Strong Bounds Consistencies and Their Application to Linear Constraints

Christian Bessiere
CNRS, University of Montpellier
Montpellier, France
bessiere@lirmm.fr

Anastasia Paparrizou
CNRS, University of Montpellier
Montpellier, France
paparrizou@lirmm.fr

Kostas Stergiou
University of Western Macedonia
Kozani, Greece
kstergiou@uowm.gr

Abstract

We propose two local consistencies that extend bounds consistency (BC) by simultaneously considering combinations of constraints as opposed to single constraints. We prove that these two local consistencies are both stronger than BC, but are NP-hard to enforce even when constraints are linear. Hence, we propose two polynomial-time techniques to enforce approximations of these two consistencies on linear constraints. One is a reformulation of the constraints on which we enforce BC whereas the other is a polynomial time algorithm. Both achieve stronger pruning than BC. Our experiments show large differences in favor of our approaches.

Introduction

Generalized Arc Consistency (GAC) and Bounds Consistency (BC) are the two local consistencies that are predominantly used for propagation by finite domain constraint solvers. Many stronger local consistencies based on GAC have been proposed, both for binary and non-binary constraints, e.g. (Debruyne and Bessière 2001; Bessière, Stergiou, and Walsh 2008). However, similar consistencies that are based on BC have been comparatively overlooked.

There is also a considerable body of work dedicated to the study of propagation methods for linear constraints (Zhang and Yap 2000; Harvey and Stuckey 2003; Apt and Zoeteweij 2007; Bordeaux et al. 2011). However, these works focus almost exclusively on BC propagation. One exception is (Trick 2003) where the idea of considering many linear constraints simultaneously to strengthen propagation was investigated in the context of knapsack constraints. Another exception is Singleton Bounds Consistency (SBC), which has been mainly studied on numerical CSPs (Lhomme 1993).

In this paper we introduce rBC2 and rBCall, two new strong local consistencies for non-binary constraints that extend BC. rBC2 is related to the domain filtering consistency relational Path Inverse Consistency (rPIC) and rBCall is related to maxRPWC (Bessière, Stergiou, and Walsh 2008). The application of rBC2 and rBCall results in the shrinking of domain bounds just like BC, but unlike BC it simultaneously considers combinations of constraints as opposed to single constraints. Naturally, this results in stronger filtering.

We present theoretical results that investigate the pruning power of the new local consistencies. For example, we show that BC is strictly weaker than rBCall and rBC2, which are incomparable to SBC, just as GAC is strictly weaker than maxRPWC and rPIC, which are incomparable to SAC. These hold even when constraints are linear inequalities where we know that GAC is equivalent to BC (Zhang and Yap 2000). However, the stronger pruning of rBC2 and rBCall comes at a cost: we prove that rBC2 and rBCall are NP-hard to enforce even when constraints are linear.

Thereafter, we focus exclusively on linear constraints. This is the most common class of constraints on which BC propagators are applied. We propose two polynomial-time techniques to enforce approximations of rBC2 and rBCall that are still stronger than BC. The first technique is a reformulation of the constraints which introduces extra variables to capture intersections (i.e., shared variables) between constraints. Then we simply enforce BC on the reformulation. This is a flexible technique that depending on the extra variables introduced achieves approximations of varying pruning strength. The drawback, that becomes evident on dense problems, is its memory requirements. The second technique is a polynomial time algorithm that operates on the original model of the problem. This algorithm exploits intersections between pairs of constraints to approximate rBC2.

Both techniques can be easily crafted in CP solvers to enhance their pruning power on linear constraints.

Finally, we make an experimental study on various types of problems. Results demonstrate that both of our techniques can outperform BC, being exponentially better in many cases. The reformulation technique is faster on sparse problems, but the algorithm is the preferred technique on dense problems with many constraint intersections.

Background

A (finite domain) Constraint Satisfaction Problem (CSP) is defined as a triplet \((X, D, C)\) where: \(X = \{x_1, \ldots, x_n\}\) is a set of \(n\) variables, \(D = \{D(x_1), \ldots, D(x_n)\}\) is a set of domains, one for each variable, with maximum cardinality \(d\), and \(C = \{c_1, \ldots, c_e\}\) is a set of \(e\) constraints. Each constraint \(c\) is a pair \((\text{var}(c), \text{rel}(c))\), where

[^1]: Funded by the EU project ICON (FP7-284715).

Copyright © 2015, Association for the Advancement of Artificial Intelligence (www.aaai.org). All rights reserved.
\[ \text{var}(c) = \{x_1, \ldots, x_m\} \] is an ordered subset of \( \mathcal{X} \) called the \textit{scope} of \( c \), and \( \text{rel}(c) \) is a subset of the Cartesian product \( D(x_1) \times \cdots \times D(x_m) \) that specifies the allowed combinations of values for the variables in \( \text{var}(c) \). Each tuple \( \tau \in \text{rel}(c) \) is an ordered list of values \( \{v_1, \ldots, v_m\} \) such that \( v_j \in D(x_j), j = 1, \ldots, m \). Given a tuple \( \tau \in \text{rel}(c) \), we denote by \( \tau[x] \) the projection of \( \tau \) on a variable \( x \in \text{var}(c) \).

An important class of non-binary constraints is the class of linear arithmetic constraints. Such a constraint \( c \) with \( \text{var}(c) = \{x_1, \ldots, x_m\} \) is of the form \( a_1 x_1 + a_2 x_2 + \cdots + a_m x_m \geq b \), where \( a_i, b \in \mathbb{Z} \) and \( \emptyset \in \{<, >, \leq, \geq\} \).

Here we are concerned with domains composed of integers. For any variable \( x_i \), \( \text{min}_D(x_i) \) will denote the smallest value in \( D(x_i) \) and \( \text{max}_D(x_i) \) the greatest one. These two values are called the \textit{bounds} of \( D(x_i) \). Given two constraints \( c_i \) and \( c_j \), if \( \text{var}(c_i) \cap \text{var}(c_j) \neq \emptyset \) then we say that the constraints \textit{intersect or overlap}.

The most commonly used local consistency is \textit{Generalized Arc Consistency} (GAC). A tuple \( \tau = (v_1, \ldots, v_m) \) is a \textit{support} on a constraint \( c \), with \( \text{var}(c) = \{x_1, \ldots, x_m\} \), iff \( \tau \in \text{rel}(c) \) and for each \( x_i, 1 \leq i \leq m, v_i \in D(x_i) \). A value \( v_i \in D(x_i) \) is GAC iff for every constraint \( c \), such that \( x_i \in \text{vars}(c) \), there exists a support \( \tau[x_i] = v_i \). A variable is GAC iff all values in its domain are GAC.

Bounds consistency (BC) has been proposed in order to overcome the prohibitive cost of GAC on some non-binary constraints. A tuple \( \tau = (v_1, \ldots, v_m) \) is a \textit{bound-support} on a constraint \( c \), with \( \text{var}(c) = \{x_1, \ldots, x_m\} \), iff \( \tau \) satisfies \( c \) and for each \( x_i \in \text{var}(c) \), \( \min_D(x_i) \leq v_i \leq \max_D(x_i) \). A CSP is BC iff \( \forall c \in \mathcal{C} \) and \( \forall x_i \in \mathcal{X} \), there exist bound-supports \( \tau_{\text{min}} \) and \( \tau_{\text{max}} \) on \( c \) s.t. \( \tau_{\text{min}}[x_i] = \min_D(x_i) \) and \( \tau_{\text{max}}[x_i] = \max_D(x_i) \) (i.e., both bounds belong to a bound-support on \( c \)).

\section*{Strong local consistencies}

Several local consistencies stronger than GAC have also been proposed. Those that only remove values from domains are particularly interesting because they do not alter the constraint (hyper)graph or the relations. Among such consistencies, we find \textsc{rPIC} and maxRPWC, proposed in (Bessiere, Stergiou, and Walsh 2008), and that have inspired our work. A CSP is \textit{relational Path Inverse Consistent} (rPIC) iff for each pair of overlapping constraints, each value of each variable in the first constraint belongs to a support that can be extended into a support of the second constraint. A CSP is \textit{max Restricted Pairwise Consistent} (maxRPWC) iff for each constraint, each value of each variable in this constraint belongs to a support that can be extended into a support of any second constraint overlapping the first.

\section*{Singleton Bounds Consistency (SBC)} (Lecoutre and Prosser 2006) is a strong local consistency based on BC. It is an adaptation of 3B Consistency (Lhomme 1993) from numerical to finite domain CSPs. A CSP is SBC iff \( \forall x_i \in \mathcal{X} \) and \( \forall v_i \in \{\min_D(x_i), \max_D(x_i)\} \) the problem derived from assigning \( v_i \) to \( x_i \) (i.e., \( N[x_i] = \{v_i\} \)) is BC.

Following (Debruyne and Bessiere 2001) we will call a local consistency \( A \) stronger than \( B \) (\( A \geq B \)) iff in any problem in which \( A \) holds then \( B \) holds, and strictly stronger (\( A \succ B \)) iff \( A \) is stronger and there is at least one problem in which \( B \) holds but \( A \) does not. Accordingly, \( A \) is incomparable to \( B \) (\( A \not\succ B \)) iff none is stronger than the other. The relation \( \succ \) is transitive.

\section*{Strong Consistencies on Bounds}

We now define two new strong bounds consistencies, which are relaxed versions of rPIC and maxRPWC.

\begin{definition}[rBC2] A CSP is relational \textit{BC on pairs} (rBC2) iff for each \( x_i \in \mathcal{X} \) and each value \( v_i \in \{\min_D(x_i), \max_D(x_i)\} \), for each \( c_j \in \mathcal{C} \), where \( x_i \in \text{var}(c_j) \), and for each \( c_j \in \mathcal{C} \) s.t. \( \text{var}(c_j) \cap \text{var}(c_i) \neq \emptyset \), there exist bound-supports \( \tau \) on \( c_j \) and \( \tau' \) on \( c_i \) s.t. \( \tau[x_i] = v_i \) and \( \tau'[\text{var}(c_j) \cap \text{var}(c_i)] = \tau'[\text{var}(c_j) \cap \text{var}(c_i)] \).
\end{definition}

\begin{definition}[rBCall] A CSP is relational \textit{BC on all} (rBCall) iff for each \( x_i \in \mathcal{X} \) and for each value \( v_i \in \{\min_D(x_i), \max_D(x_i)\} \), for each \( c_j \in \mathcal{C} \), where \( x_i \in \text{var}(c_j) \), there exists a bound-support \( \tau \) on \( c_j \) s.t. \( \tau[x_i] = v_i \), and for all \( c_j \in \mathcal{C} \) (\( c_i \not\sim c_j \)), s.t. \( \text{var}(c_j) \cap \text{var}(c_i) \neq \emptyset \), there exists a bound-support \( \tau' \) on \( c_j \) s.t. \( \tau'[\text{var}(c_j) \cap \text{var}(c_i)] = \tau'[\text{var}(c_j) \cap \text{var}(c_i)] \).
\end{definition}

\section*{Comparison of Propagation Strength}

We prove the relationships between the aforementioned local consistencies. Surprisingly, all consistencies remain distinct even when we consider linear constraints only. Hence, Propositions 1 to 5 below hold even in the case of linear constraints.

\begin{proposition}[rBCall] \( rBCall \succ rBC2 \succ BC \).
\end{proposition}

\begin{proof}
By definition, \( rBCall \succeq rBC2 \succeq BC \). Strictness is shown by the following examples.

Consider a problem where \( D(x_1) = \{0, \ldots, 10\} \), \( D(x_2) = \{0, \ldots, 2\} \) and for \( i > 2 \) \( D(x_i) = \{0, 1\} \) and the constraints:
\[
\begin{align*}
c_1: x_1 + x_2 + 8x_3 - 8x_4 + 9x_5 - 9x_6 &= 10 \\
c_2: x_1 + x_2 + 8x_5 - 8x_5 + 10x_6 - 10x_6 &= 10 \\
c_3: x_1 + x_2 + 9x_7 - 9x_7 + 10x_8 - 10x_8 &= 10
\end{align*}
\]
All bounds of all variables are rBC2 whereas value 0 of \( x_1 \) is not rBCall because it does not belong to any bound-support that satisfies all three constraints at the same time. Hence, \( rBCall \succ rBC2 \).

Now consider a problem with variables \( x_1 \) and \( x_2 \), domains \( D(x_1) = \{1, 2\} \), \( D(x_2) = \{0, 1\} \), and constraints \( c_1 : x_1 + x_2 \leq 2 \) and \( c_2 : x_1 - x_2 \geq 1 \). The problem is BC. However, value 1 of \( x_2 \) is not rBC2 because its only bound-support (1,1) on \( c_1 \) is not a bound-support on \( c_2 \). Hence, \( rBC2 \succ BC \).
\end{proof}

\begin{proposition}
\( SBC \not\succ rBC2 \) and \( SBC \not\succeq rBCall \).
\end{proposition}

\begin{proof}
We need an example that is SBC but not rBC2 and an example that is rBCall but not SBC. Consider a problem with four variables, all having domain \( \{0, \ldots, 10\} \), and two constraints: \( c_1 : x_1 + x_2 + x_3 + x_4 \leq 20 \) and \( c_2 : x_1 + x_2 + x_3 + x_4 > 20 \). rBC2 will immediately detect inconsistency whereas SBC will not remove any value.

Now consider a problem with five variables, all having domain \( \{0, 5\} \), and the constraints: \( x_1 + x_2 + x_3 \geq 10 \),
$x_1 \leq x_4$, $x_3 \leq x_5$ and $x_4 + x_5 \leq 5$. All bounds of all variables are rBCall but value 0 of $x_2$ is not SBC. Hence, SBC is incomparable to both rBC2 and rBCall.

We now compare our new consistencies to the related domain filtering consistencies.

**Proposition 3** maxRPWC $\succ$ rBCall and rPIC $\succ$ rBC2.

**Proof.** By definition maxRPWC $\geq$ rBCall and rPIC $\geq$ rBC2. Strictness is shown by the following example. Consider a problem with variables $x_1, \ldots, x_4$, domains $D(x_1) = \{0, 1, 2\}$, $D(x_2) = \{0, 2\}$, $D(x_3) = \{-2\}$, and constraints $c_1 : x_1 \leq x_2 - x_3$ and $c_2 : x_1 - x_2 \geq x_4$. The problem is rBCall and rBC2. But the only support for value 2 of $x_1$ on $c_1$ (tuple (2,3,0)) cannot be extended to $c_2$. Thus, if we apply maxRPWC or rPIC, value 2 of $x_1$ will be deleted.

**Proposition 4** rBCall $\not\succ$ GAC and rBC2 $\not\succ$ GAC.

**Proof.** Consider a problem with variables $x_1, x_2$, domains $D(x_1) = \{0, 1, 2\}$, $D(x_2) = \{0, 2\}$, and constraint $x_1 = x_2$. This problem is already rBCall and rBC2 whereas GAC will remove value 1 from $x_1$. Consider the second example in the proof of Proposition 1. It is already GAC whereas it is neither rBCall nor rBC2.

**Proposition 5** rBCall $\not\succ$ rPIC.

**Proof.** We need an example that is rPIC but not rBCall and another example that is rBCall but is not rPIC. For the former case, recall the first example in Proposition 1. If we replace the domain of $x_1$ by $\{0, 8, 9, 10\}$, then it is rPIC whereas rBCall still prunes value 0 of $x_1$. For the later, in the proof of Proposition 4 we have an example where rPIC is stronger than rBCall.

![Figure 1: Summary of relationships.](image)

### Complexity of Strong Bounds Consistencies

We now analyse the complexity of enforcing rBC2 and rBCall. As shown in (Bessiere 2006), on general constraints, even the simple BC on binary constraints is NP-hard. So we concentrate on the case of linear constraints. In this case, enforcing BC on inequalities ($\{\leq, \geq\}$) is known to be linear (Zhang and Yap 2000). But as soon as we move to stronger bounds consistencies or allow equations, enforcing consistency is hard.

**Proposition 6** ([Choi et al. 2006]) Enforcing BC on linear equations is NP-hard.

**Proposition 7** Enforcing rBC2 (or rBCall) on linear inequalities is NP-hard even when the inequalities have the same coefficients on the overlapping variables.

**Proof.** We reduce SUBSETSUM to the problem of enforcing rBC2. Consider the SUBSETSUM instance $S = \{a_1, \ldots, a_n\}$ and integer $k$. Enforcing rBC2 on the two inequalities $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq k$ and $a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq k$, with $D(x_i) = \{0, 1\}$ for all $i$ is equivalent to GAC because domains are binary. Hence, rBC2 will finish without detecting a wipe out iff there is a solution to the constraint. Now, if $I$ is a solution to the constraint, the set $\{a_i \mid I[x_i] = 1, i = 1 \ldots n\}$ is a solution to the SUBSETSUM problem.

### Enforcing Strong Bounds Consistencies

We have shown in the previous section that there is no way to enforce rBC2 and rBCall in polynomial time, unless P=NP. In this section we propose two polynomial-time techniques that enforce approximations of rBC2 and rBCall. The first one is a reformulation with extra variables whereas the second approach is a polynomial time algorithm that exploits intersections between pairs of constraints.

### Constraint Reformulation

The first approach is a modelling reformulation of the original problem through the introduction of extra variables to capture common subexpressions among constraints. BC is enforced on the reformulation. Such a technique has been introduced in (Araya, Neveu, and Trombettoni 2008) for numerical CSPs. It has also been independently investigated in Savile Row for finite domain CSPs (Nightingale et al. 2014).

Allowing extra variables opens the door to many reformulation choices. We present two versions, one that mimics rBC2 whereas the other mimics rBCall. Our first reformulation, called rBC2-Y, approximates rBC2 by introducing an extra variable $Y$ each time a pair of constraints $c_i$ and $c_j$ share more than one variable with coefficients that are proportional. For instance, consider the two constraints $a_1x_1 + a_2x_2 + a_3x_3 = b_1$ $ka_1x_1 + ka_2x_2 + a_4x_4 = b_2$

We encapsulate the variables that are common (i.e., $x_1, x_2$) in the extra variable $Y$ and we replace the common subexpression with the $Y$ variable in the two constraints. The problem is rewritten:

$Y = a_1x_1 + a_2x_2$ $Y + a_3x_3 = b_1$ $kY + a_4x_4 = b_2$

Our second reformulation, called rBCall-Y, approximates rBCall by introducing an extra variable $Y$ each time a subset of variables belongs to several (possibly more than two) constraints with proportional coefficients. Consider the constraints $x_1 + x_2 + x_3 + x_4 \leq b_1$, $x_1 + x_2 + x_3 + x_5 \leq b_2$, $x_1 + x_2 + x_4 + x_5 \geq b_3$. rBC2-Y would reformulate with three variables $Y_{12}, Y_{13}, Y_{23}$ representing the pairwise intersections among the three constraints. On top of this, rBCall-Y adds another extra variable $Y_{123} = x_1 + x_2$ to catch the subexpression common to all three constraints and rewrites all three constraints with this extra variable $Y_{123}$.

The reformulation technique used in Savile Row is more elaborated than ours, handling an abstract syntax tree to detect subexpressions. Nevertheless, it is incomparable to ours.

3719
In Savile Raw, once a subexpression is reformulated, all conflicting ones become unusable. In rBC2-Y we reformulate all pairwise common subexpressions.

We characterize the amount of pruning that our two reformulations can achieve. As expected, the reformulations achieve stronger pruning than BC. More interestingly, the second reformulation sometimes prunes more than rBC2.

**Proposition 8** \( rBC2 \gg BC(rBC2-Y) \gg BC, rBCall \gg BC(rBCCall-Y) \gg BC(rBC2-Y), \) and \( BC(rBCCall-Y) \not\gg rBC2. \)

**Proof.** By construction \( rBC2 \gg BC(rBC2-Y) \gg BC \), and \( rBCall \gg BC(rBCCall-Y) \gg BC(rBC2-Y). \)

Consider the first example in Proposition 2, which is BC. rBC2-Y introduces an extra variable \( Y = x_1 + x_2 + x_3 + x_4 \) and the constraints \( Y \leq 20 \) and \( Y \geq 21 \). BC immediately detects inconsistency on this reformulation. Hence, \( BC(rBC2-Y) \gg BC \). Now consider the second example in Proposition 1, which is BC. rBC2-Y and rBCall-Y do not change anything to the original formulation and are thus BC. However, we saw that rBC2, and therefore also rBCall, prunes value 1 from \( x_3 \). Thus, \( rBC2 \gg BC(rBC2-Y) \) and \( rBCall \gg BC(rBCCall-Y) \). Consider the first example in Proposition 1, which is already rBC2. rBCall-Y sets \( rBC2 \gg BC \) and \( rBCall \gg BC(rBCCall-Y) \). The previous example, together with the second example in Proposition 1 where rBC2 prunes value 1 from \( x_3 \) whereas rBCall-Y does nothing, prove that \( BC(rBCCall-Y) \not\gg rBC2. \)

The drawback of the reformulation approach is the extra space required for the new variables and constraints. This is prohibitive in problems with many constraints and many intersections between them. For this reason, we now present an alternative approach that is free from such requirements.

**An algorithm approximating rBC2**

As opposed to the reformulation approach, the algorithm we will now present does not explicitly represent the extra variables and does not introduce extra constraints to the problem. Instead, each time it revises a pair of intersecting constraints, it computes the interval of values for the intersecting variables "on the fly".

Following (Zhang and Yap 2000), we associate each variable \( x \) with the smallest interval \([lb(x), ub(x)]\) containing \( D(x_i)\).

Given \([x] = [lb(x), ub(x)]\) and \([y] = [lb(y), ub(y)]\), we define the following interval operations:

\[
[x] + [y] = [lb(x) + lb(y), ub(x) + ub(y)],
\]

\[
[x] - a = [lb(x) - a, ub(x) - a],
\]

\[
a[x] = \begin{cases} [a \cdot lb(x), a \cdot ub(x)], & a > 0 \\
[a \cdot ub(x), a \cdot lb(x)], & a < 0 
\end{cases}
\]

Given a constraint \( c \) defined by \( a_1 x_1 + \cdots + a_n x_n \leq b \), with \( \diamond \in \{\leq, =, \geq\} \), given intervals on all the variables in \( \text{var}(c) \), we define \( \Pi_x(c) \), the interval projection of \( c \) on \( x \):

\[
\Pi_x(c) = \frac{-1}{a_i}[a_1 x_1 + \cdots + a_i - 1 x_i - 1 + a_{i+1} x_{i+1} + \cdots + a_n x_n - b]
\]

We denote by \( \min(x_i, c) \) and \( \max(x_i, c) \) the integer values \([lb(\Pi_x(c))]\) and \([ub(\Pi_x(c))]\) respectively.

We now define the function \( \text{Proj}(x_i, c) \) that will be used in the algorithm as:

\[
\text{Proj}(x_i, c) = \begin{cases} [\min(x_i, c), \max(x_i, c)], & i \text{ if } \diamond = \leq \\
[\max(x_i, c), +\infty], & i \text{ if } \diamond = > \\
[\min(x_i, c), +\infty], & i \text{ if } \diamond = = 
\end{cases}
\]

where \( \diamond \) is the same as \( \diamond \) if \( a_i \) is positive or \( \diamond = = \), and \( \diamond \) swaps at \( \geq \) to \( > \) at \( \leq \) in \( \diamond \), otherwise.

We now extend the definition of the \( \text{Proj} \) function to subsets of variables. Given a subset \( S = \{x_i, \ldots, x_j\} \) of \( \text{var}(c) \) and \( Y = a_i x_i + \cdots + a_j x_j \) the part of \( c \) that involves variables in \( S \), we can define the subset version \( \Pi_Y(c) \) of the projection of \( c \) on \( Y \) as:

\[
[\Pi_Y(c)] = [a_1 x_1 + \cdots + a_{i-1} x_{i-1} + a_{i+1} x_{i+1} + \cdots + a_n x_n - b]
\]

The projection function \( \text{Proj}(Y, c) \) can be defined in a way similar to \( \text{Proj}(x_i, c) \).

**The algorithm**

We now present rBC2-A (from rBC2-Approximation), a filtering algorithm for linear constraints extending the BC algorithm in (Zhang and Yap 2000).

rBC2-A uses a propagation queue \( Q \) containing the variables whose domain bounds have been modified since their last propagation. If rBC2-A is used for preprocessing, \( Q \) is initialized with all variables. During search, only the variable instantiated at a given node is put in \( Q \) (lines 1-2). Once a variable \( x_i \) is extracted from \( Q \), all constraints \( c_j \) that include \( x_i \) are processed. This involves two phases. First, each variable \( x_i \) in \( \text{var}(c_j) \) other than \( x_i \), is revised. The revision of \( x_i \) is done in two steps. The first one (line 7) performs the basic BC operation, that is, it narrows the interval \([x_i]\) by intersecting it with \( \text{Proj}(x_i, c_j) \). In the second step, rBC2-A tries to further narrow \([x_i]\) by exploiting the intersection of \( c_j \) with any other constraint \( c_i \) by calling procedure \( \text{interCheck} \) with \( x_i, c_j \), and \( c_i \) as arguments (line 9).

Procedure \( \text{interCheck} \) takes a variable \( x \), a constraint \( c \), where \( x \) is involved, and another constraint \( c' \) that intersects with \( c \). It seeks the maximal subset \( S \) of \( \text{var}(c) \cap \text{var}(c') \) such that \( S \) includes more than one variable, it does not include \( x \), and the coefficients that the variables in this subset have in \( c \) and \( c' \) are proportional (line 18). It creates a temporary variable \( Y \) that replaces variables \( S \) in \( c \) and \( c' \) in the
Algorithm 1: rBC2-A

1 if Preprocessing then $Q \leftarrow V$;
2 else $Q \leftarrow \{\text{currently assigned variable}\}$;
3 while $Q \neq \emptyset$ do
4     select and remove $x_i$ from $Q$;
5     foreach $c_j \in C$, s.t. $x_i \in \text{var}(c_j)$ do
6         foreach $x_k \in \text{var}(c_j)$ s.t. $x_k \neq x_i$ do
7             $[x_k] \leftarrow [x_k] \cap \text{Proj}(x_k, c_j)$;
8         endforeach
9         foreach $c_l \in C$, s.t. $\text{var}(c_l) \cap \text{var}(c_j) > 1$ do
10            interCheck($x_k, c_j, c_l$);
11        endforeach
12         if $D(x_k) = \emptyset$ then return FAILURE;
13         if $D(x_k)$ has been filtered then $Q \leftarrow Q \cup \{x_k\}$;
14     endforeach
15     foreach $x_m \in \text{var}(c_j)$ s.t. $x_m \neq x_i$ do
16         interCheck($x_m, c_j, c_l$);
17         if $D(x_m) = \emptyset$ then return FAILURE;
18         if $D(x_m)$ has been filtered then $Q \leftarrow Q \cup \{x_m\}$;
19     endforeach
20 return SUCCESS;

procedure interCheck($x, c, c'$);
18 $S \leftarrow$ maximal subset of $\text{var}(c) \cap \text{var}(c')$, s.t. $|S| > 1$
19 AND $x \notin S$ AND the coefficients for $S$ in $c$ and $c'$ are
20 proportional;
21 if $S \neq \emptyset$ then
22     $[x] \leftarrow \sum_{x_i \in S} a_i[x_i]$;
23     $[Y] \leftarrow [Y] \cap \text{Proj}(Y, c');$
24     $[x] \leftarrow [x] \cap \text{Proj}(x, c'|Y);$
for eq-9) corresponds to the cross-over point (center of the phase transition), where the instances are the most difficult to solve. We solved 30 instances per class, with a cutoff limit of 1800 seconds. We compare BC, SBC, rBC2-wA, rBC2-A, rBC2-Y and rBCall-Y23. The latter reformulation uses extra variables to capture common subexpressions between pairs and between triplets of constraints. The pruning achieved on it is in between rBC2-Y and rBCall-Y.

Table 1 reports the results in cpu time and in number of nodes. On problems with inequalities, our proposed techniques are the most efficient in time both on ineq-6 and on ineq-9, sometimes being more than one order of magnitude faster than BC. Comparing our techniques, the reformulation rBC2-Y is the most efficient. The two algorithms display similar performance. It is notable that as the arity increases to arity 9, resulting in more intersections, the stronger reformulation and the stronger algorithm become worse than their respective weak versions. This is due to the cost of the extra variables and constraints that rBCall-Y23 builds and the extra calls to the interCheck function that slows down the performance of rBC2-A. Regarding SBC, its performance deteriorates significantly on problems of arity 9.

On problems with equations, the performance of the various techniques is similar to that on inequalities. Our techniques are again one order of magnitude faster than BC. SBC is not competitive at all. Among our techniques, rBC2-Y is the winning one, rBCall-Y23 is a close second, and the algorithms follow, being around two times slower.

When comparing the number of nodes, we observe that BC is the one that explores the largest search tree, up to two orders of magnitude more nodes than our algorithms. SBC is incomparable to our algorithms, sometimes exploring more nodes, sometimes less. This is consistent with our theoretical comparison between rBC2 and SBC. Concerning the reformulation techniques, they use extra variables, which makes a comparison of the search trees irrelevant. We thus did not report their number of nodes.

We also explored problems where our reformulation techniques face a space limitation. rBC2-Y (resp. rBCall-Y23) cannot construct the reformulation when we impose over 220 (resp. 120) and 145 (resp. 60) constraints on problems with arity 6 and 9 respectively. In contrast, the algorithms had no problem dealing with such problems and significantly outperformed BC.

Finally, we compared rBC2-wA to BC and GAC under a similar generation model as here, but this time we randomly created holes in domains. Results, which are not detailed for space reasons, showed that rBC2-wA was by far more efficient than BC and especially GAC, which reached the cutoff limit very often.

Web Services

In our second experiment, we compare the different techniques on large real instances from Web Services. Ruiz Cortés and Martín-Díaz used CP for solving matchmaking problems in Web Services. Their problems have \( n = 100 \) to 1500 variables, domains size from 0 to 255, and constraints of arity \( k = 3 \) to 20. Results showed that CP solvers were unable to solve some instances when the size increases (Ruiz Cortés et al. 2005; Kritikos and Plexousakis 2009). We reproduced their experiments.

Figure 2 displays the performance of BC, SBC, rBC2-A, rBC2-wA, and rBC2-Y, as \( n \) and \( k \) increase. The left (resp. right) plot presents the results by increasing arity (resp. increasing number of variables). We omit rBCall-Y because there is no gain compared to rBC2-Y, whereas memory is exhausted faster.

We observe that, except on small arity constraints, BC is always orders of magnitude slower than rBC2-A, rBC2-wA, and rBC2-Y. SBC is consistently the worst. Comparing our techniques, we see that when arity increases, the performance of rBC2-A deteriorates compared to rBC2-wA. This is because the satisfiable instances appear to be very easy, so rBC2-A suffers from many useless calls to function interCheck. On small problems rBC2-Y and rBC2-wA are the two winners, but when the size increases (arity or number of variables) rBC2-Y exhausts memory.
Computing Lower Bounds in Planning

As a last experiment, we solve CSPs whose solutions provide lower bounds to the number of moves in a 15-puzzle. Using lower bounds as heuristics is often exploited in planning problems (Korf 1985). The example of 15-puzzle was presented by Malte Helmert in his ECAI 2014 invited talk. We define the CSP exactly as Helmert did. Variables $U_i, D_i, L_i, R_i$ represent the minimal number of up, down, left and right moves the tile $i$ must do to reach its final location. Constraints express minimal number of moves for subsets of variables. These constraints are derived from distances to final location or from conflicting positions between tiles. A global constraint forces the sum of all the 60 variables to be less than a given upper bound $M$. As it is an optimization problem, the CSP is considered as solved once we have found $M$ such that $M − 1$ leads to inconsistency and $M$ to a solution.

Table 2 gives average CPU times for 20 instances. As in the two previous experiments, we observe that BC is significantly outperformed by our techniques (up to factor 8). Among our techniques, rBC2-wA is the best, almost twice faster than rBC2-A. As opposed to the previous experiments, the reformulations rBC2-Y and rBCall-Y23 are both slower than the algorithms rBC2-wA and rBC2-A (factor from 1.6 to 3.3). As usual, SBC is by far the worst technique (more than one order of magnitude slower than rBC2-wA and rBC2-A).

Table 2: Lower bounds for the 15-puzzle: CPU time in sec.

<table>
<thead>
<tr>
<th></th>
<th>BC</th>
<th>SBC</th>
<th>rBC2-wA</th>
<th>rBC2-A</th>
<th>rBC2-Y</th>
<th>rBCall-Y23</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>39.5</td>
<td>122.5</td>
<td>49.7</td>
<td>7.8</td>
<td>12.4</td>
<td>16.4</td>
</tr>
</tbody>
</table>

Discussion

These experiments show the efficiency of our techniques to solve CSPs with equations or inequalities that share common variables. Our techniques significantly outperform existing alternatives such as BC or SBC. These results also show the usefulness of the reformulation technique on sparse problems, and at the same time the practical need for the proposed algorithm that overcomes the space requirements of reformulations when the size of the problem increases.

Conclusion

We have proposed rBC2 and rBCall, two new strong local consistencies that extend BC by taking into account combinations of constraints. We gave a theoretical comparison of the propagation strength of the new consistencies with respect to existing ones. We have shown that these consistencies are intractable to enforce, even on linear constraints. Hence, we proposed two techniques to approximate rBC2 and rBCall in the case of linear constraints. The first one is a reformulation with extra variables. The second is a polynomial filtering algorithm. Both our two techniques achieve stronger pruning than BC. Experimental results demonstrated the potential of strong consistencies that reason on domain bounds.

Acknowledgments

We thank Malte Helmert for the useful input he provided on heuristics for solving the 15-puzzle.

References


