

The Pricing War Continues: On Competitive Multi-Item Pricing

Omer Lev

Hebrew University of Jerusalem
Jerusalem, Israel
omerl@cs.huji.ac.il

Joel Oren and Craig Boutilier

University of Toronto
Toronto, Canada
oren@cs.toronto.edu

Jeffrey S. Rosenschein

Hebrew University of Jerusalem
Jerusalem, Israel
jeff@cs.huji.ac.il

Abstract

We study a game with *strategic* vendors (the agents) who own multiple items and a single buyer with a submodular valuation function. The goal of the vendors is to maximize their revenue via pricing of the items, given that the buyer will buy the set of items that maximizes his net payoff.

We show this game may not always have a pure Nash equilibrium, in contrast to previous results for the special case where each vendor owns a single item. We do so by relating our game to an intermediate, discrete game in which the vendors only choose the available items, and their prices are set exogenously afterwards.

We further make use of the intermediate game to provide tight bounds on the price of anarchy for the subset games that have pure Nash equilibria; we find that the optimal PoA reached in the previous special cases does not hold, but only a logarithmic one. Finally, we show that for a special case of submodular functions, efficient pure Nash equilibria always exist.

Introduction

Consider a scenario in the world of e-commerce, where a single consumer is seeking to buy a set of products through an online website with multiple vendors, such as Amazon or eBay. Given the items available for sale and their prices, the buyer will purchase some subset of them, according to his valuation of the items and their prices.

Naturally, the vendors (our agents) strive to maximize their profits.¹ A vendor can both competitively tailor the set of items it offers and adjust the prices of these items to react to their competitors (pricing an item sufficiently high can be regarded as not offering it). Indeed, automatic mechanisms for rapid online price optimization exist in many markets and industries (Angwin and Mattioli 2012). This practice, sometimes called *competitive price intelligence* (Skorupa 2014), is a growing phenomenon within online retail. The specific question it addresses is how companies should price products in this competitive environment.

Furthermore, as argued by Babaioff et al. 2014, such a setting introduces subtle algorithmic questions, since changing

the prices of the products may affect the resulting revenues in a complex fashion, which may induce responses by one's competitors. Therefore, studying the convergence properties of such pricing dynamics is of interest.

In this paper, we take a game-theoretic approach to price competition among multiple sellers, each with a set of items to sell. As in Babaioff et al. 2014, we study a setting with a single buyer with a (combinatorial) valuation function, taken to be a monotone and submodular set function over the set of items, which is fully known to the vendors. However, unlike that earlier work, we examine the more general case where each of the k vendors controls a disjoint set of items A_i , rather than a single item. Given the prices of all of the items, the buyer will buy the set with the highest net-payoff (valuation minus the total price). Our model induces a game in which the vendors' strategy is a pricing of their items.

Contributions We begin by discussing a related two-phase game, that serves as a way-station in our study of the main game. In this intermediate game, vendors can only modify the sets of items being offered, whereas their prices for these items are subsequently set by a specific pricing mechanism. We show that this game, which results only from this modification of game dynamics (without changing its parameters), has key properties that its equilibria share with those of the original game. This allows us to reduce the pricing decision to a decision over what items to sell, thereby significantly simplifying the problem.

We next study basic game theoretic properties of our game. We first show that there are games which admit no pure Nash equilibrium. To do so, we show that our two-phase game admits no pure Nash equilibria, which then implies the nonexistence of a pure Nash equilibrium in the original game, using Theorem 4 (see Proposition 6). This result suggests the following question: suppose we restrict attention to instances of the game that have some pure Nash equilibria—can we then say anything about their value? To accomplish this, we analyze the price of anarchy (PoA) of this subset of games, where the objective function in question is the social welfare value, taken to be simply the buyer's valuation of the set of items that he purchased. We provide a tight bound of $\Theta(\log m)$, where m is the maximal number of items controlled by any of the vendors.

Finally, as an additional way of dealing with the conse-

Copyright © 2015, Association for the Advancement of Artificial Intelligence (www.aaai.org). All rights reserved.

¹For ease of exposition we assume production costs are zero, hence profits can be equated with revenues, or the sum of the prices of their sold items.

quences of Proposition 6, we investigate a special class of valuation functions, which we call *category-substitutable*, that, informally, partition products into “equivalence classes” or categories, such that only a single item will be chosen within a specific category, while different categories do not influence one another. We show not only that *efficient* pure Nash equilibria always exist given such buyer valuations, but provide a precise characterization of this equilibria.

Previous Work

Multi-item pricing has been a significant topic of research for many years (Hart and Nisan 2012; Hart and Reny 2012), including analyses of the price of anarchy (see Christodoulou et al. 2008 and follow-up papers). The work of Babaioff et al. 2014 is the most directly related to the model developed here; the game that they study is a special case of our game, in which each vendor sells only a single item. They show that for a buyer with a general valuation function, pure Nash equilibria may not exist, and they prove several properties of the equilibria for games where one does. Furthermore, for *submodular* valuation functions (the focus of our paper), they show not only that pure Nash equilibria always exist, but also that they are unique (they give a closed-form characterization of the prices of the items that are sold) and efficient—i.e., have a *price of anarchy (PoA)* of one. In contrast to their setting, we show in our more general case there exist games with no pure Nash equilibria. In cases where they exist, we provide a characterization similar to theirs, though in our more general case, PoA is significantly higher.

Non-competitive (i.e., single-vendor) optimal-pricing problems have been studied in the theoretical computer science community. (Guruswami et al. 2005) study a number of settings with *multiple* buyers possessing various valuation functions. They show that even with unit-demand buyers and an unlimited supply of each item, selecting the optimal price vector is APX-hard; they then provide a logarithmic approximation algorithm for the same case. It should be noted that Babaioff et al. 2014 also provide a $\log n$ approximation algorithm for the case of a single vendor, and for that of a single buyer with a submodular valuation function.

In a recent paper, Oren et al. 2014 analyze a model in which fixed prices are given exogenously, and there are multiple unit-demand buyers. As above, their model assumes an unlimited supply of each item. The strategies of the vendors are which **sets** of items to sell. Having the vendor make decisions only about the set of items to sell has traditionally been studied in the field of operation research. In particular, *assortment optimization* (Schön 2010) deals with optimizing a seller’s “assortment” (e.g., his catalog, or shelf), under various circumstances. Although our game does not fall directly into this category, we do define a discrete game in which the vendors’ decisions are similar to those in assortment optimization (although the pricing procedure differs).

Preliminaries — The Vendor Competition (VC) Game

We consider the following setting: there is a set of k vendors, with a corresponding vector of pairwise-disjoint sets of items $\mathbf{A} = (A_1, \dots, A_k)$, such that $|A_i| = n_i$, and $n = \sum_{i=1}^k n_i$. We let $A^* = \bigcup_{i=1}^k A_i$.

A *strategy profile* of the vendors is a price vector $\mathbf{p} \in \mathbb{R}_+^n$, where $p(a)$ denotes the price of item a according to \mathbf{p} . For a set $S \subseteq A^*$, we let $p(S) = \sum_{a \in S} p(a)$. For a vendor $i \in [k]$, we let $p_i \in \mathbb{R}_+^{n_i}$ denote vendor i ’s price vector for the items in A_i , and as before, for an item $a \in A_i$, $p_i(a)$ denotes vendor i ’s price for item a according to p_i . For convenience, we will let \mathbf{p}_{-i} denote the price vector corresponding to the items not in A_i (of all other vendors).

The buyer’s valuation function We assume there is a single buyer with a valuation function $v : 2^{A^*} \rightarrow \mathbb{R}_+$; i.e., the function $v(\cdot)$ assigns a non-negative value to every *bundle* (or subset) of items. We let $m_a(S) = v(S \cup \{a\}) - v(S)$, where $S \subseteq A^*$, denote the *marginal contribution* of item a to the set S . Following (Babaioff, Nisan, and Leme 2014), we assume that $v(\cdot)$ is *non-decreasing*: for $S \subseteq T \subseteq A^*$, $v(S) \leq v(T)$ (implying that $v(\cdot)$ is maximized at A^*). Furthermore, we assume that the valuation function is *submodular*: for $S \subseteq T \subseteq A^*$ and $a \in A^* \setminus T$, we have that $m_a(S) \geq m_a(T)$. Both of these assumptions are central in the model proposed by Babaioff et al. (2014) (although additional discussion and results are provided for non-submodular functions). Note that $v(\cdot)$ is said to be submodular if the following, equivalent property holds:

$$v(S) + v(T) \geq v(S \cup T) + v(S \cap T), \quad \text{for all } S \subseteq T \subseteq A^*.$$

Finally, slightly abusing notation, let the valuation be defined over vectors of item sets as follows: for $S^* \subseteq A^*$, define $\mathbf{S} = (S_1, \dots, S_k)$ where $S_i = S^* \cap A_i$, for $i = 1, \dots, k$. Then $v(\mathbf{S}) = v(S^*)$. We adapt the rest of our function definitions in an analogous fashion.

The buyer is assumed to have a *quasi-linear* utility function: given a vector of prices $\mathbf{p} \in \mathbb{R}_+^n$, the buyer’s utility for a bundle $S \subseteq A^*$ is $u_b(S, \mathbf{p}) = v_b(S) - \sum_{a \in S} p(a)$. The *demand correspondence* of the buyer is the family of sets that maximizes his utility:

$$D(v; \mathbf{p}) = \{S \subseteq A^* : u_b(S) \geq u_b(S'), \forall S' \subseteq A^*\}.$$

The buyer’s *decision function* $X(v; \mathbf{p}) \subseteq 2^{A^*}$ must satisfy $X(v; \mathbf{p}) \in D(v; \mathbf{p})$. That is, given the price vector \mathbf{p} , the buyer buys the bundle $X(v; \mathbf{p})$. The buyer’s decision is said to be *maximal* (or simply, the buyer is maximal), if there does not exist a set $\tilde{X} \in D(v; \mathbf{p})$ such that $X(v; \mathbf{p}) \subsetneq \tilde{X}$. Babaioff et al. showed that this property is critical to ensure the existence of pure Nash equilibria in their setting. In our work, we will explicitly state where this is required.

Vendor payoffs Given the buyer’s decision function X , and a (fixed) price vector $\mathbf{p} = (p_i, \mathbf{p}_{-i})$, vendor i ’s utility is $u_i^X(\mathbf{p}) = \sum_{a \in X(v; \mathbf{p}) \cap A_i} p(a)$. If the vendors select mixed strategies, then a vendor’s utility is defined to be his *expected* utility. Vendor j ’s *best response* to the other agents’ mixed

strategies is a distribution over prices for A_j that maximizes his expected utility.

This setup defines a game, parameterized by the vector \mathbf{A} and the valuation function v , in which each of the vendors prices his items to maximize his utility. We will refer to such a game as a *vendor competition game*, or simply a *VC game*.

When discussing our special case, we will also use the following theorem, which was proved by Babaioff et al.:

Theorem 1 ((Babaioff, Nisan, and Leme 2014)). *Consider the case where each vendor owns a single item, that is $A_i = \{a_i\}$, for $i = 1, \dots, k$ (and $n_i = 1$). Then if the buyer's valuation function $v(\cdot)$ is non-decreasing and submodular, then there exists a pure Nash equilibrium, $\mathbf{p} \in \mathbb{R}_+^k$, of the following form: for every vendor i , such that $m_{a_i}(A^* \setminus a_i) > 0$, $p(a_i) = m_{a_i}(A^* \setminus a_i)$, and $a_i \in X(v; \mathbf{p})$. Also, the payoff of each vendor i is precisely $m_{a_i}(A^* \setminus a_i)$.*

The objective function Given a VC game $G = (v, \mathbf{S})$ and pricing vector $\mathbf{p} \in \mathbb{R}_+^n$, we use the standard definition of social welfare, namely, the total payoff of all the parties in the game, including the (non-strategic) buyer. Notice that by this definition, social welfare is simply the valuation of the set bought by the buyer, $v(X(v; \mathbf{p}))$, since all payments are simply transferred from the buyer to the vendors. Let $f(\mathbf{p})$ denote the social welfare resulting from a price vector \mathbf{p} .

A Related Discrete Game

The game in its current formulation may seem somewhat hard to reason about, due to large (continuous) strategy spaces.² To simplify our analysis, we use the following discrete game, which can be thought of as imposing a specific pricing mechanism given vendors' selection of items.

Definition 2 (The price-moderated VC game). *Given a buyer valuation function over the vendors' items, consider the following two-round process:*

1. Each vendor $i \in [k]$ commits to offering a subset of $S_i \subseteq A_i$ of items; this is its (discrete) strategy;
2. Given strategy vector $\mathbf{S} = (S_1, \dots, S_k)$, item prices are set to their marginal values. I.e., if we set $S^* = \bigcup_{i=1}^k S_i$, then for each $a \in S^*$, the mechanism will set $\tilde{p}(a) = m_a(S^* \setminus \{a\})$. For each item $a' \notin S^*$ the mechanism sets $\tilde{p}(a') = v(A^*) + 1$. Let $\tilde{\mathbf{p}}$ be the resulting price vector.

The consumer then buys the set $X(v; \tilde{\mathbf{p}})$, as before. We call the resulting game a price-moderated vendor competition game, or more succinctly, a PMVC game.

By analogy to our definitions for the original game, let $X'(v; \mathbf{S})$ denote the set of items sold, given the strategy profile \mathbf{S} ; i.e., given the price vector $\tilde{\mathbf{p}}$ imposed by the pricing mechanism in the second round, $X'(v; \mathbf{S}) = X(v; \tilde{\mathbf{p}})$. Similarly, define a vendor's utility to be $u'_i(S_i, \mathbf{S}_{-i})$, for $i \in [k]$.

Note that the specified pricing ($v(A^* + 1)$) of items not offered (i.e., not in S^*) ensures that the consumer will never buy them (i.e., $X(v; \tilde{\mathbf{p}}) \subseteq S^*$). Further observe that the set

²In particular, the game is clearly not normal form. Hence, we cannot directly apply Nash's theorem about existence of a mixed equilibrium. We defer treatment of such equilibria to future study.

of price vectors $\tilde{\mathbf{p}}$ that correspond to the discrete strategy profiles \mathbf{S} in the PMVC game is a strict subset of the strategy space in the original VC game. We justify our use of this game in our analysis by establishing the relationship between the original VC game and the proposed PMVC game, using a number of straightforward results.

Assumption For ease of exposition, we assume that the buyer is maximal. As we shall see, this implies that $X'(v; \mathbf{S}) = X(v; \mathbf{p})$. However, we can adapt the pricing mechanism by judiciously setting the prices to be slightly below the marginal contributions to ensure maximality (we leave the details of such a modification to an expanded version of the paper).

We now describe an important relationship between the VC and PMVC games that simplifies our subsequent analysis by relating our original model to a simpler discrete game:

Proposition 3. *For every strategy profile \mathbf{p} in the VC game and valuation v , there is a strategy profile \mathbf{S} in the PMVC game such that $X'(v; \mathbf{S}) = X(v; \mathbf{p})$, and $u'_i(S_i, \mathbf{S}_{-i}) \geq u_i(p_i, \mathbf{p}_{-i})$ for each vendor i .*

Proof. Let \mathbf{p} be a strategy profile in the VC game, and let $T = X(v; \mathbf{p})$. Consider the strategy profile \mathbf{S} where $S_i = X(v; \mathbf{p}) \cap A_i$, for $i = 1, \dots, k$, and let $\tilde{\mathbf{p}}$ be the resulting price vector imposed by the pricing mechanism. Furthermore, we let $\tilde{T} = X'(v; \mathbf{S})$. We begin by showing that $T = \tilde{T}$. First, notice that, as for all $a \notin T$, $\tilde{p}(a) = v(A^*) + 1$, and hence item a is not sold, and $\tilde{T} \subseteq T$. Next, suppose for the sake of contradiction that $\tilde{T} \subsetneq T$, and let $a \in T \setminus \tilde{T}$. By the submodularity of the function $v(\cdot)$, we have that $m_a(\tilde{T}) \geq m_a(T) \geq 0$. This implies that

$$\begin{aligned} u_b(\tilde{T} \cup a, \tilde{\mathbf{p}}) &= v(\tilde{T} \cup a) - \sum_{a' \in \tilde{T}} \tilde{p}(a') - m_a(T \setminus a) \\ &\geq v(\tilde{T} \cup a) - \sum_{a' \in \tilde{T}} \tilde{p}(a') - m_a(\tilde{T}) = u_b(\tilde{T}, \tilde{\mathbf{p}}). \end{aligned}$$

By maximality, the buyer would rather buy item a as well, resulting in a contradiction.

We now claim that $u'_i(S_i, \mathbf{S}_{-i}) \geq u_i(p_i, \mathbf{p}_{-i})$. This follows from the fact that marginal contributions are the maximal prices at which the buyer still buys $X'(v; \tilde{\mathbf{p}})$. That is, any increase in the price would result in the buyer not buying the product:

$$\begin{aligned} u_b(\tilde{T}, \tilde{\mathbf{p}}) &= v(\tilde{T}) - \sum_{a' \in \tilde{T} \setminus a} \tilde{p}(a') - m_a(T \setminus a) \\ &= v(\tilde{T} \setminus a) - \sum_{a' \in \tilde{T} \setminus a} \tilde{p}(a') = u_b(\tilde{T} \setminus a, \tilde{\mathbf{p}}) \end{aligned}$$

□

Similarly to the previous proposition, which offered a mapping of strategy profiles in a way that does not cause the vendor's utilities to deteriorate, we now show that the same mapping also preserves Nash equilibria in cases where such equilibria exist. We note that the following result uses similar arguments to those given by Babaioff et al. for proving a related characterization of pure Nash equilibria.

Theorem 4. For every pure Nash equilibrium \mathbf{p} of a VC game there is a pure Nash equilibrium $\mathbf{S} = (S_1, \dots, S_k)$ in the corresponding PMVC game, such that: (1) $X'(v; \mathbf{S}) = X(v; \mathbf{p})$; and (2) for all $a \in X(v; \mathbf{p})$, $\tilde{p}(a) = p(a)$, where $\tilde{\mathbf{p}}$ is the induced price vector for \mathbf{S} .

Proof. For convenience, let $B = X(v; \mathbf{p})$. As before, we let the strategy profile in the corresponding PMVC game be $\mathbf{S} = (S_1, \dots, S_k)$, where $S_i = B \cap A_i$, for $i = 1, \dots, k$.

We begin by proving part (2) of the theorem. Suppose that there is an item $a \in B$ such that $p(a) \neq m_a(B \setminus a)$. If $p(a) > m_a(B) = v(B) - v(B \setminus a)$, then $v(B \setminus a) - p(B \setminus a) > v(B) - p(B)$, implying that the buyer would not buy item a , contradicting our assumption that $a \in B$.

Assume now that $p(a) < m_a(B)$. Letting \mathbf{p}' denote the vector resulting by replacing $p(a)$ in \mathbf{p} with $m_a(B \setminus a)$, we clearly have that $u_b(B, \mathbf{p}') = u_b(B \setminus a, \mathbf{p}')$. We now prove the following claim:

Claim 5. $u_b(B \setminus a, \mathbf{p}') \geq u_b(T, \mathbf{p}')$, for all $T \subseteq B \setminus a$.

(Proof omitted due to space constraints.)

Therefore, the vendor who owns a can increase his payoff by setting the price of item a to any value between $p(a)$ and $m_a(B)$, contradicting the equilibrium state.

What is left to prove is that \mathbf{S} is a Nash equilibrium in the PMVC game. Note that we can assume w.l.o.g. that the price of all products which are not sold is $v(A^*) + 1$, as they remain unsold and continue to contribute nothing to the buyer or seller. Now, suppose \mathbf{S} is not a Nash equilibrium, and that there is a player i , which can benefit from changing his set of sold items from S_i to S'_i , which would result in a different vector of induced prices $\tilde{\mathbf{p}}' = (\tilde{p}'_i, \tilde{\mathbf{p}}'_{-i})$. We now argue that vendor i can make an identical improvement in his revenue by changing his price vector from p_i to \tilde{p}'_i , contradicting p being a Nash equilibrium. For convenience, we let $B' = (B \setminus S_i) \cup S'_i$, and $B'' = X(v; \tilde{p}'_i, \mathbf{p}_{-i})$.

To show this, first notice that no other vendor would sell any previously unsold items as a result; that is, $X(v; \tilde{p}'_i, \mathbf{p}_{-i}) \setminus A_i \subseteq X(v; p_i, \mathbf{p}_{-i}) \setminus A_i$ (since prices of items in $(A^* \setminus A_i) \cap X(v; p_i, \mathbf{p}_{-i})$ are still $v(A^*) + 1$). So $B'' = X(v; \tilde{p}'_i, \mathbf{p}_{-i}) \subseteq X'(v; S'_i, \mathbf{S}'_{-i}) = B'$. Thanks to submodularity, we have that for every $a \in S'_i$, $m_a(B') < m_a(B'')$. Arguments similar to the ones given above (on $p(a) = m_a(B)$) imply that player i would sell all the items in S'_i , and as the prices are unchanged from the PMVC game, will make the same profit as in the PMVC game. As this increases the player's profit in the PMVC game, it would increase its profit in the VC game as well, in contradiction to p being a Nash equilibrium. \square

Discussion Note that we have not shown an exact equivalence between the two games: the set of Nash equilibria in the VC game is a subset of the equilibria in the PMVC game. However, Proposition 3 and Theorem 4 allow us to reason about our original game to a considerable extent.

In contrast to the original model of Babaioff et al. in which $n_i = 1$ for all $i = 1, \dots, k$, we can show that in our more general game, there may not always be a pure Nash equilibrium. In order to do so, we provide an example of a VC game

in the next section with two vendors who each control two items. We show that this game does not admit any pure Nash equilibrium by relating to its corresponding PMVC game, using Theorem 4. Moreover, if we restrict ourselves to VC games that do admit pure Nash equilibria, we can provide quantitative bounds on their quality. Specifically, when restricting ourselves to VC games that have pure Nash equilibria, we provide a lower bound on the price of stability of the PMVC game by analyzing an instance of the game. As the optimal objective value (the valuation of the set that is bought by the buyer) is always $v(A^*)$, Theorem 4 immediately implies that the same lower bound applies to the VC game. To complement lower bound, we provide an upper bound for the price of anarchy, also ensuring tightness of bounds.

Equilibrium Analysis

Previously we outlined several properties of the discrete PMVC game. We now describe how the PMVC game can serve as a surrogate to help analyze the stability of the VC game, and its quality of equilibria when they exist.

Existence of pure Nash equilibria

We begin by showing that, as opposed to the special case where each vendor owns a single item, some instances of our game may not actually admit pure Nash equilibria.

Proposition 6. *There exists an instance of the VC game with two vendors, where $n_1 = n_2 = 2$, that does not admit a pure Nash equilibrium.*

Proof. Let $A_1 = \{a, b\}$ and $A_2 = \{c, d\}$. We define the buyer's valuation function v according to Table 1 (the value in each cell is the valuation of the union of the sets given at the head of the entry's row and column).

	\emptyset	$\{c\}$	$\{d\}$	$\{c, d\}$
\emptyset	0	2.8	2.7	4.1
$\{a\}$	3.2	5.4	5.3	6.5
$\{b\}$	2.5	5.3	5.2	6.6
$\{a, b\}$	4.4	6.6	6.5	7.6

Table 1: The buyer's valuation function

It is easy to verify that v is (strictly) non-decreasing and submodular. Now, consider the PMVC game with the same item sets and valuation function v . For each strategy profile (S_1, S_2) , the mechanism prices items according to their marginal contributions (Definition 2). Therefore, vendor payoffs are the sum of the prices of their offered items. The vendors' payoffs for each strategy profile are easily calculable from the table (omitted due to space constraints), and it is evident from them that there is no pure Nash equilibrium in the PMVC game. Theorem 4 then implies our proposition. \square

How bad can equilibria be?

Given the negative nature of Proposition 6, we now restrict attention to the subclass of VC games that *do* admit pure

Nash equilibria, and ask whether reasonable guarantees on social welfare in such equilibria can be derived

More formally, let $\mathcal{G} = \{G = (v, (A_1, \dots, A_k)) : \exists \text{ a pure Nash equilibrium in } G\}$ be the set of VC games which admit a pure Nash equilibrium. Define the *price of anarchy (PoA)* as follows:

$$PoA_G = \max_{G \in \mathcal{G}} \frac{\max_{\mathbf{p}^*} f(\mathbf{p}^*)}{\min_{\mathbf{p}: \mathbf{p} \text{ is a pure Nash equilibrium}} f(\mathbf{p})}$$

PoA is a commonly used worst-case measure of the efficiency of the equilibria, and in our case reflects the efficiency loss in \mathcal{G} resulting from the introduction of strategic pricing, as opposed to using a ‘‘centrally coordinated’’ pricing policy.

Theorem 7. *Define the set of VC games \mathcal{G}_m , such that $G \in \mathcal{G}_m$ iff (1) G has a pure Nash equilibrium, and (2) $\max_{i=1}^k |A_i| = m$. Then the PoA of \mathcal{G}_m is at most $H_m + 1$, where H_m is the m 'th harmonic number.*

Proof. Consider a game $G = (v, \mathbf{A} = (A_1, \dots, A_k))$ in \mathcal{G}_m . It is enough to provide a lower bound on the minimal social welfare of a pure Nash equilibrium in the corresponding PMVC game: by Theorem 4, this will establish a lower bound on the social welfare of a pure Nash equilibrium in G as well. So let $\mathbf{S} = (S_1, \dots, S_k)$ be a pure Nash equilibrium of the PMVC game. As $v(\cdot)$ is non-decreasing, we can assume w.l.o.g. that $|S_i| = \{a_i\}$, for some $a_i \in A_i$.

Again by the assumption that $v(\cdot)$ is non-decreasing, we know that optimal social welfare is obtained when all of A^* is sold, so it is enough to upper bound $v(\mathbf{A})$ in terms of $v(\mathbf{S})$.

We now show the following straightforward bound on the social welfare resulting from switching from S_i to A_i :

Lemma 8. $v(A_i, \mathbf{S}_{-i}) \leq v(\emptyset, \mathbf{S}_{-i}) + H_{n_i}(v(S_i, \mathbf{S}_{-i}) - v(\emptyset, \mathbf{S}_{-i}))$, for all $i = 1, \dots, k$.

Proof. As \mathbf{S} is a Nash equilibrium, the profit from selling A_i is higher than selling any set $B \subseteq A_i$. Using the definition of the pricing mechanism of the PMVC game, we know

$$v(S_i, \mathbf{S}_{-i}) - v(\emptyset, \mathbf{S}_{-i}) \geq \sum_{b \in B} m_b(B \setminus b, \mathbf{S}_{-i}), \quad \text{for all } B \subseteq A_i$$

By an averaging argument, this means that for all $B \subseteq A_i$, there exists an item $b \in B$ such that

$$\frac{1}{|B|} (v(S_i, \mathbf{S}_{-i}) - v(\emptyset, \mathbf{S}_{-i})) \geq m_b(B \setminus b, \mathbf{S}_{-i}) \quad (1)$$

The above implies that there is a relabelling of the items in A_i , so that: (1) $A_i = \{b_1, \dots, b_{n_i}\}$, (2) $b_1 = a_1$, and (3) if set $P_t = \{b_1, \dots, b_t\} \cup \mathbf{S}_{-i}$ and $P_0 = \mathbf{S}_{-i}$, the following holds:

$$\begin{aligned} v(A_i, \mathbf{S}_{-i}) &\leq v(\emptyset, \mathbf{S}_{-i}) + \sum_{i=1}^{n_i} \frac{1}{t} (v(S_i, \mathbf{S}_{-i}) - v(\emptyset, \mathbf{S}_{-i})) \\ &= v(\emptyset, \mathbf{S}_{-i}) + H_{n_i} (v(S_i, \mathbf{S}_{-i}) - v(\emptyset, \mathbf{S}_{-i})) \end{aligned}$$

where the first equality follows from a simple telescopic series, and the first inequality follows from Eq. 1. \square

Next, we show the following useful bound:

Lemma 9. $\sum_{i=1}^k v(A_i, \mathbf{S}_{-i}) \geq v(A_i, \mathbf{A}_{-i}) + (k - 1)v(S_i, \mathbf{S}_{-i})$

Proof. $L^{(t)} = (A_1, \dots, A_t, S_{t+1}, \dots, S_k)$, for $t = 1, \dots, k$, and $L^{(0)} = \mathbf{S}$. That is, $L^{(t)}$ is the strategy profile resulting from replacing the length- t prefix of \mathbf{S} with that of \mathbf{A} . We prove by induction that

$$\sum_{i=1}^t v(A_i, \mathbf{S}_{-i}) \geq v(L^{(t)}) + (t - 1)v(\mathbf{S})$$

and the lemma would follow by setting $t = k$.

The inequality clearly holds for $t = 1$, due to the monotonicity of $v(\cdot)$. Assume that the inequality holds for $t < k$. Thus, for $t + 1$, we have:

$$\sum_{i=1}^{t+1} v(A_i, \mathbf{S}_{-i}) \geq v(L^{(t)}) + (t - 1)v(\mathbf{S}) + v(A_{t+1}, \mathbf{S}_{-(t+1)})$$

By the second definition of submodularity, we know $v(L^{(t)}) + v(A_{t+1}, \mathbf{S}_{-(t+1)}) \geq v(L^{(t+1)}) + v(\mathbf{S})$. Putting this in the preceding inequality concludes the proof. \square

We can also prove an upper bound on the optimal social welfare in terms of the social welfare of \mathbf{S} . By the above two lemmas, we get:

$$\begin{aligned} v(\mathbf{A}) &\leq \sum_{i=1}^k v(A_i, \mathbf{S}_{-i}) - (k - 1)v(\mathbf{S}) \\ &\leq \sum_{i=1}^k v(S_i, \mathbf{S}_{-i}) + \sum_{i=1}^k H_{n_i} (v(S_i, \mathbf{S}_{-i}) - v(\emptyset, \mathbf{S}_{-i})) - (k - 1)v(\mathbf{S}) \\ &= v(\mathbf{S}) + \sum_{i=1}^k H_{n_i} (v(S_i, \mathbf{S}_{-i}) - v(\emptyset, \mathbf{S}_{-i})), \end{aligned}$$

where third inequality follows from monotonicity of $v(\cdot)$. By submodularity and that $n_i \leq m$ for $i = 1, \dots, k$,

$$\begin{aligned} v(\mathbf{A}) &\leq v(\mathbf{S}) + H_m \sum_{i=1}^k (v(S_i, \mathbf{S}_{-i}) - v(\emptyset, \mathbf{S}_{-i})) \\ &\leq v(\mathbf{S}) + H_m v(\mathbf{S}) = v(\mathbf{S})(H_m + 1), \end{aligned}$$

which establishes our upper bound on the PoA. \square

We also give an example of a game with a pure Nash equilibrium that matches the above bound.

Theorem 10. *There exists a game in \mathcal{G}_m with a price of anarchy of H_m .*

Proof. Our counter-example is obtained by making the bound of Lemma 8 tight. Consider a game $G = (v, \mathbf{A} = (A_1, \dots, A_k))$, in which $|A_i| = m$, for $i = 1, \dots, k$.

We define the valuation function as follows. For a strategy profile $\mathbf{T} = (T_1, \dots, T_k)$, we set $v(\mathbf{T}) = \sum_{i=1}^k \ell(T_i)$, where $\ell(T_i) = 0$ if $|T_i| = 0$, and otherwise we set $\ell(T_i) = H_{|T_i|}$. Observe that the vendors are all symmetric, and that furthermore, the payoffs only depend on their own prices.

We now consider the following strategy profile \mathbf{p} . Pick an arbitrary item a_i from each A_i , for $i = 1, \dots, k$, and set $p(a_i) = 1$. Price the remaining items at $v(A^*) + 1$. Note that the payoff of each vendor is precisely 1 (for non-maximal buyers, a_i prices can be decreased by a small ϵ).

It is easy to see that \mathbf{p} is a pure Nash equilibrium. Indeed, suppose that it is not, and let i be an arbitrary vendor. Then he has an alternative pricing $p'_i \neq p_i$, such that deviating to it

would improve his payoff of 1. Suppose that the set of items being bought under a deviation to p'_i is $B = X(v; p'_i, \mathbf{p}'_{-i})$, such that $\sum_{a \in B} p(a) > 1$. Then there exists an item $b \in B$, such that $p(b) > 1/|B|$. But then by the definition of the valuation function we have:

$$\ell(B) - p(B) = H_{|B|} - p(B \setminus b) - p(b) < H_{|B|-1} - p(B \setminus b)$$

contradicting the assumption that the buyer buys the set B . \square

Note that the above construction can be extended to show that the price of stability (PoS) is identical:

Corollary 11. *The price of stability of \mathcal{G}_m is $\Omega(H_m)$.*

Sketch of Proof Use the construction in the proof for Proposition 10, but set $\ell(T_i) = 1$ if $|T_i| = 1$, and $\ell(T_i) = H_{|T_i|} - \epsilon$, if $|T_i| > 1$, for sufficiently small ϵ . It is not hard to show that the aforementioned pure Nash equilibrium is the *only* Nash equilibrium. A similar bound follows. \square

Special Case: Product Categories

A particular VC game of interest is one in which we have classes of items that are roughly equivalent; as such the buyer is interested in at most one item from each class (e.g., TV sets of a certain size, with different manufacturers and sets of feature). Items in different classes however are “unrelated” so the buyer’s valuation for any set of items is additive *across* these classes. This scenario reflects the case of shops selling very similar products, of which the buyer only needs one, and we seek to try to understand the model’s pricing behavior.

Definition 12. *A Category-Divided Substitutable-Product Vendor Competition game (CDSP-VC) is a VC game with a buyer with a category-product-substitutable valuation:*

- A^* is partitioned into r pairwise-disjoint sets, $T^{(1)}, \dots, T^{(r)}$. That is, $T^{(i)} \cap T^{(j)} = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^r T^{(i)} = A^*$. We refer to each set $T^{(j)}$ for $j = 1, \dots, r$, as a category.
- For $S \subseteq A^*$, $v(S) = \sum_{i=1}^r v(S \cap T^{(i)})$.
- For $S \subseteq A^*$ and $1 \leq i \leq r$, $v(S \cap T^{(i)}) = \max_{a \in S \cap T^{(i)}} v(a)$.

The additivity allows us to focus on the pricing dynamic within a specific category and easily generalize the results.

Observation 13. *For a category $T^{(j)}$, regardless of the other vendors’ strategies, no vendor can profit by selling any items other than his most valuable one in category $T^{(j)}$.*

(Proof omitted due to space constraints.)

Observation 13 implies that within every category, each of the vendors is better off effectively trying to sell his highest valued item. In other words, we can assume w.l.o.g. that for every vendor i and category $T^{(j)}$, the vendor can pick an item $a_i^{(j)} \in \arg \max_{a \in T^{(j)} \cap A_i} v(a)$ (if such item exists) and set $p(b) = v(A^*) + 1$, for all $b \neq a_i^{(j)}$, without incurring a loss as a result. Therefore, this reduces our game to r independent special cases of the VC game, in which each vendor owns a *single* item.

We turn to the result given by Babaioff et al. 2014 (Theorem 1 in the preliminaries). Their result implies the following characterization of the prices in a pure Nash equilibrium.

Corollary 14. *Every CDSP-VC game has a pure Nash equilibrium of the following form. For every category $T^{(j)}$, let $c_i^{(j)} = \arg \max_{a \in (T^{(j)} \cap A_i)} v(a)$, and $w = \arg \max_i c_i^{(j)}$. Let $b^{(j)} = \arg \max_{a \in (T^{(j)} \setminus A_w)} v(a)$ or $b^{(j)} = 0$ if $|T^{(j)}| = 1$. Then $p(c_w^{(j)}) = v(c_w^{(j)}) - b^{(j)}$, and for every player $i \neq w$, $p(c_i^{(j)}) = 0$. For all other items $a \in T^{(j)}$, $p(a) = v(A^*) + 1$.*

Proof. Once we know that each player sells only a single product, unsold products need to be priced high, and the rest of the result stems from Theorem 1, as the above difference constitutes the marginal contribution of item $c^{(j)}$. \square

Conclusions and Future Work

The multi-item, multi-vendor problem is a practical instance of rival agents’ problems, with their actions directly affecting actions to be taken by the others. This relationship among the agents is, in many cases, intractable to handle. However, in the simplified model, which is robust enough to incorporate realistic limitations, we were able to analyze the effects of each agent’s moves.

We defined a discrete game that allowed us to consider a related game that was instrumental in analyzing our original game. The main property of the discrete game was to transform player strategies from pricing, to selecting what items to sell. To paraphrase Clausewitz’s famous dictum, displaying (what to sell) became pricing by other means. Utilizing this discrete game, we were able to prove that a multi-item, multi-vendor game with submodular buyers valuations does not necessarily have a Nash equilibrium (unlike the “single item per vendor” model). Furthermore, even when equilibria exist, it may provide only a logarithmic price of anarchy. Building on these results, we showed that in a particular category-substitute model, there will always be an efficient pure Nash equilibrium.

Many open problems remain, even before the “holy grail” of pricing multi-item multi-buyer scenarios. We believe that there is a need to establish the characteristics of valuation functions that guarantee the existence of Nash equilibria.

Adding more buyers changes the model significantly, as vendors do not simply construct some “buyer in expectation” and act according to it, but rather have a wider range of options to pursue (primarily bundling). Perhaps using a metric to define a set of similar, yet not identical, buyers, it might be possible to build on our results, and construct extensions to the current model incorporating multiple buyers.

Acknowledgements This research has been partly funded by Microsoft Research through its PhD Scholarship Programme, Israel Science Foundation grant #1227/12, and the Israel Ministry of Science and Technology — Knowledge Center in Machine Learning and Artificial Intelligence grant #3-9243. Oren and Boutilier acknowledge the support of NSERC.

References

- Angwin, J., and Mattioli, D. 2012. Coming soon: Toilet paper priced like airline tickets. *The Wall Street Journal*.
- Babaioff, M.; Nisan, N.; and Leme, R. P. 2014. Price competition in online combinatorial markets. *CoRR* abs/1401.1559.
- Christodoulou, G.; Kovács, A.; and Schapira, M. 2008. Bayesian combinatorial auctions. In *Proceedings of the 35th International Colloquium on Automata, Languages and Programming (ICALP)*, 820–832.
- Guruswami, V.; Hartline, J. D.; Karlin, A. R.; Kempe, D.; Kenyon, C.; and McSherry, F. 2005. On profit-maximizing envy-free pricing. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '05*, 1164–1173. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics.
- Hart, S., and Nisan, N. 2012. Approximate revenue maximization with multiple items. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC)*, 656.
- Hart, S., and Reny, P. J. 2012. Maximizing revenue with multiple goods: Nonmonotonicity and other observations. DP 630, Hebrew University of Jerusalem, Center for Rationality.
- Oren, J.; Narodytska, N.; and Boutilier, C. 2014. A game-theoretic analysis of catalog optimization. In *Proceedings of the 28th AAAI Conference On Artificial Intelligence (AAAI)*, 1463–1470.
- Schön, C. 2010. On the product line selection problem under attraction choice models of consumer behavior. *European Journal of Operational Research* 206(1):260–264.
- Skorupa, J. 2014. Pricing intelligence goes to war. *Retail Infor Systems News*.