Proof. Consider a game with 2 vendors, $C = \{c_1, \ldots, c_m\}$, and prices: $p(c_1) = 1$, $p(c_i) = \epsilon, i \geq 1$. Consider a PNE where all vendors select C. The revenue in is $\frac{\sum_{i=1}^{m} p(c_i)}{m} = \frac{1}{m} + \frac{(m-1)\epsilon}{m}$. The optimal total revenue is 1. □

If the number of vendors is assumed to be constant, then PoS is a logarithmic factor smaller than PoA.

Theorem 17. *Given identical item sets, if preferences are drawn from IC, then PoS is* $\Theta(m \cdot k / \log m)$.

Let $r_j(X, \mathbf{Y^{k-1}})$ be vendor j's utility when selecting set X in response to competitor profile Y. Furthermore, let $P_i = \{c_1, \ldots, c_i\}$. Alg. 4 is a simple procedure for computing a PNE, as the following lemma shows:

Algorithm 4: Finding a Nash equilibrium

Input: k vendors, items $C = \{c_1, \ldots, c_m\}$, price vector $\mathbf{p}$ such that $p(c_1) \geq \cdots \geq p(c_m)$

for $i \leftarrow 2$ *to* m **do**
  if $r_1(P_{i-1}, \mathbf{P^{k-1}_{i-1}}) \geq r_1(P_{i-1} \cup \{c_i\}, \mathbf{P^{k-1}_{i-1}})$ **then**
    return P_{i-1}
return C

Lemma 18. *If the algorithm halts at step* $i < m$*, then* $(P_i, \ldots, P_i)$ *is a Nash equilibrium.*

Proof. As the items are ordered in a non-increasing order of price, it suffices to show that no (arbitrary, by symmetry) vendor would deviate by selecting a prefix P_j, for $j > i$. We show inductively that if a vendor improves by deviating to such a P_j, then she can do so by deviating to P_{i+1} too. Assume w.l.o.g. the first vendor deviates. First, we show that if a vendor improves her revenue by selecting P_j then she can improve it by deviating to P_{j-1}. Suppose by way of contradiction that $r_1(P_j, \mathbf{P^{k-1}_i}) > r_1(P_i, \mathbf{P^{k-1}_i})$, for $j > i$, but $r_1(P_{j-1}, \mathbf{P^{k-1}_i}) \leq r_1(P_i, \mathbf{P^{k-1}_i})$. Then by definition

$$\frac{\sum_{t=1}^{i} p(c_t)}{k \cdot i} \geq \frac{\sum_{t=1}^{i} p(c_t)}{k \cdot (j-1)} + \frac{\sum_{t=i+1}^{j-1} p(c_t)}{j-1},$$

which implies $\sum_{t=i+1}^{j-1} p(c_t) \leq \frac{j-i-1}{i \cdot k} \sum_{t=1}^{i} p(c_t)$. Then:

$$\begin{aligned}
r_1(P_j, \mathbf{P^{k-1}_i}) &= \frac{\sum_{t=1}^{i} p(c_t)}{k \cdot j} + \frac{\sum_{t=i+1}^{j-1} p(c_t)}{j} + \frac{p(c_j)}{j} \\
&\leq \frac{\sum_{t=1}^{i} p(c_t)}{k \cdot j} + \frac{(j-i-1)\sum_{t=1}^{i} p(c_t)}{j \cdot k \cdot i} + \frac{\sum_{t=i+1}^{j-1} p(c_t)}{j \cdot (j-i-1)} \\
&\leq \frac{\sum_{t=1}^{i} p(c_t)}{k \cdot j} + \frac{(j-i-1)\sum_{t=1}^{i} p(c_t)}{j \cdot k \cdot i} + \frac{\sum_{t=1}^{i} p(c_t)}{j \cdot k \cdot i} \\
&= \frac{\sum_{t=1}^{i} p(c_t)}{k \cdot i} = r_1(P_i, \mathbf{P^{k-1}_i})
\end{aligned}$$

where the first inequality follows from the bound above and an averaging argument on $p(c_j)$, a contradiction. Hence, deviating to P_{j-1} also improves vendor revenue. Repeating this process until $i + 1$ contradicts the stopping condition of the for-loop of the algorithm. □

Next, we bound the rate of decrease in prices to construct a lower bound on expected social welfare.

Lemma 19. *Suppose Alg. 4 returns set* $P_i = \{c_1, \ldots, c_i\}$*. Then* $p(c_j) \geq \frac{1}{k \cdot (j-1)} + \Theta(\frac{1}{k^2})$*, for* $2 \leq j \leq i$.

Proof. Alg. 4 stops when $r_1(P_i, \mathbf{P^{k-1}_{i-1}}) \leq r_1(P_{i-1}, \mathbf{P^{k-1}_{i-1}})$. Using the definitions of $r_1(P_i, \mathbf{P^{k-1}_{i-1}})$ and $r_1(P_{i-1}, \mathbf{P^{k-1}_{i-1}})$, and rearranging the terms, we get that for every $1 < j \leq i$,

$$\frac{\sum_{t=1}^{i} p(c_t)}{k \cdot (j-1)} < \frac{\sum_{t=1}^{j-1} p(c_t)}{k \cdot j} + \frac{p(c_j)}{j}$$

which implies the recursive inequality: $p(c_j) > \frac{\sum_{t=1}^{j-1} p(c_t)}{k \cdot (j-1)}$. The statement of the lemma can be then be shown to be the solution of this inequality, using induction. □

Proof of Thm. 17. The worst case execution of Alg. 4 occurs when it reaches the last item. By Lemma 19, expected welfare is bounded below by $\frac{1}{m}(1 + \sum_{i=2}^{m} \frac{1}{k \cdot (i-1)}) = \Omega(\frac{\ln m}{m \cdot k})$. The fact that $p(c_1) = 1$ implies the upper bound on PoS. We can construct a matching worst-case price vector using the bound on the $p(c_i)$'s given in Lemma 19. □

Conclusions

We have presented a model of competition among vendors who offer slates or catalogs of products to their consumers using rank-based models of preferences that have connections to models in computational social choice and algorithmic game theory. We studied both best response computation (and equlibrium finding in some cases) and various equilibrium properties under two different informational assumptions w.r.t. consumer preferences.

There are a number of directions remaining to be explored. The possibility of approximating best responses in the full information setting remains open. This problem doesn't appear to have any of the usual "nice" properties often used for devising efficient optimization algorithms (e.g., symmetry, monotonicity, submodularity). The study of our model where the strategies are required to satisfy certain combinatorial constraints (e.g., matroid or knapsack) reflecting limits on individual catalogs would be of interest. Under such restrictions, our worst case PoA and PoS ratios might be improved. Connections to other game-theoretic models also bear exploration. For instance, allowing endogenous prices requires vendors to set prices, e.g., in a multivendor platform, offering a competitive extension of profit-maximizing, envy-free mechanisms (see e.g., (Guruswami et al. 2005)). Endogenous pricing has been considered in a recent competitive model related to ours, but where each vendor has a single item (Babaioff, Nisan, and Leme 2014).

Acknowledgements

This research was supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

References

Babaioff, M.; Nisan, N.; and Leme, R. P. 2014. Price competition in online combinatorial markets. In *Proceedings of the Twenty-Third International World Wide Web Conference (WWW-14)*, 711–722.

Black, D. 1948. On the rationale of group decision-making. *Journal of Political Economy* 56(1):23–34.

Chamberlin, J. R., and Courant, P. N. 1983. Representative deliberations and representative decisions: Proportional representation and the Borda rule. *American Political Science Review* 77(3):718–733.

Feige, U. 1998. A threshold of ln *n* for approximating set cover. *Journal of the ACM* 45(4):634–652.

Gaertner, W. 2002. Domain restrictions. In *Handbook of Social Choice and Welfare*, Vol. 1, 131–170. Arrow, K. J.; Sen, A. K.; and Suzumura, K. (eds.). North Holland.

Goldberg, A. V.; Hartline, J. D.; Karlin, A. R.; Saks, M.; and Wright, A. 2006. Competitive auctions. *Games and Economic Behavior* 55(2):242–269.

Guruswami, V.; Hartline, J. D.; Karlin, A. R.; Kempe, D.; Kenyon, C.; and McSherry, F. 2005. On profit-maximizing envy-free pricing. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA-05)*, 1164–1173.

Honhon, D.; Jonnalagedda, S.; and Pan, X. A. 2012. Optimal algorithms for assortment selection under ranking-based consumer choice models. *Manufacturing & Service Operations Management* 14(2):279–289.

Kleinberg, J.; Papadimitriou, C.; and Raghavan, P. 2004. Segmentation problems. *Journal of the ACM* 51:263–280.

Li, X.; Nukala, S.; and Mohebbi, S. 2013. Game theory methodology for optimizing retailers' pricing and shelf-space allocation decisions on competing substitutable products. *International Journal of Advanced Manufacturing Technology* 68(1-4):375–389.

Lu, T., and Boutilier, C. 2011. Budgeted social choice: From consensus to personalized decision making. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence (IJCAI-11)*, 280–286.

Mallows, C. L. 1957. Non-null ranking models. *Biometrika* 44:114–130.

Marden, J. I. 1995. *Analyzing and Modeling Rank Data*. London: Chapman and Hall.

Martínez-de Albéniz, V., and Roels, G. 2011. Competing for shelf space. *Production and Operations Management* 20(1):32–46.

Monroe, B. L. 1995. Fully proportional representation. *American Political Science Review* 89(4):pp. 925–940.

Schön, C. 2010. On the product line selection problem under attraction choice models of consumer behavior. *European Journal of Operational Research* 206(1):260–264.

Skowron, P.; Faliszewski, P.; and Slinko, A. 2013. Fully proportional representation as resource allocation: Approximability results. In *Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence (IJCAI-13)*, 353–359.