta} \to [0, 1]$ is a mass function if $m(\emptyset) = 0$ and $\sum_{A \subseteq \Theta} m(A) = 1$.*

For Definition 1, if for any $m(A) > 0$ ($A \subseteq \Theta$), we have $|A| = 1$ (i.e., A is a singleton), then m is probability function. Thus, probability function is actually a special case of mass function.

Now, based on Definition 1, we can consider the formal definition of ambiguity degree for a given choice.

Definition 2 *Let m_c be a mass function over a frame Θ_c and $|A|$ be the cardinality of set A. Then the ambiguity degree of m_c, denoted as δ_c, is given by:*

$$\delta_c = \frac{\sum\limits_{A \subseteq \Theta_c} m_c(A) \log_2 |A|}{\log_2 |\Theta_c|}. \tag{1}$$

Definition 2 is a normalised version of the generalised Hartley measure for non-specificity (Dubois and Prade 1985). Such a normalised version can guarantee the ambiguity degree is in the range of $[0, 1]$.

Also, by the concept of mass function in Definition 1, the point-valued expected utility function (Neumann and Morgenstern 1944) can be extended to an expected utility interval (Strat 1990):

Definition 3 *For choice c specified by mass function m_c over $\Theta_c = \{u_1, ..., u_n\}$, where u_i is a real number indicating a possible utility of choice c, its expected utility interval is $EUI(c) = [\underline{E}(c), \overline{E}(c)]$, where*

$$\underline{E}(c) = \sum_{A \subseteq \Theta} m_c(A) \min\{u_i \mid u_i \in A\}, \tag{2}$$

$$\overline{E}(c) = \sum_{A \subseteq \Theta} m_c(A) \max\{u_i \mid u_i \in A\}. \tag{3}$$

In Definition 3, when $|A| = 1$ for any $m_c(A) > 0$ ($A \subseteq \Theta$), we have $\underline{E}(c) = \overline{E}(c)$. Thus, the interval-valued expected utility degenerates to the point-valued one. In other words, the interval-valued expected utility covers the point-valued expected utility as a special case.

Problem Definition

Now we give a formal definition for the problem of decision making under ambiguity as follows:

Definition 4 *A decision problem under ambiguity is a 6-tuple $(S, X, C, U, \Theta, M, \succeq)$, where*

(i) S is the state space that contains all possible states of nature (exactly one state is true but a decision-maker is uncertain which state that is);

(ii) X is the set of possible outcomes;

(iii) $C = \{c : S \to X\}$ is the choice set of all options;

(iv) $U = \{u(c(s)) \mid \forall c \in C, \forall s \in S, u(c(s)) \in \mathbb{R}\}$, where $u(c(s)) \in \mathbb{R}$ (real number) is the utility of the outcome $x \in X$ that is caused by choice $c \in C$ in state $s \in S$;

(v) $\Theta = \{\Theta_c \mid c \in C\}$, where Θ_c is the utility set for all possible outcomes of choice c;

(vi) $M = \{m_c \mid c \in C\}$, where $m_c : 2^{\Theta_c} \to [0, 1]$ is a mass function to represent the decision maker's uncertainty about the utility that choice c could cause; and

(vii) $\succeq \subseteq C \times C$ is a binary relation to represent the preference ordering of decision-maker over choices (as usual, $\succ$ and $\sim$ respectively denote the asymmetric and symmetric parts of $\succeq$).

Here we must notice that the mass distribution, which represents the decision maker's uncertainty about the outcomes that choice c could cause, is not based on the state space as the expected utility theory does, but based on the possible utilities of a choice c. The reason for such a change is that the decision maker cares more about the ambiguity of possible utilities rather than the ambiguity of the states of the world and in some cases, the ambiguity of the states of the world does not mean the ambiguity of possible utilities. For instance, suppose a decision maker obtains \$10 when the result of rolling a dice is odd, but obtains \$5 when the result of rolling a dice is even. Now, the dice is unfair, where the chance of points 1 or 3 or 5 are 1/2 , and the chance of other points is 1/2. Clearly, in this example, the states of the world are ambiguity, and multiple probability-values can be assigned to points 1 or 3 or 5. However, the possible utilities are unambiguity since one unique probability distribution can be assigned to $u(\$5)$ with 1/2 and $u(\$10)$ with 1/2.

Hence, although we define mass function over the set of possible utilities for convenience, essentially such a mass function still is obtained based on the uncertainty of states of world. Thus, we can transform such a mass function over the state space as well.

Basic Principles

Now, based on Definition 4, we consider how to set the preference ordering for the choice set in a decision problem under ambiguity. More specifically, we present some basic principles the decision maker should obey when he sets a preference ordering in order to compare two choices in decision making under ambiguity according to their interval-valued expected utilities. Formally, we have:

Definition 5 *Let C be a finite choice set and the interval-valued expected utility of choice $c \in C$ be $EUI(c) = [\underline{E}(c), \overline{E}(c)]$, the ambiguity degree of choice $c \in C$ be $\delta(c)$, and S be a state space. Then a binary relation, denoted as $\succeq$, over C is a preference ordering over C if it satisfies that for all $c_1, c_2, c_3 \in C$:*

1. ***Weak order.*** *(i) Either $c_1 \succeq c_2$ or $c_2 \succeq c_1$; and (ii) if $c_1 \succeq c_2$ and $c_2 \succeq c_3$, then $c_1 \succeq c_3$.*
2. ***Archimedean axiom.*** *If $c_1 \succ c_2$ and $c_2 \succ c_3$, then there exist $\lambda, \mu \in (0, 1)$ such that*
$$\lambda c_1 + (1 - \lambda)c_3 \succ c_2 \succ \mu c_1 + (1 - \mu)c_3.$$
3. ***Monotonicity.*** *If $c_1(s) \succeq c_2(s)$ for all $s \in S$, then $c_1 \succeq c_2$.*
4. ***Risk independence.*** *If $\underline{E}(c_3) = \overline{E}(c_3)$, then $\forall \lambda \in (0, 1]$,*
$$c_1 \succeq c_2 \Leftrightarrow \lambda c_1 + (1 - \lambda)c_3 \succeq \lambda c_2 + (1 - \lambda)c_3.$$

5. ***Representation axiom.*** *There exists a function* Q : $EUI \rightarrow \Re$ *such that*
$$c_1 \succeq c_2 \Leftrightarrow Q(EUI(c_1)) \geq Q(EUI(c_2)).$$
6. ***Range determinacy.*** *If* $\underline{E}(c_1) > \overline{E}(c_2)$*, then* $c_1 \succ c_2$.
7. ***Ambiguity Aversion.*** *If* $EUI(c_1) = EUI(c_2)$ *and* $\delta(c_1) < \delta(c_2)$*, then* $c_1 \succ c_2$.

Principles 1-3 are the standard axioms in the subjective expected utility theory (Savage 1954). The *first* means a preference ordering can compare any pair of choices and satisfies the law of transitivity. The *second* works like a continuity axiom on preferences. It asserts that no choices are either infinitely better or infinitely worse than any other choices. In other words, this principle means no lexicographic preferences for certainty.[1] The *third* is a monotonicity requirement, asserting that if the decision maker considers a choice not worse than the utility of another choice on each state of world, then the former choice is conditionally preferred to the latter.

The rest of principles are our own. The *fourth* means when comparing two choices, it is unnecessary to consider states of nature in which the probability value is determined and these choices yield the same outcome. Moreover, since in decision making under risk, the probability value of each state is determined, under this condition, the fourth principle in fact means when comparing two decisions, it is unnecessary to consider states of nature in which these choices yield the same outcome. That is, this principle requires the preference ordering to obey the *sure thing principle* (Savage 1954) in case of decision making under risk. The *fifth* means the preference ordering can be represented by the quantitative relation of the real number evaluations based on expected utility intervals.

By principles 1-5, we can find that the Q function is monotonic, linear function that satisfies:
$$Q(EUI(c_1) + EU(c_2)) = Q(EUI(c_1)) + EU(c_2),$$
where $EU(c_2) = \underline{E}(c_2) = \overline{E}(c_2)$ is the unique expected utility of choice c_2. Clearly, such function also satisfies:

(i) $Q(EUI(c_1) + a) = Q(EUI(c_1)) + a$; and

(ii) $Q(k \times EUI(c_1)) = k \times Q(EUI(c_1))$,

where $a \in \Re$ and $k \geq 0$.

The *sixth* principle means if the worst situation of a choice is better than the best situation of another, we should choose the first one. In fact, together with the *fifth* principle, it restricts
$$\underline{E}(c) \leq Q(EUI(c)) \leq \overline{E}(c).$$
The *seventh* principle reveals the relation of ambiguity degree and the preference ordering. That is, a decision maker will select a choice with less ambiguity, *ceteris paribus*.

[1] Lexicographic preferences describe comparative preferences, where an economic agent prefers any amount of one good (X) to any amount of another (Y).

The Ambiguity-Aversion Model

Now we turn to construct an apparatus that considers the impact of ambiguity aversion effect in decision making under ambiguity. Accordingly, we will be able to set a proper preference ordering over a set of choices with different expected utility intervals.

Definition 6 *Let* $EUI(c) = [\underline{E}(c), \overline{E}(c)]$ *be the expected utility intervals of choices c, and* $\delta(c)$ *be the ambiguity degree of choice c. Then the ambiguity-aversion expected utility of choice c, denoted as* $\varepsilon(c)$*, is defined as*
$$\varepsilon(c) = \overline{E}(c) - \delta(c)(\overline{E}(c) - \underline{E}(c)). \quad (4)$$
For any two choices c_1 *and* c_2*, the preference ordering* $\succeq$ *is defined as follows:*
$$c_1 \succeq c_2 \Leftrightarrow \varepsilon(c_1) \geq \varepsilon(c_2). \quad (5)$$
When binary relation $\succeq$ *expresses preference ordering, we can use it to define two other binary relations:*

(i) $c_1 \sim c_2$ *if* $c_1 \succeq c_2$ *and* $c_2 \succeq c_1$*; and*

(ii) $c_1 \succ c_2$ *if* $c_1 \succeq c_2$ *and* $c_1 \not\sim c_2$.

From formula (4), we can see that $\underline{E}(c) \leq \varepsilon(c) \leq \overline{E}(c)$, and the higher ambiguity degree δ_c is, the more $\overline{E}(c)$ (the upper bound of the expected utility interval of a choice) will be discounted. In particular, when $\delta = 1$ (meaning absolute ambiguity, *i.e.*, $m(\Theta) = 1$), $\overline{E}(c)$ is discounted to $\underline{E}(c)$ (the lower bound of the expected utility interval of a choice). And when $\delta = 0$ (meaning no ambiguity at all, *i.e.*, all focal elements are singletons and thus the mass function degenerates to a probability function), $\varepsilon(c) = \underline{E}(c) = \overline{E}(c)$ (*i.e.*, no discount is possible).

Hence, by Definition 6, we can easily obtain that
$$\varepsilon(c) = \delta_c \underline{E}(c) + (1 - \delta_c)\overline{E}(c).$$
In other words, if we consider the parameter δ_c as the interpretation of an ambiguity attitude. Then, our ambiguity-aversion expected utility exactly is a special case of the α-$maxmin$ model (Ghirardato, Maccheroni, and Marinacci 2004), which extends the well-known Hurwicz criterion (Jaffray and Jeleva 2007) for ambiguity. In other words, we show a method to obtain the exactly value of ambiguity attitude of a decision maker based on the ambiguity degree of the potentially obtained utility. Nonetheless, we must notice that the α-$maxmin$ model does not really give a method to calculate the ambiguity degree, thus it is arbitrary to assume different choices have different ambiguity degrees in their model. As a result, the α-$maxmin$ model somehow requires a decision maker to have the same ambiguity attitude to all the choices in a decision making problem, while in our model, since we give a method to calculate the ambiguity degree, it allows the case that the decision maker has different ambiguity attitudes to different choices.

In the following, we will check whether or not the preference ordering $\succeq$ in Definition 6 enables us to compare any two choices properly based on the basic principles presented in Definition 5.

Theorem 1 *The preference ordering that is set in Definition 6 satisfies the principles listed in Definition 5.*

Proof: We check the theorem according to the principles listed in Definition 5 one by one.

(i) Since $\varepsilon(c_1) \geq \varepsilon(c_2)$ or $\varepsilon(c_1) \leq \varepsilon(c_2)$ holds for any $c_1, c_2 \in C$, by Definition 6, we have $c_1 \succeq c_2$ or $c_2 \succeq c_1$. Moreover, suppose $c_1 \succeq c_2$ and $c_2 \succeq c_3$. Then by Definition 6, we have $\varepsilon(c_1) \geq \varepsilon(c_2)$ and $\varepsilon(c_2) \geq \varepsilon(c_3)$. As a result, $\varepsilon(c_1) \geq \varepsilon(c_3)$. Thus, by Definition 6 we have $c_1 \succeq c_3$. So, principle 1 in Definition 5 holds for the preference ordering.

(ii) Suppose $c_1 \succeq c_2$ and $c_2 \succeq c_3$. Then by Definition 6, we have $\varepsilon(c_1) \geq \varepsilon(c_2)$ and $\varepsilon(c_2) \geq \varepsilon(c_3)$. Since $\varepsilon(c_i) \in \Re$ for $i = 1, 2, 3$, by continuity of the real numbers, we can always find some $\lambda, \mu \in (0, 1)$ such that

$$\lambda c_1 + (1-\lambda)c_3 \succ c_2,\ c_2 \succ \mu c_1 + (1-\mu)c_3.$$

Thus, principle 2 in Definition 5 holds for the preference ordering.

(iii) Suppose $c_1(s) \succeq c_2(s)$ for all $s \in S$, then by Definitions 4 and 6, we have $u(c_1(s)) \geq u(c_2(s)), \forall s \in S$. Hence, by Definitions 2, 3 and 6, and the fact that $\underline{E}(c) \leq \varepsilon(c) \leq \overline{E}(c)$, we can obtain $c_1 \succeq c_2$. Thus, principle 3 in Definition 5 holds for the preference ordering.

(iv) By $EU(c_3) = \underline{E}(c_3) = \overline{E}(c_3)$ and $\lambda \in (0, 1]$, we have $(1-\lambda)EU(c_3) \in \Re$. Thus, by Definition 6 and $\lambda \in (0, 1]$, we have

$$\begin{aligned}
& c_1 \succeq c_2 \\
\Leftrightarrow\ & \varepsilon(c_1) \geq \varepsilon(c_2) \\
\Leftrightarrow\ & \lambda\varepsilon(c_1) + (1-\lambda)EU(c_3) \geq \lambda\varepsilon(c_2) + (1-\lambda)EU(c_3) \\
\Leftrightarrow\ & \lambda c_1 + (1-\lambda)c_3 \succeq \lambda c_2 + (1-\lambda)c_3.
\end{aligned}$$

Thus, principle 4 in Definition 5 holds for the preference ordering.

(v) Principle 5 can be obtained by Definition 5 directly.

(vi) By the fact that $\underline{E}(c) \leq Q(EUI(c)) \leq \overline{E}(c)$, principle 6 always holds for the preference ordering.

(vii) By Definitions 2 and 6, it is straightforward that principle 7 holds for the preference ordering. □

Properties

Now we release some insightful properties of our model.

In the theory of rational choice (Hindmoor and Taylor 2015; Suzumura 2016), there are some axioms that we can impose on our choice rules. One of them is well-known Houthakker Axiom of Revealed Preference (Houthakker 1950):

> *"Suppose choices x and y are both in choice sets A and B. If $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$ (and, by symmetry, $y \in c(A)$ as well), where $c(A)$ and $c(B)$ is an optimal choice set for set A or set B."*

The above axiom means that if x is chosen when y is available, then x is also chosen whenever y is chosen and x is available. For example, suppose the quality of food in all the branches of McDonald restaurant is the same in China and so is that of KFC. Now McDonald and KFC both have restaurants in Beijing and Shanghai in China. If a girl likes to have meals in McDonald in Beijing and have meals in KFC in Shanghai, the axiom states that she should also like to have meals in KFC in Beijing and in McDonald in Shanghai.

Actually, many researchers in the field of the decision-making (*e.g.*, Suzumura (2016)) advocate that Houthakker Axiom of Revealed Preference provides a necessary and sufficient condition for observing whether a set of choice functions can be generated by a set of preferences that are *well-behaved* (meaning that these preferences satisfy the reflexive, transitive, complete, monotonic, convex and continuous axioms). Fortunately, we have:

Theorem 2 *The preference ordering that is set in Definition 6 satisfies the Houthakker axiom.*

Proof: Let choices x and y are in both choice sets A and B. Suppose that $x \in c(A)$ and $y \in c(B)$ ($c(A)$ and $c(B)$ are the optimal choice sets of choice sets A and B, respectively). Then, by Definition 6, we have $\varepsilon(x) \geq \varepsilon(y)$ and $\varepsilon(y) \geq \varepsilon(z)$, where $z \in B$ ($z \neq y$). Hence, we have $\varepsilon(x) \geq \varepsilon(z)$. Thus, $x \in c(B)$. □

Now, we reveal the relation of our models and the expected utility theory with the following theorem:

Theorem 3 *For a decision problem under ambiguity $(S, X, C, U, \Theta, M, \succeq)$, if for two choices $c_1, c_2 \in C$, we have $|A| = 1$ for any $m_{c_i}(A) > 0$ ($A \subseteq \Theta$) and $i = 1, 2$. Then the ambiguity-aversion expected utility of choice c_i is an expected utility $EU(c_i)$ and the preference ordering satisfies:*

$$c_1 \succeq c_2 \Leftrightarrow EU(c_1) \geq EU(c_2). \tag{6}$$

Proof: By $|A| = 1$ for any $m_{c_i}(A) > 0$ ($A \subseteq \Theta$) and $i = 1, 2$, we have m_{c_i} is probability function. Suppose $A = \{u_j\}$, then by Definition 3, we have

$$EU(c_i) = \underline{E}(c_i) = \overline{E}(c_i) = \sum m_{c_i}(A)u_j.$$

Thus, by Definition 6, we have

$$c_1 \succeq c_2 \Leftrightarrow EU(c_1) \geq EU(c_2).$$

□

Theorem 3 shows that preference ordering defined in the expected utility theory is a special case of our model. In other words, for a choice c under risk (*i.e.*, $|A| = 1$ for all $m_c(A) > 0$, where $A \subseteq \Theta$), the ambiguity-aversion expected utility $\varepsilon(c)$ is the same as the expected utility $EU(c)$. That is, $\varepsilon(c) = EU(c)$.

Also, some properties related to the comparison of two choices based on the relation of expected utility intervals are shown as follows:

Theorem 4 *Let C be a finite choice set and the interval-valued expected utility of choice $c_i \in C$ be $EUI(c_i) = [\underline{E}(c_i), \overline{E}(c_i)]$, and its ambiguity degree be $\delta(c_i)$. A binary relation $\succ$ over C, which is defined by Definition 6, satisfies:*

(i) if $\underline{E}(c_1) > \underline{E}(c_2)$, $\overline{E}(c_1) > \overline{E}(c_2)$, and $\delta(c_1) \leq \delta(c_2)$, then $c_1 \succ c_2$; and

(ii) if $\underline{E}(c_1) \geq \underline{E}(c_2)$ and $\delta(c_1) = \delta(c_2) = 1$, then $c_1 \succeq c_2$.

Table 1: Three Color Ellsberg Paradox

	30 balls	60 balls	
	R: red	B: blue	G: green
c_1	\$100	\$0	\$0
c_2	\$0	\$100	\$0
c_3	\$100	\$0	\$100
c_4	\$0	\$100	\$100

Proof: (i) When $\underline{E}(c_1) > \underline{E}(c_2)$, $\overline{E}(c_2) > \overline{E}(c_2)$ and $0 \leq \delta(c_1) \leq \delta(c_2) \leq 1$, by Definition 6, we have

$$\begin{aligned} & \varepsilon(c_1) - \varepsilon(c_2) \\ \geq\ & (1-\delta(c_2))(\overline{E}(c_1)-\overline{E}(c_2))+\delta(c_2)(\underline{E}(c_1)-\underline{E}(c_2)) \\ >\ & 0. \end{aligned}$$

Thus, we have $c_1 \succ c_2$.

(ii) By $\delta(c_1) = \delta(c_2) = 1$ and Definition 6, we have

$$\varepsilon(c_i)=\underline{E}(c_i)+(1-\delta_{c_i})(\overline{E}(c_i)-\underline{E}(c_i))=\underline{E}(c_i)\ for\ i=1,2.$$

Thus, by $\underline{E}(c_1) \geq \underline{E}(c_2)$, we have $c_1 \succ c_2$. □

Item (i) of Theorem 4 means that if the ambiguity degree of a choice is not more than that of the other choices, and the worst and best situations of this choice is better than those of the other choices respectively, we should make this choice. And item (ii) of Theorem 4 means that in the case of absolute ambiguity, the decision maker should take the maximin attitude (*i.e.*, compare their worst cases and choose the best one).

Paradoxes Analyses

Now we validate our ambiguity-aversion model by solving two paradoxes (Ellsberg 1961; Machina 2009).

The Ellsberg paradox is a well-known, long-standing example about ambiguity (Ellsberg 1961). Suppose in an urn containing 90 balls, among which 30 are red, and the rest are either blue or green. Table 1 shows two pairs of decision problems, each involving a decision between two choices: c_1 and c_2, or c_3 and c_4. A ball is randomly selected from the urn. And the return of selecting a ball for each choice is \$100 or \$0. Ellsberg (1961) found that a very common pattern of responses to these problems is: $c_1 \succ c_2$ and $c_4 \succ c_3$, which actually violates the expected utility theory.

Now we use our model to analyse this paradox. Firstly, since the possible utilities of all the choices could be 100 or 0 (here we regard the monetary prizes as the utility values to simplify the problem), we have

$$\Theta_{c_1} = \Theta_{c_2} = \Theta_{c_3} = \Theta_{c_4} = \{0, 100\}.$$

Further, since the number of red balls is 30 out of 90 balls, the probability of that the selected ball is red is $\frac{30}{90} = \frac{1}{3}$; and although the number of blue or green balls is unknown, it is known that their total number is 60, and so the probability that the selected ball is blue or green is $\frac{200}{300} = \frac{2}{3}$. Then, by Definition 4, we have

$$\begin{aligned} & m_{c_1}(\{100\}) = \frac{1}{3},\ m_{c_1}(\{0\}) = \frac{2}{3}; \\ & m_{c_2}(\{0\}) = \frac{1}{3},\ m_{c_2}(\{0,100\}) = \frac{2}{3}; \\ & m_{c_3}(\{100\}) = \frac{1}{3},\ m_{c_3}(\{0,100\}) = \frac{2}{3}; \\ & m_{c_4}(\{0\}) = \frac{1}{3},\ m_{c_4}(\{100\}) = \frac{2}{3}. \end{aligned}$$

Hence, by Definition 2, the ambiguity of each choice is

$$\delta_{c_1} = 0,\ \delta_{c_2} = \frac{2}{3},\ \delta_{c_3} = \frac{2}{3},\ \delta_{c_4} = 0.$$

Thus, by Definition 3, we have

$$\begin{aligned} & EUI_{c_1} = \frac{100}{3},\ EUI_{c_2} = [0, \frac{200}{3}], \\ & EUI_{c_3} = [\frac{100}{3}, 100],\ EUI_{c_4} = \frac{200}{3}. \end{aligned}$$

Finally, by Definition 6, we have

$$\begin{aligned} \varepsilon(c_1) &= \frac{100}{3}, \\ \varepsilon(c_2) &= \frac{200}{3}-\frac{2}{3}\times(\frac{200}{3}-0)=\frac{200}{9}, \\ \varepsilon(c_3) &= 100-\frac{2}{3}\times(100-\frac{100}{3})=\frac{500}{9}, \\ \varepsilon(c_4) &= \frac{200}{3}. \end{aligned}$$

Thus, we have $c_1 \succ c_2$ and $c_4 \succ c_3$. Since Ellsberg Paradox shows that decision maker exhibits a systematic preference for the choices with a deterministic probability distribution for the potentially obtained utility over the choices with undetermined probability distribution for the potentially obtained utility, a phenomenon known as ambiguity aversion and our method can solve such a type of paradox. Thus, our ambiguity-aversion expected utility indeed shows the decision maker's ambiguity aversion. Actually, by the structure of Ellsberg paradox, we will find that it is constructed based on the uncertainty of the probability values that are assigned to some possible utilities and the violation of the sure thing principle in the subjective expected utility theory. That is, by changing the utilities of the states of world that two choices yield the same outcome, the preference of the decision maker might be changed (*e.g.*, the utility of green ball have changed from 0 in c_1 and c_2 to 100 in c_3 and c_4). However, it does not mean such phenomenon will happen for any state. In fact, by the fourth principle about *risk independence* and our model, we can reveal the essence of Ellsberg paradox by the following claim:

Claim 1 *A paradox of Ellsberg type will not exist if we only change the utility of all the choices in a state satisfying two conditions:*

(i) the probability value assigned to the state is determined; and

(ii) all the choices yield the same outcome in such a state.

Table 2: 50:51 Example

	50 balls		51 balls	
	E_1	E_2	E_3	E_4
c_1	\$80	\$80	\$40	\$40
c_2	\$80	\$40	\$80	\$40
c_3	\$120	\$80	\$40	\$0
c_4	\$120	\$40	\$80	\$0

Table 3: Reflection Example

	50 balls		51 balls	
	E_1	E_2	E_3	E_4
c_1	\$40	\$80	\$40	\$0
c_2	\$40	\$40	\$80	\$0
c_3	\$0	\$80	\$40	\$40
c_4	\$0	\$40	\$80	\$0

Recently, Machina (2009) posed the following questions regarding the ability of Choquet expected utility theory (Coletti, Petturiti, and Vantaggi 2014) and some well-known decision making models under ambiguity to cover variations of the Ellsberg paradox that appear plausible and even natural. We refer to these variations as examples of the Machina paradox.

More specifically, considering the first example with two pairs of choice selections as shown in Table 2. In this 50:51 example, the comparison of c_1 and c_2 vs. that of c_3 and c_4 differ only in whether they offer the higher prize \$80 on the event E_2 or E_3. As argued by Machina, an ambiguity averse decision maker will prefer c_1 to c_2, but the decision maker may feel that the tiny ambiguity difference between c_3 and c_4 does not offset c_4 objective advantage (a slight advantage due to that the 51st ball may yield \$80), and therefore prefers c_4 to c_3.

In our method, since the possible utilities of c_1 or c_2 could be 40 or 80 and that for c_3 or c_4 could be 120 or 80 or 40 or 0, we have $\Theta_{c_1} = \Theta_{c_2} = \{40, 80\}$ and $\Theta_{c_3} = \Theta_{c_4} = \{0, 40, 80, 120\}$. Then, by Definitions 4, 2, 3 and 6, we have

$$\varepsilon(c_1)=80\times\frac{50}{101}+40\times\frac{51}{101}=59.8,$$
$$\varepsilon(c_2)=80-(\frac{50}{101}+\frac{51}{101})\times(80-40)\times\frac{\log_2 2}{\log_2 2}=40,$$
$$\varepsilon(c_3)=79.6-\frac{\log_2 2}{\log_2 4}\times(\frac{50}{101}+\frac{51}{101})\times(79.6-39.6)=59.6,$$
$$\varepsilon(c_4)=99.8-\frac{\log_2 2}{\log_2 4}\times(\frac{50}{101}+\frac{51}{101})\times(99.8-19.8)=59.8.$$

Thus, we have $c_1 \succ c_2$ and $c_4 \succ c_3$. Moreover, we will find that for c_1 and c_2, although c_2 has a small advantage due to that the 51st ball may yield \$80, the decision maker still take choice c_1 for the reason of ambiguity aversion. Whilst, for c_3 and c_4, although the ambiguity degree for both choices are the same, the slight advantage of c_4 can still influence the decision maker's selection. In other words, our method can consider the ambiguity aversion of the decision maker as well as the advantage of higher utility for a decision making under ambiguity. So, our method covers well the 50:51 example. Thus, we can make the following claim:

Claim 2 *A 50:51 type Machina paradox can be solved by a decision model that considers both the expected utility interval and the ambiguity aversion attitude of a given decision maker.*

Now we turn to the second type of Machina paradoxes called the reflection example in Table 3. In this example, since c_4 is an informationally symmetric left-right reflection of c_1 and c_3 is a left-right reflection of c_2, any decision maker who prefers c_1 to c_2 should have the "reflected" ranking, *i.e.*, c_4 is preferred to c_3. And if $c_2 \succ c_1$ then $c_3 \succ c_4$. In recent experimental analyses, some authors found that over 90 percent of subjects expressed strict preference in the reflection problems, and that roughly 70 percent with the structure $c_1 \succ c_2$ and $c_4 \succ c_3$ or $c_2 \succ c_1$ and $c_3 \succ c_4$. And among those subjects, roughly $\frac{2}{3}$ prefer "packaging" the two extreme outcomes together. That is, $c_2 \succ c_1$ and $c_3 \succ c_4$ (L'Haridon and Placido 2008). And such a result cannot be explained well by five famous ambiguity models (Baillon and Placido 2011).

Similarly, in our method, since the possible utilities for all choices could be 80 or 40 or 0, we have

$$\Theta_{c_1} = \Theta_{c_2} = \Theta_{c_3} = \Theta_{c_4} = \{0, 40, 80\}.$$

Then, by Definitions 4, 2, 3 and 6, we have:

$$\varepsilon(c_1) = 60-(\frac{1}{2}+\frac{1}{2})\times\frac{\log_2 2}{\log_2 3}\times(60-20)=34.8,$$
$$\varepsilon(c_2) = 60-\frac{1}{2}\times\frac{\log_2 2}{\log_2 3}\times(60-20)=47.4,$$
$$\varepsilon(c_3) = 60-\frac{1}{2}\times\frac{\log_2 2}{\log_2 3}\times(60-20)=47.4,$$
$$\varepsilon(c_4) = 60-(\frac{1}{2}+\frac{1}{2})\times\frac{\log_2 2}{\log_2 3}\times(60-20)=34.8.$$

Thus, we have $c_2 \succ c_1$ and $c_3 \succ c_4$. So our method covers the reflect example in Machina paradoxes well. In fact, based on the structure of the reflect example in Machina paradoxes, we will find that it considers two factors to construct the paradoxes. (i) The ambiguity attitude of different choices will be different. Thus, even both c_1 and c_2 have same expected utility interval, most of the decision makers still expressed strict preference in the reflection problems. (ii) The decision maker will be indifferent between the choices that have the same mass function over the same set of possible obtained utilities. In this vein, we can make the following claim to reveal the essence of such a paradox as follows:

Claim 3 *An reflect type Machina paradox can be solved by a decision model satisfying two conditions:*

(i) the decision model can distinguish the ambiguity degrees of different choices; and

(ii) the evaluation of two choices should be the same if they have the same mass function over the same set of utilities.

Related Work

There are three strands of literature related to our research.

First, there have been various decision models that consider the impact of ambiguity in decision making under uncertainty. For example, Choquet expected utility model (Chateauneuf and Tallon 2002; Coletti, Petturiti, and Vantaggi 2014; Schmeidler 1989) successfully captures the type of ambiguity aversion displayed in Ellsberg paradox by applying the Choquet integral to model the expected utility. Maxmin expected utility (Gilboa and Schmeidler 1989) considers the worst outcome in a decision making under ambiguity to find a robust choice. And the α-MEU model (Ghirardato, Maccheroni, and Marinacci 2004) distinguishes different ambiguity attitudes of the decision makers. Also, a general model called variational preferences (VP) has been proposed (Maccheroni, Marinacci, and Rustichini 2006) to capture both MEU and multiplier preferences. Hence, the smooth ambiguity model (Klibanoff, Marinacci, and Mukerji 2005) enables decision makers to separate the effect of ambiguity aversion from that of risk aversion. However, they cannot provide an explanation for both two paradoxes that we analysed in this paper (Baillon and Placido 2011).

Second, some decision models based on D-S theory have been proposed to solve decision problems under ambiguity. Generally speaking, various decision models based on D-S theory can be regarded as the Unique Ranking Order Satisfaction Problem (UROSP) where the point-value probability of each single outcome is unknown or is unavailable. Ordered weighted averaging model (Yager 2008), Jaffray–Wakker approach (Jaffray and Wakker 1993), and partially consonant belief functions model (Giang 2012) have developed different formulas based on a decision maker's attitude to search for solutions to UROSP. However, these models are based on the assumption that a decision maker's attitude can be elicited accurately. Unfortunately, it is difficult to obtain a determinate point value of a decision maker's attitude in a uncertain environment. Therefore, to this end some authors apply some basic notions and terminology of D-S theory without any additional parameter. TBM (Smets and Kennes 2008) and MRA (Liu, Liu, and Liu 2013) are this kind. Nevertheless, both cannot solve the two famous paradoxes so well as our model proposed in this paper.

Third, in recently years, some decision models (Gul and Pesendorfer 2014; Siniscalchi and Marciano 2008) have been proposed with the claim of solving both of these two paradoxes with some parameter variations. For example, the expect uncertainty utility (EUU) theory (Gul and Pesendorfer 2014) will solve the paradoxes based on the interval-valued expected utility and the utility indexes (a parameter variation) applied to the upper and lower bounds of the interval. However, such a model has the following limitations: (i) It is unclear why a decision maker needs to choose different utility indexes in different decision problems. (ii) Since such utility indexes will be obtained after we know the human selection of a given decision problem or a paradox, it is questionable whether or not such utility indexes have the same predication power for any new problems. (iii) Such a model is somehow descriptive rather than normative. As a result, it has few benefits for the decision support issues. Nevertheless, our ambiguity-aversion model can work as a normative decision making model solving two paradoxes without setting any additional parameters setting.

Finally, for the belief expected utility model in (Jaffray 1989; Jaffray and Philippe 1997), which can be recovered the mathematical form considered in our model as

$$V(c) = \alpha V^{+}(c) + (1-\alpha)V^{-}(c),$$

where $\alpha \in [0,1]$ is a constant, and

$$V^{+}(c) = \sum_{B} \phi(B)u(M_B),$$
$$V^{-}(c) = \sum_{B} \phi(B)u(m_B),$$

where ϕ is the Mobius transform of a belief function that acts as mass function m, B is a set of outcomes, m_B and M_B are respectively the worse and best outcomes in set B. However, since the value of α is the value of a pessimism index function, determined by the risk attitude or ambiguity attitude of decision-maker, it should be not changed for the same paradox. Thus, for the 50:51 example of Machina paradox, we find that it is hard to determine a value of α that satisfies the preference ordering. And if we change the utilities of some outcomes in Ellsbergs Paradox, the range of α will change. Therefore, as mentioned in (Jaffray and Philippe 1997), they just give a descriptive model to accommodate the paradoxes. And since such α is a parameter, the Jaffray's model also has the limitations as the decision model with additional parameters. Rather, in our method, since we give a way to calculate the ambiguity degree, which can change accordingly, we can avoid the issues of Jaffray's model.

Conclusion

This paper proposed an ambiguity aversion decision model based on D-S theory and normalised version of the generalised Hartley measure for nonspecificity for decision making under ambiguity. First, we introduce the basic principles that should be followed when setting the preference ordering in decision making under ambiguity. Then, we proposed an ambiguity-aversion model to instantiate these principles and revealed some insightful properties of our model. Finally, we validated our models by solving the two famous paradoxes (*i.e.*, Ellsberg paradox and Machina paradox).

There are many possible extensions to our work. Perhaps the most interesting one is the axiomatisation and some psychological related experimental studies of our method. To achieve this goal, we need identify further axioms to derive the ambiguity degree as well as construct a representation theorem based on the axioms proposed. Another tempting avenue is to apply our decision model into game theory as (Zhang et al. 2014) did. Finally, although our method can model the general human behaviour in decision making under ambiguity as shown by solving some well-known paradoxes in the research field of decision-making, it is still worth discussing the construction of a parameterised model based on our ambiguity aversion model to describe the individual's behaviour for real world application.

Acknowledgements

This work was partially supported by Bairen plan of Sun Yat-sen University, the National Natural Science Foundation of China under Grant No. 61272066, the Program for New Century Excellent Talents in University in China under Grant No. NCET-12-0644, the Natural Science Foundation of Guangdong Province of China under Grant No. S2012030006242, and the Project of Science and Technology in Guangzhou in China under Grant No. 2014J4100031 and 201604010098.

References

Baillon, A., and Placido, L. 2011. Ambiguity models and the Machina paradoxes. *American Economic Review* 101(4):1547–60.

Chateauneuf, A., and Tallon, J. M. 2002. Diversification, convex preferences and non-empty core in Choquet expected utility model. *Economic Theory* 19(3):509–523.

Coletti, G.; Petturiti, D.; and Vantaggi, B. 2014. *Choquet Expected Utility Representation of Preferences on Generalized Lotteries*. Springer International Publishing.

Dubois, D., and Prade, H. 1985. A note on measures of specificity for fuzzy sets. *International Journal of General Systems* 10(4):279–283.

Dubois, D.; Fargier, H.; and Perny, P. 2003. Qualitative decision theory with preference relations and comparative uncertainty: An axiomatic approach. *Artificial Intelligence* 148(1):219–260.

Ellsberg, D. 1961. Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics* 75(4):643–669.

Ghirardato, P.; Maccheroni, F.; and Marinacci, M. 2004. Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory* 118(2):133–173.

Giang, P. H. 2012. Decision with Dempster-Shafer belief functions: Decision under ignorance and sequential consistency. *International Journal of Approximate Reasoning* 53(1):38–53.

Gilboa, I., and Schmeidler, D. 1989. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics* 18(2):141–153.

Gul, F., and Pesendorfer, W. 2014. Expected uncertain utility theory. *Econometrica* 82(1):1–39.

Hindmoor, A., and Taylor, B. 2015. *Rational Choice*. Palgrave Macmillan Ltd.

Houthakker, H. S. 1950. Revealed preference and the utility function. *Economica* 17(17):159–174.

Jaffray, J. Y., and Jeleva, M. 2007. Information processing under imprecise risk with the hurwicz criterion. *In Proceedings of the Fifth International Symposium on Imprecise Probability: Theories and Applications* 49(1):117–129.

Jaffray, J. Y., and Philippe, F. 1997. On the existence of subjective upper and lower probabilities. *Mathematics of Operations Research* 22(1):165–185.

Jaffray, J. Y., and Wakker, P. 1993. Decision making with belief functions: Compatibility and incompatibility with the sure-thing principle. *Journal of Risk & Uncertainty* 7(3):255–71.

Jaffray, J. Y. 1989. Linear utility theory for belief functions. *Operations Research Letters* 8(2):107–112.

Klibanoff, P.; Marinacci, M.; and Mukerji, S. 2005. A smooth model of decision making under ambiguity. *Econometrica* 73(6):1849–1892.

L'Haridon, O., and Placido, L. 2008. Betting on Machina's reflection example: An experiment on ambiguity. *Theory & Decision* 69(3):375–393.

Liu, H. C.; Liu, L.; and Liu, N. 2013. Risk evaluation approaches in failure mode and effects analysis: A literature review. *Expert Systems with Applications* 40(2):828–838.

Liu, W. 2006. Analyzing the degree of conflict among belief functions. *Artificial Intelligence* 170(11):909–924.

Maccheroni, F.; Marinacci, M.; and Rustichini, A. 2006. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica* 74(6):1447–1498.

Machina, M. J. 2009. Risk, ambiguity, and the rank-dependence axioms. *The American Economic Review* 99(1):385–392.

Neumann, J. v., and Morgenstern, O. 1944. *Theory of games and economic behavior*. Princeton University Press.

Savage, L. J. 1954. *The Foundations of Statistics*, volume 1. John Wiley & Sons Inc.

Schmeidler, D. 1989. Subjective probability and expected utility without additivity. *Econometrica* 57(3):571–87.

Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton University Press.

Siniscalchi, and Marciano. 2008. Vector expected utility and attitudes toward variation. *Econometrica* 77(3):801–855.

Smets, P., and Kennes, R. 2008. The transferable belief model. *Artificial Intelligence* 66(94):191–234.

Strat, T. M. 1990. Decision analysis using belief functions. *International Journal of Approximate Reasoning* 4(5-6):391–417.

Suzumura, K. 2016. *Choice, Preferences, and Procedures: A Rational Choice Theoretic Approach*. Harvard University Press.

Tversky, A., and Kahneman, D. 1992. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk & Uncertainty* 5(4):297–323.

Yager, R. R., and Alajlan, N. 2015. Dempster-Shafer belief structures for decision making under uncertainty. *Knowledge-Based Systems* 80(C):58–66.

Yager, R. R. 2008. *Decision Making Under Dempster-Shafer Uncertainties*. Springer Berlin Heidelberg.

Zhang, Y.; Luo, X.; Ma, W.; and Ho−Fung, L. 2014. Ambiguous bayesian games. *International Journal of Intelligent Systems* 29(12):1138–1172.