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Price of Pareto Optimality in Hedonic Games

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Abstract

Price of Anarchy measures the welfare loss caused by selfish behavior: it is defined as the ratio of the social welfare in a socially optimal outcome and in a worst Nash equilibrium. A similar measure can be derived for other classes of stable outcomes. In this paper, we argue that Pareto optimality can be seen as a notion of stability, and introduce the concept of Price of Pareto Optimality: this is an analogue of the Price of Anarchy, where the maximum is computed over the class of Pareto optimal outcomes, i.e., outcomes that do not permit a deviation by the grand coalition that makes all players weakly better off and some players strictly better off. As a case study, we focus on hedonic games, and provide lower and upper bounds of the Price of Pareto Optimality in three classes of hedonic games: additively separable hedonic games, fractional hedonic games, and modified fractional hedonic games; for fractional hedonic games on trees our bounds are tight.

1 Introduction

The prisoners' dilemma and the tragedy of commons (Osborne and Rubinstein 1994) are two prominent examples where selfishness causes significant loss of social welfare. By now, a standard measure of disutility caused by selfish behavior is the *Price of Anarchy* (Koutsoupias and Papadimitriou 1999): this is the ratio of the social welfare in a socially optimal outcome of the game and in a worst (social welfare-minimizing) Nash equilibrium of the game. Good upper and lower bounds on the Price of Anarchy have been obtained for many classes of games (see, e.g., Roughgarden and Tardos, 2007); researchers have also considered the related concept of *Price of Stability* (Correa, Schulz, and Moses 2004; Anshelevich et al. 2008), which compares socially optimal outcomes and *best* Nash equilibria.

Importantly, the concept of the Price of Anarchy is defined for a specific notion of stability, namely, Nash equilibrium. Nevertheless, its analogues can be defined for other solution concepts: e.g., Strong Price of Anarchy (Andelman, Feldman, and Mansour 2007) measures the worst-case welfare loss in strong Nash equilibria. Indeed, one can extend this concept beyond normal-form games and explore the worst-case efficiency loss caused by strategic behavior in other types of games.

In this paper, we are interested in exploring Price of Anarchy-like measures in hedonic games. These are games where players form coalitions, and each player has preferences over coalitions that she can be a part of (Drèze and Greenberg 1980; Banerjee, Konishi, and Sönmez 2001; Bogomolnaia and Jackson 2002). While the standard model of hedonic games assumes that players' preferences over coalitions are ordinal, there are several prominent classes of hedonic games where players assign cardinal utilities to coalitions (e.g., a player may assign utilities to individual players, and lift them to coalitions by computing the sum or average of her utilities for players in a coalition), and in such settings it is desirable to have a measure of welfare loss caused by stability considerations. This agenda was recently pursued by Bilò et al. (2014; 2015) who have analyzed the analogues of Price of Anarchy and Price of Stability for the concept of Nash stability in the context of *fractional hedonic games*—a simple, but expressive class of hedonic games that was recently proposed by Aziz et al. (2014).

Now, Nash stability is a well-known notion of stability for hedonic games, and can be seen as the closest analogue of Nash equilibrium for such games. However, in contrast with normal-form games, where Nash equilibrium is clearly the most prominent solution concept, there are several notions of stability that are commonly studied for hedonic games. Indeed, Nash stability focuses on individual deviations and assumes that any player can join any existing coalition, without asking permission of the coalition members. Given that hedonic games are intended to model group formation, we may consider modifying the notion of a permissible deviation along two dimensions: first, we can allow for group deviations, and second, we can allow (some of) the non-deviators to veto the deviators' moves. These two modifications have the opposite effect: the former enriches the set of actions available to the deviators, while the latter shrinks it. By combining these ideas and their variants, one arrives at the well-known notions of individual stability, contractual individual stability, core, strict core, and several others (see an overview by Aziz and Savani, 2015; Sung and Dimitrov, 2007, propose a somewhat different classification).

The classic notion of Pareto optimality has a natural interpretation within this framework. Indeed, according to the standard definition, an outcome is Pareto optimal if there is no other outcome that makes all players weakly better off and some players strictly better off. In the language of deviations and vetoes, this can be restated as follows: an outcome is Pareto optimal if there is no group of players that can deviate (possibly by forming several pairwise disjoint coalitions) so that all of the deviators are weakly better off, some of them are strictly better off, and no non-deviating player is negatively affected by the deviation (and therefore does not want to veto it). Indeed, Pareto optimality is recognized as a valid notion of stability for hedonic games (Morrill 2010; Aziz and Savani 2015). We remark that it can be viewed as a refinement of *contractual strict core* (Sung and Dimitrov 2007): the latter is defined similarly, the only difference being that the deviating players should form a single coalition.

In this paper, we introduce and study the *Price of Pareto Optimality (PPO)*: this is the ratio of the social welfare in a social welfare-maximizing outcome of the game and the social welfare in a worst Pareto optimal outcome of that game. This concept is a direct analogue of the Price of Anarchy—the only difference is that we maximize over all Pareto optimal outcomes rather than all Nash equilibria. (Note that defining an analogue of the Price of Stability with respect to Pareto optimal outcomes is meaningless: every social welfare-maximizing outcome is Pareto optimal, and therefore the respective quantity would always be 1). While viewing Pareto optimality as a notion of stability is motivated by the analysis of solution concepts in hedonic games, and our technical results pertain to hedonic games, the definition of PPO applies equally well to arbitrary noncooperative games. This concept has the following intuitive interpretation. Consider a society that is strongly motivated by egalitarian fairness ideas, and is unwilling to change the status quo if the change will harm any of its members. Price of Pareto Optimality is exactly the worst-case loss of total welfare that such a society may experience because of its principles.

While similar measures can be defined for other notions of stability for hedonic games, we believe that PPO is particularly appealing, because every hedonic game admits a Pareto optimal outcome; in contrast, many well-known classes of hedonic games (including the ones considered in this paper) may fail to have Nash stable outcomes. Thus, PPO is immune to an important critique of the Price of Anarchy, namely, that it is not clear how to interpret bounds on welfare loss in welfare-pessimal Nash stable outcomes: even if such bounds are not too bad, when a Nash stable outcome does not exist, players may cycle among outcomes with arbitrarily bad social welfare. Indeed, Pareto optimality appears to be the most decisive solution concept for hedonic games that has this property: the set of individually stable or core stable outcomes may be empty, and, while the contractual strict core is always non-empty, the argument above shows that the set of Pareto optimal outcomes is a subset of the contractual strict core.

Our technical contribution in this paper is the study of PPO in three classes of hedonic games: additively separable hedonic games, fractional hedonic games, and a variant of fractional hedonic games, which we call modified fractional hedonic games. We focus on these classes of games for three reasons. First, these classes capture a broad range of coalition formation scenarios (see Aziz et al. 2014, for a discussion of applications of fractional hedonic games). Second, for fractional hedonic games bounds on the Price of Nash Stability are known (Bilò et al. 2014; 2015), which enables us to directly compare the quality of Pareto optimal outcomes and that of Nash stable outcomes. Finally, the analysis of PPO for these classes of games presents an interesting technical challenge: we obtain non-trivial upper and lower bounds for simple symmetric (modified) fractional hedonic games and show that our bounds are tight when the underlying network is a tree. However, perhaps the best way to view our results is as a proof of concept, showing that PPO is a reasonable measure, which can also be investigated in other scenarios (including, but not limited to, other classes of hedonic games).

2 Preliminaries

We consider games with a finite set of players $N = \{1, \ldots, n\}$. A *coalition* is a non-empty subset of N. The set of all players N is called the *grand coalition*, and a coalition of size 1 is called a *singleton coalition*. Given a player i, let $\mathcal{N}_i = \{S \subseteq N : i \in S\}$. For the purposes of this paper, it will be convenient to define a *hedonic game* as a pair $(N, (v_i)_{i \in N})$, where $v_i : \mathcal{N}_i \to \mathbb{R}$ is the *utility function* of player i (traditionally, a hedonic game is defined by endowing each player i with a weak order on $\mathcal{N}_i$; in contrast, our definition assumes that we are given cardinal representations of these orders). We assume that $v_i(\{i\}) = 0$ for all $i \in N$. For every $S, S' \in \mathcal{N}_i$, we say that i *strictly prefers* S *to* S' if $v_i(S) > v_i(S')$; if $v_i(S) = v_i(S')$, we say that i is *indifferent* between S and S'. The *value* of a coalition S is defined as $V(S) = \sum_{i \in S} v_i(S)$. A *coalition structure* (also called *partition*, or *outcome*), is a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ of N into $m \geq 1$ coalitions. We denote by $\mathcal{P}(i)$ the coalition of $\mathcal{P}$ that includes player i; the value $v_i(\mathcal{P}(i))$, also denoted by $v_i(\mathcal{P})$, is the *utility of* i *in* $\mathcal{P}$.

The *social welfare of a partition* $\mathcal{P}$ is defined as

$$SW(\mathcal{P}) = \sum_{1 \leq k \leq m} V(P_k) = \sum_{i \in N} v_i(\mathcal{P}(i)).$$

A partition $\mathcal{P}$ is *optimal* if $SW(\mathcal{P}) \geq SW(\mathcal{P}')$ for every other partition $\mathcal{P}'$. A partition $\mathcal{P}$ *Pareto dominates* another partition $\mathcal{P}'$ if $v_i(\mathcal{P}) \geq v_i(\mathcal{P}')$ for every $i \in N$ and $v_i(\mathcal{P}) > v_i(\mathcal{P}')$ for some $i \in N$. A partition $\mathcal{P}$ is *Pareto optimal* if there is no partition $\mathcal{P}'$ that Pareto dominates $\mathcal{P}$. In other words, a Pareto optimal partition is an outcome that does not permit a deviation by the grand coalition that makes all players weakly better off and some players strictly better off. Note that an optimal partition is necessarily Pareto optimal.

Let $\mathbb{P}$ be the set of all Pareto optimal partitions and let $\mathcal{P}^*$ be an optimal partition. Note that $SW(\mathcal{P}^*) \geq 0$, as the players can form the partition that consists of n singletons.

Definition 1. *Given a hedonic game* $(N, (v_i)_{i \in N})$*, the* Price of Pareto Optimality *(*PPO*) is defined as*

$$\mathsf{PPO} = \max_{\mathcal{P} \in \mathbb{P}} \frac{SW(\mathcal{P}^*)}{SW(\mathcal{P})}$$

if $SW(\mathcal{P}) > 0$ for all $\mathcal{P} \in \mathbb{P}$ and $\mathsf{PPO} = +\infty$ *otherwise.*

We will consider several classes of hedonic games defined on graphs. Let $G = (N, E, w)$ be a weighted directed graph, where N is the node set, E is the edge set, and $w : E \to \mathbb{R}$ is a real-valued edge weight function. We denote a generic edge of G by (i, j) and denote its weight by $w_{i,j}$. We say that G is *unweighted* and write $G = (N, E)$ if $w_{i,j} = 1$ for every $(i, j) \in E$; otherwise we say that G is *weighted*. We say that G is *symmetric* if $(i, j) \in E$ if and only if $(j, i) \in E$ and $w_{i,j} = w_{j,i}$ for all $(i, j) \in E$. The *degree* of node $i \in N$ a symmetric unweighted graph G is the number of nodes $j \in N$ with $(i, j) \in E$; it is denoted by $\delta_G(i)$. We let $\Delta_G = \max_{i \in N}\{\delta_G(i)\}$. The subgraph of G induced by a subset $S \subseteq N$ is denoted by $G_S = (S, E_S)$. We refer to a tree as a symmetric and unweighted graph which is acyclic and connected. A tree with one internal node and $d \geq 0$ leaves is called a d-*star*; its only internal node is called the *center*. Note that a 0-star consists of a single node (the center), while a 1-star has two nodes and we can arbitrarily take one of them as the center and the other as the leaf. A *multi-degree* star is any d-star with $d \geq 2$. A tree G is called a (d, e)-*superstar*, where $d, e \geq 2$, if it has a node of degree d (the *center*) that is adjacent to d internal nodes, and each of these nodes is adjacent to $e - 1$ leaves. Note that a (d, e)-superstar has diameter 4 and admits a vertex cover of size d (which consists of all internal nodes other than the center).

We say that a hedonic game is *unweighted* or *symmetric* if its underlying graph has these properties, and use other graph-theoretic terminology when speaking of players and coalitions.

Some proofs are omitted due to space constraints and can be found in the appendix.

3 Additively Separable Hedonic Games

In this section, we consider a well-studied class of hedonic games known as *additively separable hedonic games*. The analysis of PPO for this class of games turns out to be fairly straightforward, and can be seen as a warm-up for the more sophisticated analysis in subsequent sections.

An *additively separable hedonic game* is defined by a weighted directed graph $G = (N, E, w)$. In this game, the set of players corresponds to the set of nodes and the utility of player i from a coalition $S \ni i$ is given by $v_i(S) = \sum_{j \in S, (i,j) \in E} w_{i,j}$. We denote the additively separable hedonic game that corresponds to a graph G by $\mathtt{H}(G)$.

Our first observation is that if all weights are non-negative, for any pair of players i, j with $w_{i,j} + w_{j,i} > 0$ in any Pareto optimal outcome i and j belong to the same coalition. Thus, in this case any Pareto optimal outcome maximizes the social welfare and the Price of Pareto Optimality is 1. On the other hand, if edge weights may be negative, the Price of Pareto Optimality may be $+\infty$, even if the game is symmetric.

Example 2. Let $N = \{1, 2, 3\}$, $w_{1,2} = w_{2,1} = w_{1,3} = w_{3,1} = 1$, $w_{2,3} = w_{3,2} = -3$. Then the grand coalition is Pareto optimal, as any deviation will lower the utility of player 1. However, its social welfare is negative.

The game in Example 2 contains a cycle. We will now show that Pareto optimal partitions with negative social welfare may exist even in the absence of cycles as long as we allow asymmetric weights.

Example 3. Let $N = \{1, 2\}$, $w_{1,2} = 1$, $w_{2,1} = -2$. The grand coalition is Pareto optimal, as player 1 prefers being with player 2 to being on her own. However, its social welfare is negative.

In contrast, if the game is acyclic and weights are symmetric, every Pareto optimal partition maximizes the social welfare.

Proposition 4. *For every additively separable hedonic game* $\mathtt{H}(G)$ *where* $G = (V, E, w)$ *is symmetric and does not contain cycles it holds that* $\mathsf{PPO}(\mathtt{H}(G)) = 1$.

Proof. We obtain an optimal partition by removing all negative-weight edges from G, and placing nodes in each connected component in a coalition of their own (note that the notion of a negative-weight edge is well-defined, since the graph is symmetric). Indeed, as the graph is acyclic, the social welfare of this partition is $2 \sum_{(i,j) \in E : w_{i,j} > 0} w_{i,j}$, which is an upper bound of the social welfare of any partition of N.

It remains to argue that for any Pareto optimal partition it holds that if $w_{i,j} > 0$ then i and j belong to the same part of the partition and if $w_{i,j} < 0$ then i and j belong to different parts of the partition. Indeed, consider a Pareto optimal partition $\mathcal{P}$, and suppose that $w_{i,j} > 0$, but $\mathcal{P}(i) \neq \mathcal{P}(j)$. Then the deviation where $\mathcal{P}(i)$ and $\mathcal{P}(j)$ merge increases the utility of i and j and does not affect other players. The argument for the case where $w_{i,j} < 0$, but $\mathcal{P}(i) = \mathcal{P}(j)$ is symmetric: splitting this coalition along the edge (i, j) increases the utility of i and j and does not affect other players. $\square$

4 Fractional Hedonic Games

We will now consider fractional hedonic games. In these games, a player's utility is its average value for the members of its coalition (including itself). Formally, a weighted directed graph $G = (N, E, w)$ defines a *fractional hedonic game*, where the set of players corresponds to the set of nodes and the utility of player i from a coalition $S \ni i$ is given by $v_i(S) = \frac{1}{|S|} \sum_{j \in S, (i,j) \in E} w_{i,j}$. We denote the fractional hedonic game that corresponds to a graph G by $\mathtt{F}(G)$.

We first observe that for weighted games PPO can be unbounded, even if all weights are positive, the game is symmetric, and the underlying graph is a tree.

Proposition 5. *For any $M > 0$ there is a symmetric fractional hedonic game* $\mathtt{F}(G)$ *where $G = (N, E, w)$ is a tree and all weights are positive such that* $\mathsf{PPO}(\mathtt{F}(G)) > M$.

Proof. Let $N = \{1, 2, 3, 4\}$, $E = \{(1, 2), (2, 3), (3, 4)\}$, $w_{1,2} = w_{3,4} = 1$, $w_{2,3} = 2M$. The grand coalition maximizes the social welfare, but the reader can check that $\{\{1, 2\}, \{3, 4\}\}$ is Pareto optimal. $\square$

Therefore from now on we will focus on symmetric unweighted graphs. We collect a few useful observations about games on such graphs in the following proposition.

Proposition 6. *Let $G = (N, E)$ be a symmetric unweighted graph with $|N| \geq 2$, and let $\mathcal{P}$ be a Pareto optimal partition for $\mathtt{F}(G)$. Then*

(a) every coalition in $\mathcal{P}$ is connected,

(b) if $E \neq \emptyset$, then $\mathcal{P}$ contains at least one non-singleton coalition.

The following theorem upper-bounds PPO in terms of the maximum degree of the graph.

Theorem 7. *Let $G = (N, E)$ be a symmetric unweighted graph with $|N| \geq 2$. Then $\mathsf{PPO}(\mathtt{F}(G)) \leq 2\Delta_G(\Delta_G + 1)$.*

Proof. Let N_i denote the set of all neighbors of a node i in G. Let $\mathcal{P}$ be a Pareto optimal partition. Note that the size of each coalition in $\mathcal{P}$ is at most $\Delta_G(\Delta_G + 1)$. Indeed, if there is a coalition $P_k \in \mathcal{P}$ with $|P_k| > \Delta_G(\Delta_G + 1)$, then the utility of each player in P_k is strictly less than $\frac{1}{\Delta_G+1}$. On the other hand, we can take a spanning tree of P_k (P_k is connected by Proposition 6), split it into stars, and obtain a utility of at least $\frac{1}{\Delta_G+1}$ for everyone in P_k. This means, in particular, that the utility of each player in a non-singleton coalition in $\mathcal{P}$ is at least $\frac{1}{\Delta_G(\Delta_G+1)}$.

Let $\mathcal{P}^*$ be an optimal partition. Let S denote the set of all players that form singleton coalitions in $\mathcal{P}$; by Proposition 6 S forms an independent set in G. Consider an arbitrary player i in $N \setminus S$. Let $D(i) = \mathcal{P}^*(i) \cap S \cap N_i$, and let $d(i) = |D(i)|$. We define the following payment scheme: we pay 1 to node i to keep for itself, and also give it another $\frac{d(i)}{d(i)+1}$ units of currency, and ask it to pass on $\frac{1}{d(i)+1}$ to each of the nodes in $D(i)$. In this way, we directly give at most 2 units of payoff to each node in $N \setminus S$ and 0 to nodes in S, but the nodes in S will then receive some transfers from their "neighbors" in the optimal partition $\mathcal{P}^*$.

We will now argue that under this payment scheme each player gets at least as much utility as in the optimal partition $\mathcal{P}^*$. Consider first a player in $N \setminus S$. It gets to keep 1 unit of payoff, and it gets at most that much utility in $\mathcal{P}^*$ (in unweighted fractional hedonic games the utility of every player is at most 1). Now, consider a player $j \in S$. If in $\mathcal{P}^*$ player j also forms a singleton coalition, we are done. Otherwise, let $F(j) = N_j \cap \mathcal{P}^*(j)$. By Proposition 6 S forms an independent set in G, so all nodes in $F(j)$ belong to $N \setminus S$. Pick $r \in F(j)$ so that $d(r) \geq d(\ell)$ for all $\ell \in F(j)$. By construction, r belongs to $\mathcal{P}^*(j)$, and all $d(r)$ nodes in $D(r)$ also belong to $\mathcal{P}^*(j)$. Thus, $|\mathcal{P}^*(j)| \geq d(r) + 1$, and therefore the utility of j in $\mathcal{P}^*$ is at most $|F(j)|/(d(r) + 1)$. On the other hand, if t is some other node in $F(j)$, then it transfers $\frac{1}{d(t)} \geq \frac{1}{d(r)}$ units of payoff to j (where the inequality holds by our choice of r), so the sum of transfers received by j is at least $|F(j)|/(d(r) + 1)$, which is exactly what we wanted to prove.

To conclude, under our payment scheme we paid at most 2 to each node in $N \setminus S$, and, after the transfers, each node received at least as much as in $\mathcal{P}^*$. Therefore we have $2|N \setminus S| \geq SW(\mathcal{P}^*)$, and since each node in $N \setminus S$ earns at least $\frac{1}{\Delta_G(\Delta_G+1)}$ in $\mathcal{P}$, the Pareto optimal solution has social welfare at least $\frac{|N \setminus S|}{\Delta_G(\Delta_G+1)} \geq \frac{SW(\mathcal{P}^*)}{2\Delta_G(\Delta_G+1)}$. Thus, the Price of Pareto Optimality is at most $2\Delta_G(\Delta_G + 1)$. $\square$

For fractional hedonic games on unweighted trees, we can show a better bound on PPO. We use the following characterization of the structure of optimal partitions.

Lemma 8 (Bilò et al., 2014)**.** *Let $G = (N, E)$ be an unweighted tree with $|N| \geq 2$, and let $\mathcal{P}^*$ be an optimal partition for the fractional hedonic game $\mathtt{F}(G)$. Then every $P_k^* \in \mathcal{P}^*$ is a d_k-star for some $d_k \geq 1$.*

Using the notion of a *labelling function* (Bilò et al. 2014), we obtain the following upper bound on the social welfare of an optimal partition.

Lemma 9. *Let $G = (N, E)$ be an unweighted tree with $|N| \geq 2$, and let C be a minimum vertex cover of G. The social welfare of any optimal partition for the fractional hedonic game $\mathtt{F}(G)$ is at most $\left(\frac{2\Delta_G}{\Delta_G+1}\right) \cdot |C|$.*

Lemma 8 states that every coalition in an optimal partition is a star. In contrast, Pareto optimal partitions consist of stars and superstars.

Lemma 10. *Let $G = (N, E)$ be an unweighted tree with $|N| \geq 2$, and let $\mathcal{P}$ be a Pareto optimal partition for the fractional hedonic game $\mathtt{F}(G)$. Then every coalition in $\mathcal{P}$ is either a star or a superstar.*

Our next lemma describes neighbors of singleton coalitions in Pareto optimal partitions.

Lemma 11. *Let $G = (N, E)$ be an unweighted tree with $|N| \geq 2$. Let $\mathcal{P}$ be a Pareto optimal partition for the fractional hedonic game $\mathtt{F}(G)$. If $\mathcal{P}$ contains a singleton $P = \{i\}$ then every j such that $(i, j) \in E$ satisfies the following conditions:*

(a) j is not in a singleton,

(b) j is not the center of a superstar,

(c) j is not the leaf of a superstar.

A direct consequence of Lemma 11 is that, in a Pareto optimal partition, a player in a singleton coalition can be adjacent only to the center or a leaf of a d-star with $d \geq 1$, or to the internal nodes of a superstar.

We are now ready to present our upper bound on PPO of fractional hedonic games on trees.

Theorem 12. *Let $G = (N, E)$ be an unweighted tree with $|N| \geq 2$. Then $\mathsf{PPO}(\mathtt{F}(G)) \leq \Delta_G + 2$.*

Proof. Let C be a minimum vertex cover of G and let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a Pareto optimal partition with $m \geq 1$ coalitions. Let Q_1 be the set of indices of the d-stars in $\mathcal{P}$ with $d \geq 1$, and let Q_2 be the set of indices of the superstar coalitions in $\mathcal{P}$. Let $Q = Q_1 \cup Q_2$.

Note that the number of singleton coalitions is $m - |Q|$ and by Proposition 6 we have $|Q| \geq 1$. Fix a $k \in Q_1$ and let P_k be a d_k-star with center r_k. Since $d_k = \delta_{P_k}(r_k) \leq \Delta_G$, the value of P_k can be bounded from below as

$$V(P_k) = \frac{2d_k}{d_k + 1} \geq \frac{2}{\Delta_G + 1}\delta_{P_k}(r_k). \tag{1}$$

Now, fix a $k \in Q_2$, and let P_k be a (d_k, e_k)-superstar with center r_k. Since $d_k = \delta_{P_k}(r_k) \leq \Delta_G$ and $e_k > 1$, the value of P_k can be bounded from below as

$$V(P_k) = \frac{2d_k e_k}{d_k e_k + 1} \geq \frac{2d_k e_k}{d_k e_k + e_k} > \frac{2\delta_{P_k}(r_k)}{\Delta_G + 1}. \quad (2)$$

For every $k \in Q$, define the set C_k as follows: if P_k is a d_k-star with $d_k \geq 1$, then $C_k = P_k$, and if P_k is a (d_k, e_k)-superstar with center r_k, then $C_k = \{j \in P_k : (r_k, j) \in E\}$. By construction, C_k is a vertex cover of P_k. Let $C' = \bigcup_k C_k$. Clearly, since $|Q| \geq 1$, we have $C' \neq \emptyset$. Moreover, as C_k covers P_k, every edge inside any coalition in Q is covered by C'. By Lemma 11 a player in a singleton coalition can be adjacent only to the center or a leaf of a d_k-star with $d_k \geq 1$, or to the internal nodes (other than the center) of a superstar coalition. Thus, each player in a singleton coalition is adjacent to some player in C'. As a consequence, all the edges incident to a player in a singleton coalition are covered by C'. Thus, since G is a tree, there are at most $|Q_2|$ edges that are uncovered by C', which lie outside of any coalition: these are edges connecting a subgraph induced by a coalition in Q_2 to the rest of the tree. Hence, a cover of N can be obtained by adding to C' at most another $|Q_2|$ nodes. Thus, if C is a minimum vertex cover of G,

$$\begin{aligned}
|C| &\leq |C'| + |Q_2| \\
&= \Big(\sum_{k \in Q_1} (\delta_{P_k}(r_k) + 1) + \sum_{k \in Q_2} \delta_{P_k}(r_k)\Big) + |Q_2| \\
&= \Big(\sum_{k \in Q_1 \cup Q_2} \delta_{P_k}(r_k)\Big) + |Q|. \quad (3)
\end{aligned}$$

Combining (1) and (2) we obtain

$$\begin{aligned}
SW(\mathcal{P}) &= \sum_{k \in Q_1} V(P_k) + \sum_{k \in Q_2} V(P_k) \\
&\geq \frac{2}{\Delta_G + 1}\Big(\sum_{k \in Q_1} \delta_{P_k}(r_k) + \sum_{k \in Q_2} \delta_{P_k}(r_k)\Big) \\
&= \frac{2}{\Delta_G + 1}\Big(\sum_{k \in Q_1 \cup Q_2} \delta_{P_k}(r_k)\Big) \\
&\geq \frac{2}{\Delta_G + 1}\Big(|C| - |Q|\Big) \quad (4) \\
&\geq \frac{2}{\Delta_G + 1}\Big(|C| - SW(\mathcal{P})\Big), \quad (5)
\end{aligned}$$

where (4) follows from (3), and (5) follows from observing that each non-singleton coalition contributes at least 1 to the social welfare, and hence $SW(\mathcal{P}) \geq |Q|$. From (5) we obtain

$$SW(\mathcal{P}) \geq \Big(\frac{2}{\Delta_G + 3}\Big) \cdot |C|. \quad (6)$$

Let $\mathcal{P}^*$ be an optimal partition. Combining (6) and Lemma 9, we get

$$\frac{SW(\mathcal{P}^*)}{SW(\mathcal{P})} \leq \frac{\Big(\frac{2\Delta_G}{\Delta_G+1}\Big) \cdot |C|}{\Big(\frac{2}{\Delta_G+3}\Big) \cdot |C|} = \Delta_G + \frac{2\Delta_G}{\Delta_G + 1} \leq \Delta_G + 2.$$

□

The following proposition shows that the upper bound given by Theorem 12 is optimal up to a small additive factor.

Proposition 13. *There exists a fractional hedonic game on an unweighted tree $G = (N, E)$ for which the Price of Pareto Optimality is strictly greater than $\Delta_G - 1/3$.*

Proof sketch. Let G be a (d, d)-superstar with center r. Let $\{i_1, i_2, \dots, i_d\}$ be the neighbors of r, and for $k = 1, \dots, d$ let S_{i_k} be the set consisting of i_k and the leaves of G that are adjacent to i_k. It can be shown that the partition $\mathcal{P} = \{N\}$ is Pareto optimal; its social welfare is $SW(\mathcal{P}) = \frac{2d^2}{d^2+1} < 2$.

We now give a lower bound on the social welfare of an optimal partition. Let $R = S_{i_1} \cup \{r\}$. Let us consider the partition $\mathcal{P}' = \{R, S_{i_2}, S_{i_3}, \dots, S_{i_d}\}$. It is easy to see that the value of R is $\frac{2d}{d+1}$, and the value of every coalition S_{i_k}, with $2 \leq k \leq d$, is $\frac{2(d-1)}{d}$. Thus, the social welfare of $\mathcal{P}'$ is

$$SW(\mathcal{P}') = \frac{2d}{d+1} + \frac{2(d-1)^2}{d} > \frac{2d}{d+1} + 2(d-1).$$

Hence, the Price of Pareto Optimality is strictly greater than

$$\frac{\frac{2d}{d+1} + 2(d-1)}{2} = d - (1 - \frac{d}{d+1}) \geq \Delta_G - 1/3.$$

□

It is interesting to compare our bounds on PPO with bounds on the Price of Nash Stability obtained by Bilò et al. (2014; 2015). Bilò et al. argue that the social welfare in Nash stable outcomes of simple symmetric fractional hedonic games may be arbitrarily bad, simply because players may be stuck in the grand coalition. In contrast, PPO allows group deviations, and therefore it can be reasonably low in low-degree graphs.

5 Modified Fractional Hedonic Games

In fractional hedonic games, the value that a player i assigns to a coalition is averaged over all members of that coalition, including i itself. Arguably, it is more natural to compute the average value of all *other* members of the coalition. This approach gives rise to a new class of hedonic games, which we call *modified fractional hedonic games*; to the best of our knowledge, this class of games has not been considered in prior work. Formally, a weighted directed graph $G = (N, E, w)$ defines a *modified fractional hedonic game*, where the set of players corresponds to the set of nodes and the utility of player i from a coalition $S \ni i$, $|S| \geq 2$, is given by $v_i(S) = \frac{1}{|S|-1} \sum_{j \in S, (i,j) \in E} w_{i,j}$; as we assumed for any hedonic game, the utility of player i from the singleton coalition $\{i\}$ is 0. (Equivalently, we say that the utility of player i from a coalition $S \ni i$, $|S| \geq 1$, is given by $v_i(S) = \frac{1}{\max\{1,|S|-1\}} \sum_{j \in S, (i,j) \in E} w_{i,j}$). We denote the modified fractional hedonic game that corresponds to a graph G by $\mathtt{MF}(G)$.

Modified fractional hedonic games share many properties of fractional hedonic games; for instance, the example in Proposition 5 can be adapted to show that for weighted graphs PPO may be unbounded, observations in Proposition 6 also apply to modified fractional hedonic games, and so does the upper bound on PPO for general symmetric unweighted graphs (Theorem 7). However, for general symmetric unweighted graphs and unweighted bipartite graphs we can obtain a much stronger upper bound on the PPO.

We first derive a bound on the optimal social welfare.

Lemma 14. *Let $G = (N, E)$ be a symmetric unweighted graph with $|N| \geq 2$. Let C be a minimum vertex cover of G. The social welfare of any optimal partition for the modified fractional hedonic game $\mathtt{MF}(G)$ is at most $2|C|$.*

Proof. Let $\mathcal{P} = \{P_1^*, P_2^*, \ldots, P_m^*\}$ be an optimal partition for $\mathtt{MF}(G)$. Consider some $k \in \{1, \ldots, m\}$. If $|P_k^*| = 1$, we simply have $V(P_k^*) = 0$. Thus, suppose that $|P_k^*| \geq 2$. Let $C_k = P_k^* \cap C$ and let $I_k = P_k^* \cap (N \setminus C)$; note that $P_k^* = C_k \cup I_k$, $C = \bigcup_{1 \leq k \leq m} C_k$, and I_k is an independent set. We have

$$\begin{aligned} V(P_k^*) &= \frac{\sum_{i \in P_k^*} \delta_{P_k^*}(i)}{|C_k| + |I_k| - 1} \\ &\leq \frac{2\Big(|C_k||I_k| + \frac{1}{2}|C_k|(|C_k| - 1)\Big)}{|C_k| + |I_k| - 1} \\ &= 2|C_k| \frac{\Big(|I_k| + \frac{1}{2}(|C_k| - 1)\Big)}{|C_k| + |I_k| - 1} \leq 2|C_k|. \quad (7) \end{aligned}$$

The last innequality in (7) follows from observing that $|I_k| + \frac{1}{2}(|C_k| - 1) \leq |C_k| + |I_k| - 1$ whenever $|C_k| \geq 1$, and that (7) is equal to 0 when $|C_k| = 0$.

Finally, we bound the social welfare of $\mathcal{P}$: $SW(\mathcal{P}) = \sum_{1 \leq k \leq m} V(P_k^*) \leq \sum_{1 \leq k \leq m} 2|C_k| = 2|C|$. □

A crucial difference between fractional hedonic games and modified fractional hedonic games is the structure of Pareto optimal solutions: in modified fractional hedonic games, we are able to show that Pareto optimal solutions consist of stars and triangles only.

Lemma 15. *Let $G = (N, E)$ be a symmetric unweighted graph with $|N| \geq 2$. Let $\mathcal{P}$ be a Pareto optimal partition for the modified fractional hedonic game $\mathtt{MF}(G)$. Then every coalition in $\mathcal{P}$ is either a star or a clique.*

The next lemma describes the structure of Pareto optimal partitions in more detail.

Lemma 16. *Let $G = (N, E)$ be a symmetric unweighted graph with $|N| \geq 2$. Let $\mathcal{P}$ be a Pareto optimal partition for the modified fractional hedonic game $\mathtt{MF}(G)$. For every edge $(i, j) \in E$ with $\mathcal{P}(i) \neq \mathcal{P}(j)$, it holds that if i in $\mathcal{P}$ forms a singleton, is a leaf of a multi-degree star or a node in a triangle, then j in $\mathcal{P}$ is either the center of a multi-degree star or a node in a 1-star.*

Another important observation is that Proposition 6 extends to modified fractional hedonic games.

Proposition 17. *Let $G = (N, E)$ be a symmetric unweighted graph with $|N| \geq 2$, and let $\mathcal{P}$ be a Pareto optimal partition for $\mathtt{MF}(G)$. Then*

(a) every coalition in $\mathcal{P}$ is connected,

(b) if $E \neq \emptyset$, then $\mathcal{P}$ contains at least one non-singleton coalition.

The following upper bound on the Price of Pareto Optimality is optimal up to a small multiplicative factor.

Theorem 18. *Let $G = (N, E)$ be a symmetric unweighted graph with $|N| \geq 2$. Then $\mathsf{PPO}(\mathtt{MF}(G)) \leq 2$.*

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a Pareto optimal partition with $m \geq 1$ coalitions. From Lemma 15, $\mathcal{P}$ contains only stars and cliques. Let Q_1 be the set of the indices of the multi-degree stars in $\mathcal{P}$, let Q_2 be the set of the indices of the 1-stars in $\mathcal{P}$, and let Q_3 be the indices of cliques with more than 3 nodes in $\mathcal{P}$. Let $Q = Q_1 \cup Q_2 \cup Q_3$. Note that the number of singleton coalitions is $m - |Q|$ and by Proposition 17 we have $|Q| \geq 1$.

In the remainder of this proof, we define a vertex cover C' and compare its size with the social welfare of $\mathcal{P}$. For every $k \in Q_1$, let r_k be the center of the multi-degree star P_k. For every $k \in Q$, define the set C_k as follows: if $k \in Q_1$ then $C_k = \{r_k\}$, if $k \in Q_2$ then $C_k = P_k$, while if $k \in Q_3$ then C_k is an arbitrary subset of P_k of size $|P_k| - 1$. Note that C_k is a vertex cover of the subgraph induced by P_k. Let $C'_{Q_1} = \bigcup_{k \in Q_1} C_k$, $C'_{Q_2} = \bigcup_{k \in Q_2} C_k$, $C'_{Q_3} = \bigcup_{k \in Q_3} C_k$. Define $C' = C'_{Q_1} \bigcup C'_{Q_2} \bigcup C'_{Q_3}$. Since $|Q| \geq 1$, C' is non-empty. We will show that C' is a vertex cover for G. Since C_k is a vertex cover of the subgraph induced by P_k, every edge inside any coalition in Q is covered by C'.

Now consider an agent $i \in N$ and an edge $(i, j) \in E$ with $\mathcal{P}(i) \neq \mathcal{P}(j)$; we will argue that (i, j) is covered by C'. First observe that C' contains the centers of all multi-degree stars (C'_{Q_1}) and the nodes of all 1-stars (C'_{Q_2}). Hence, if i is the center of a multi-degree star or any node in a 1-star then $(i, j) \in E$ is covered. Otherwise, if i is a singleton, a leaf of a multi-degree star or a node in a triangle, then again every $(i, j) \in E$ is covered because j belongs to C'. In fact, from Lemma 16, j can only be either the center of a multi-degree star or a node in a 1-star. We can conclude that C' is a cover for G. Thus, let C be a minimum vertex cover of G. We get

$$\begin{aligned} SW(\mathcal{P}) &= \sum_{k \in Q_1} V(P_k) + \sum_{k \in Q_2} V(P_k) + \sum_{k \in Q_3} V(P_k) \\ &= 2|Q_1| + 2|Q_2| + \sum_{k \in Q_3} |P_k| \\ &= 2|C'_{Q_1}| + |C'_{Q_2}| + |C'_{Q_3}| + |Q_3| \\ &\geq \big(|C'_{Q_1}| + |C'_{Q_2}| + |C'_{Q_3}|\big) = |C'| \geq |C|. \end{aligned}$$

Here the second equality follows from the fact that $|C'_{Q_1}| = |Q_1|$, $|C'_{Q_2}| = 2|Q_2|$ and $|C'_{Q_3}| = \sum_{k \in Q_3}(|P_k| - 1) = \sum_{k \in Q_3} |P_k| - |Q_3|$. Combining this bound and Lemma 14, we get $\mathsf{PPO}(\mathtt{MF}(G)) \leq 2$. □

For symmetric unweighted bipartite graphs, we can use Lemma 15 to show a stronger result.

Theorem 19. *Let $G = (N, E)$ be an symmetric unweighted bipartite graph with $|N| \geq 2$. Then $\mathsf{PPO}(\mathtt{MF}(G)) \leq 1$.*

6 Conclusion

We have introduced the notion of Price of Pareto Optimality (PPO) and obtained upper and lower bounds on PPO in three classes of hedonic games; one of these classes (modified fractional hedonic games) is new, and, as shown by our results, it is substantially different from the class of fractional hedonic games. There are many open problems suggested by our work. For instance, it is not clear if the upper bound in Theorem 7 is tight; in fact, we do not have examples of fractional hedonic games on symmetric unweighted graphs whose PPO exceeds Δ_G. More broadly, we believe that PPO is a useful measure, and it would be interesting to compute or bound it for other classes of (cooperative and non-cooperative) games.

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