(a) With 2 actions, the uniform team plays better than the diverse team

Agents	Action 1	Action 2
Agent 1	0.6	0.4
Agent 2	0.55	0.45
Agent 3	0.55	0.45
Uniform p_{best}:	0.648	
Diverse p_{best}:	0.599	

(b) When we add one more action, the diverse team plays better than the uniform team

Agents	Action 1	Action 2	Action 3
Agent 1	0.6	0.4	0
Agent 2	0.55	0.25	0.2
Agent 3	0.55	0.15	0.3
Uniform p_{best}:	0.648		
Diverse p_{best}:	0.657		

Table 1: The performance of a diverse team increases when we increase the number of available actions.

copies of agent 1) and the diverse team (one copy of each agent). We assume agent 1 is an *NST* agent, while agent 2 and 3 are *ST* agents. In this situation the uniform team plays better than the diverse team. Now let's add one more action to the problem. Because agent 2 and 3 are *ST* agents, the probability mass on action 2 scatters to the newly added action (Table 1(b)). Hence, while before the *ST* agents would always agree on the same suboptimal action if they both did not vote for the optimal action, now they might vote for different suboptimal actions, creating a tie between each suboptimal action and the optimal one. Because ties are broken randomly, when this happens there will be a $1/3$ chance that the tie will be broken in favor of the optimal action. Hence, p_{best} increases when the probability of the *ST* agents agreeing on the same suboptimal actions decreases, and the diverse team now plays better than the uniform team, even though individually agents 2 and 3 are weaker than agent 1.

We now present our theoretical work. First we show that the performance of a diverse team converges when $m \to \infty$, to a value that is higher than the performance for any other m.

Theorem 1. *$p_{best}(m)$ of a diverse team of n agents converges to a certain value $\tilde{p}_{best}$ as $m \to \infty$. Furthermore, $\tilde{p}_{best} \geq p_{best}(m)$, $\forall m$.*

Proof. Let $p_{i,min} = \min_{j \in \mathbf{D_m}} p_{i,j}$, $p_{i,max} = \max_{j \in \mathbf{D_m}} p_{i,j}$ and $\mathbf{T}$ be the set of agents in the team. By our assumptions, there is a constant α such that $p_{i,max} \leq \alpha p_{i,min}$ for all agents i. Then, we have that $1 \geq 1 - p_{i,0} = \sum_{j \in \mathbf{D_m}} p_{i,j} \geq d_m p_{i,min}$. Therefore, $p_{i,min} \leq \frac{1}{d_m} \to 0$ as d_m tends to ∞ with m. Similarly, $\alpha p_{i,min} \to 0$ as $d_m \to \infty$. As $p_{i,j} \leq \alpha p_{i,min}$ we have that $\forall j \; p_{i,j} \to 0$ as $d_m \to \infty$. We show that this implies that when $m \to \infty$, weak agents never agree on the same suboptimal action. Let i_1 and i_2 be two arbitrary agents. Without loss of generality, assume i_2's d_m ($d_m^{(i_2)}$) is greater than or equal i_1's d_m ($d_m^{(i_1)}$). The probability (σ_{i_1,i_2}) of i_1 and i_2 agreeing on the same suboptimal action is upper bounded by $\sigma_{i_1,i_2} = \sum_{a_j \in \mathbf{A} \setminus a_0} p_{i_1,j} p_{i_2,j} \leq d_m^{(i_2)} p_{i_1,max} p_{i_2,max} \leq d_m^{(i_2)} \alpha p_{i_2,min} p_{i_1,max} \leq \alpha p_{i_1,max}$ (as $d_m^{(i_2)} p_{i_2,min} \leq 1$). We have that $\alpha p_{i_1,max} \to 0$ as $p_{i_1,max} \to 0$, because α is a constant. Hence the probability of any two agents agreeing on a suboptimal action is $\frac{\sum_{i_1 \in \mathbf{T}} \sum_{i_2 \in \mathbf{T}, i_2 \neq i_1} \sigma_{i_1,i_2}}{2} \leq \frac{n(n-1)}{2} \max_{i_1,i_2} \sigma_{i_1,i_2} \to 0$, as n is a constant.

Hence, when $m \to \infty$, the diverse team only chooses a suboptimal action if all agents vote for a different suboptimal action or in a tie between the optimal action and suboptimal actions (because ties are broken randomly). Therefore, p_{best} converges to:

$$\tilde{p}_{best} = 1 - \prod_{i=1}^{n}(1-p_{i,0}) - \sum_{i=1}^{n}(p_{i,0} \prod_{j=1, j \neq i}^{n}(1-p_{j,0})) \frac{n-1}{n}, \quad (1)$$

that is, the total probability minus the cases where the best action is not chosen: the second term covers the case where all agents vote for a suboptimal action and the third term covers the case where one agent votes for the optimal action and all other agents vote for suboptimal actions.

When m is finite, the agents might choose a suboptimal action by agreeing over that suboptimal action. Therefore, we have that $p_{best}(m) \leq \tilde{p}_{best} \; \forall m$. □

Let $p_{best}^{uniform}(m)$ be p_{best} of the uniform team, with m actions. A uniform team is not affected by increasing m, as the pdf of an *NST* agent will not change. Hence, $p_{best}^{uniform}(m)$ is the same, $\forall m$. If $\tilde{p}_{best}$ is high enough so that $\tilde{p}_{best} \geq p_{best}^{uniform}(m)$, the diverse team will overcome the uniform team, when $m \to \infty$. Therefore, the diverse team will be better than the uniform team when m is large enough.

In practice, a uniform team made of copies of the best agent (the one with highest expected utility) might not behave exactly like a team of *NST* agents, as the best agent could also increase its d_m as m gets larger. We discuss this situation in Section Experimental Analysis. In order to perform that study, we derive in the following corollary how fast p_{best} converges to $\tilde{p}_{best}$, as a function of d_m.

Corollary 1. *$p_{best}(m)$ of a diverse team increases to $\tilde{p}_{best}$ in the order of $O(\frac{1}{d_m^{min}})$ and $\Omega(\frac{1}{d_m^{max}})$, where d_m^{max} is the highest and d_m^{min} the lowest d_m of the team.*

Proof. We assume here the notation that was used in the previous proof. First we show a lowerbound on $p_{best}(m)$. We have that $p_{best}(m) = 1 - \psi_1$, where ψ_1 is the probability of the team picking a suboptimal action. $\psi_1 = \psi_2 + \psi_3$, where ψ_2 is the probability of no agent agreeing and the team picks a suboptimal action and ψ_3 is the probability of at least two agents agreeing and the team picks a suboptimal action. Hence, $p_{best}(m) = 1 - \psi_2 - \psi_3 = \tilde{p}_{best} - \psi_3 \geq \tilde{p}_{best} - \psi_4$, where ψ_4 is the probability of at least two agents agreeing. Let $\sigma^{max} = \max_{i_1,i_2} \sigma_{i_1,i_2}$, and i_1^* and i_2^* are the agents whose $\sigma_{i_1^*,i_2^*} = \sigma^{max}$. We have that $p_{best}(m) \geq \tilde{p}_{best} - \frac{n(n-1)}{2}\sigma^{max} \geq \tilde{p}_{best} - \frac{n(n-1)}{2} d_m^{(i_2^*)} p_{i_1^*,max} p_{i_2^*,max} \geq$

$\tilde{p}_{best} - \frac{n(n-1)}{2} d_m^{(i_2^*)} \alpha p_{i_1^*,min} \alpha p_{i_2^*,min} \geq \tilde{p}_{best} - \frac{n(n-1)}{2}\alpha^2 \frac{1}{d_m^{(i_1^*)}}$ (as $p_{i,min} \leq \frac{1}{d_m}$). Hence, $p_{best}(m) \geq \tilde{p}_{best} - \frac{n(n-1)}{2}\alpha^2 \frac{1}{d_m^{min}} \rightsquigarrow \tilde{p}_{best} - p_{best}(m) \leq O(\frac{1}{d_m^{min}})$.

Now we show an upper bound: $p_{best}(m) = \tilde{p}_{best} - \psi_3 \leq \tilde{p}_{best} - \psi_5$, where ψ_5 is the probability of at least two agents agreeing and no agents vote for the optimal action. Let $\sigma^{min} = \min_{i_1,i_2} \sigma_{i_1,i_2}$; i_1^* and i_2^* are the agents whose $\sigma_{i_1^*,i_2^*} = \sigma^{min}$; and $p_{max,0} = \max_{i \in \mathbf{T}} p_{i,0}$. Without loss of generality, we assume that $d_m^{(i_2^*)} \geq d_m^{(i_1^*)}$. Hence, $p_{best}(m) \leq \tilde{p}_{best} - \frac{n(n-1)}{2}\sigma^{min}(1 - p_{max,0})^{n-2} \leq \tilde{p}_{best} - \frac{n(n-1)}{2} d_m^{(i_1^*)} p_{i_1^*,min} p_{i_2^*,min} (1 - p_{max,0})^{n-2} \leq \tilde{p}_{best} - \frac{n(n-1)}{2} d_m^{(i_1^*)} \frac{p_{i_1^*,max} p_{i_2^*,max}}{\alpha^2} (1 - p_{max,0})^{n-2} \leq \tilde{p}_{best} - \frac{n(n-1)}{2}\alpha^{-2}\frac{1}{d_m^{i_2^*}}(1 - p_{max,0})^{n-2} \leq \tilde{p}_{best} - \frac{n(n-1)}{2}\alpha^{-2}\frac{1}{d_m^{max}}(1 - p_{max,0})^{n-2} \rightsquigarrow \tilde{p}_{best} - p_{best}(m) \geq \Omega(\frac{1}{d_m^{max}})$. □

Hence, agents that change their d_m faster will converge faster to $\tilde{p}_{best}$. This is an important result when we consider later more complex scenarios where the d_m of the agents of the uniform team also change.

Note that $\tilde{p}_{best}$ depends on the number of agents n (Equation 1). Now we show that the diverse team tends to always play the optimal action, as $n \to \infty$.

Theorem 2. *$\tilde{p}_{best}$ converges to 1, as $n \to \infty$. Furthermore, $1 - \tilde{p}_{best}$ converges exponentially to 0, that is, $\exists$ constant c, such that $1 - \tilde{p}_{best} \leq c(1 - \frac{\epsilon}{2})^n$, $\forall n \geq \frac{2}{\epsilon}$. However, the performance of the uniform team improves as $n \to \infty$ only if $p_{s,0} = \max_j p_{s,j}$, where s is the best agent.*

Proof. By the previous proof, we know that when $m \to \infty$ the diverse team plays the optimal action with probability given by $\tilde{p}_{best}$. We show that $1 - \tilde{p}_{best} \to 0$ exponentially as $n \to \infty$ (this naturally induces $\tilde{p}_{best} \to 1$). We first compute an upper bound for $\sum_{i=1}^n (p_{i,0} \prod_{j=1, j\neq i}^n (1 - p_{j,0}))$: $\sum_{i=1}^n p_{i,0} \prod_{j=1,j\neq i}^n (1 - p_{j,0}) \leq \sum_{i=1}^n p_{i,0}(1 - p_{min,0})^{n-1} \leq n p_{max,0}(1 - p_{min,0})^{n-1} \leq n(1-\epsilon)^{n-1}$ for $p_{max,0} = max_i p_{i,0}$, $p_{min,0} = min_j p_{j,0}$.

Since $\prod_{i=1}^n (1 - p_{i,0}) \leq (1-\epsilon)^n$, thus we have that $1 - \tilde{p}_{best} \leq (1-\epsilon)^n + n(1-\epsilon)^{n-1}$. So we only need to prove that there exists a constant c such that $(1-\epsilon)^n + n(1-\epsilon)^{n-1} \leq c(1 - \frac{\epsilon}{2})^n$, as follows: $\frac{(1-\epsilon)^{n+1} + (n+1)(1-\epsilon)^n}{(1-\epsilon)^n + n(1-\epsilon)^{n-1}} = (1-\epsilon)\frac{1-\epsilon+n+1}{1-\epsilon+n} = 1 - \epsilon + \frac{1-\epsilon}{1-\epsilon+n} \leq 1 - \frac{1}{2}\epsilon$, if $n \geq \frac{2}{\epsilon}$ (by setting $\frac{1-\epsilon}{1-\epsilon+n} \leq \frac{\epsilon}{2}$). Hence, $\exists c$, such that $(1-\epsilon)^n + n(1-\epsilon)^{n-1} \leq c(1-\frac{\epsilon}{2})^n$ when $n \geq \frac{2}{\epsilon}$. Therefore, the performance converges exponentially.

For the uniform team, the probability of playing the action that has the highest probability in the pdf of the best agent converges to 1 as $n \to \infty$ (List and Goodin 2001). Therefore, the performance only increases as $n \to \infty$ if the optimal action is the one that has the highest probability. □

Now we show that we can achieve further improvement in a diverse team by breaking ties in favor of the strongest agent.

Theorem 3. *When $m \to \infty$, breaking ties in favor of the strongest agent is the optimal tie-breaking rule for a diverse team.*

The proof is available in the appendix (available at http://teamcore.usc.edu/people/sorianom/aaai14-ap.pdf). We just state here that if we break ties in favor of an agent s, the probability of voting for the optimal action is given by:

$$\tilde{p}_{best} = 1 - \prod_{i=1}^{n}(1 - p_{i,0}) - (1 - p_{s,0})(\sum_{\substack{i=1 \\ i\neq s}}^{n} p_{i,0} \prod_{\substack{j=1 \\ j\neq i, j\neq s}}^{n}(1 - p_{j,0})) \tag{2}$$

It is clear that Equation 2 is maximized by choosing agent s with the highest $p_{s,0}$. In our proof we show that Equation 2 is always higher than Equation 1 if s is the strongest agent. That is, we show that $\tilde{p}_{best}$ is higher if we break ties in favor of the strongest agent than breaking ties randomly.

Next we show that with one additional assumption, not only the diverse team converges to $\tilde{p}_{best}$, but also p_{best} monotonically increases with m. Our additional assumption is that higher utility actions have higher probabilities, i.e., if $U(a_j) \geq U(a_{j'})$, then $p_{i,j} \geq p_{i,j'}$.

Theorem 4. *The performance of a diverse team monotonically increases with m, if $U(a_j) \geq U(a_{j'})$ implies that $p_{i,j} \geq p_{i,j'}$.*

The proof of this theorem is also available in the appendix.

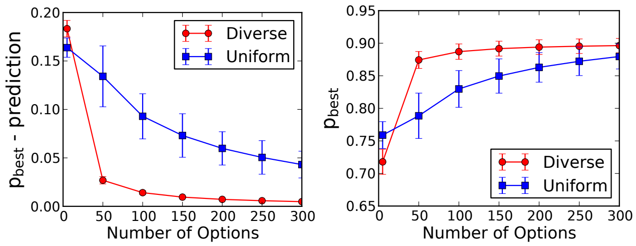
Generalizations

In the previous theorems we focused on the probability of playing the best action, assuming that $U(a_0) \gg U(a_j) \forall j \neq 0$. We show now that the theorems still hold in more general domains where r actions ($\mathbf{A_r} \subset \mathbf{A}$) have a significant high utility, i.e., $U(a_{j_1}) \gg U(a_{j_2}) \forall j_1 < r, j_2 \geq r$. Hence, we now focus on the probability of playing any action in $\mathbf{A_r}$. We assume that our assumptions are also generalized, i.e., $p_{i,j} \geq \epsilon \; \forall j < r$, and the number d_m of suboptimal actions (a_j, $j \geq r$) in the $\mathbf{D_m}$ set increases with m for *ST* agents.

Theorem 5. *The previous theorems generalize to settings where $U(a_{j_1}) \gg U(a_{j_2}) \; \forall j_1 < r, j_2 \geq r$.*

Proof Sketch. We give here a proof sketch. We just have to generate new pdfs $p'_{i,j}$, such that $p'_{i0} = \sum_{j=0}^{r-1} p_{i,j}$, and $p'_{i,b} = p_{i,b+r-1}$, $\forall b \neq 0$. We can then reapply the proofs of the previous theorems, but replacing $p_{i,j}$ by $p'_{i,j}$. Note that this does not guarantee that all agents will tend to agree on the same action in $\mathbf{A_r}$; but the team will still tend to pick any action in $\mathbf{A_r}$, since the agents are more likely to agree on actions in $\mathbf{A_r}$ than on actions in $\mathbf{A} \setminus \mathbf{A_r}$. □

Now we discuss a different generalization: what happens when $p_{i,0}$ decreases as m increases ($\forall$ agents i). If $p_{i,0} \to \tilde{p}_{i,0}$ as $m \to \infty$, the performance in the limit for a diverse team will be $\tilde{p}_{best}$ evaluated at $\tilde{p}_{i,0}$. Moreover, even if $p_{i,0} \to 0$, our conclusions about relative team performance are not affected as long as we are comparing two *ST* teams that have similar $p_{i,0}$: the same argument as in Corollary 1 implies that the team with faster growing d_m will perform better.



(a) Convergence of p_{best} to predicted value.

(b) Actual performance value.

Figure 1: Comparing diverse and uniform when uniform also increases d_m.

Experimental Analysis

Synthetic Experiments

We present synthetic experiments, in order to better understand what happens in real systems. We generate agents by randomly creating pdfs and calculate the p_{best} of the generated teams. The details are available in the appendix.

As we said earlier, uniform teams composed by *NST* agents is an idealization. In more complex domains, the best agent will not behave exactly like an *NST* agent, its d_m will also increase. We perform synthetic experiments to study this situation. We consider that the best agent is still closer to an *NST* agent, therefore it increases its d_m at a slower rate than the agents of the diverse team. We can see the average result for 200 random teams in Figure 1, where in Figure 1(a) we show the difference between the performance in the limit ($\tilde{p}_{best}$) and the actual $p_{best}(m)$ for the diverse and the uniform teams; in Figure 1(b) we show the average $p_{best}(m)$ of the teams. As can be seen, when the best agents increase their d_m at a slower rate than the agents of the diverse team, the uniform teams converge slower to $\tilde{p}_{best}$. Even though they play better than the diverse teams for a small m, they are surpassed by the diverse teams as m increases. However, because $\tilde{p}_{best}$ of the uniform teams is actually higher than the one of the diverse teams, eventually the performance of the uniform teams get closer to the performance of the diverse teams, and will be better than the one of the diverse teams again for a large enough m.

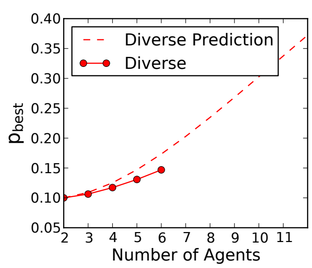


Figure 2: p_{best} of a diverse team as the number of agents increases.

This situation is expected according to Theorem 1. If the d_m of the best agent also increases as m gets larger, the uniform team will actually behave like a diverse team and also converge to $\tilde{p}_{best}$. $\tilde{p}^{uniform}_{best} \geq \tilde{p}^{diverse}_{best}$, as the best agent has a higher probability of playing the optimal action. Hence, in the limit the uniform team will play better than the diverse team. However, as we saw in Corollary 1, the speed of convergence is in the order of $1/d_m$. Therefore, the diverse team will converge faster, and can overcome the uniform team for moderately large m.

As Theorem 2 only holds when $m \to \infty$, we also explore the effect of increasing the number of agents for a large m. The $\tilde{p}_{best}$ of a team of agents is shown as the dashed line in Figure 2. We are plotting for agents that have a probability of playing the best action of only 10%, but as we can see the probability quickly grows as the number of agents increases. We also calculate p_{best} for random teams from 2 to 6 agents (shown as the continuous line), when there are 300 available actions. As can be seen, the teams have a close performance to the expected.

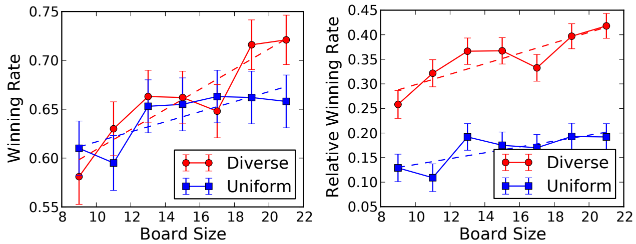
Computer Go

We show now results in a real system. We use 4 different Go software: Fuego 1.1 (Enzenberger et al. 2010), GnuGo 3.8, Pachi 9.01 (Baudiš and Gailly 2011), MoGo 4 (Gelly et al. 2006), and two (weaker) variants of Fuego (FuegoΔ and FuegoΘ), in a total of 6 different, publicly available, agents. Fuego is considered the strongest agent among all of them. The description of FuegoΔ and FuegoΘ is available in the appendix. All our results are obtained by playing either 1000 games (to evaluate individual agents) or 2000 games (to evaluate teams), in a HP dl165 with dual dodeca core, 2.33GHz processors and 48GB of RAM. We compare results obtained by playing against a fixed opponent. Therefore, we evaluate systems playing as white, against the original Fuego playing as black. We removed all databases and specific board size knowledge of the agents, including the opponent. We call Diverse as the team composed of all 6 agents, and Uniform as the team composed of 6 copies of Fuego. Each agent is initialized with a different random seed, therefore they will not vote for the same action all the time in a given world state, due to the characteristics of the search algorithms. In the graphs we show in this section, the error bars show the confidence interval (99% of confidence).

We evaluate the performance of the teams over 7 different board sizes. We changed the time settings of individual agents as we increased the board size, in order to keep their strength as constant as possible. The average winning rates of the team members, and also the winning rates of the individual agents, is available in the appendix. [1]

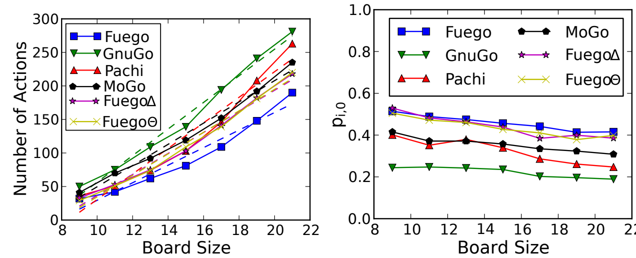
We can see our results in Figure 3 (a). Diverse improves from 58.1% on 9x9 to 72.1% on 21x21, an increase in winning rate that is statistically significant with $p < 2.2 \times 10^{-16}$. This result is expected according to Theorem 1. The Uniform team changes from 61.0% to 65.8%, a statistically significant improvement with $p = 0.0018$. As we saw before, an increase in the performance of Uniform can also be expected, as the best agent might not be a perfect *NST* agent. A linear regression of the results of both teams gives

[1] In our first experiment, Diverse improved from 56.1% on 9x9 to 85.9% on 19x19. We noted, however, that some of the diverse agents were getting stronger in relation to the opponent as the board size increased. Hence, by changing the time setting to keep the strength constant, we are actually making our claims harder to show, not easier.



(a) Absolute winning rates. (b) Relative to the average strength of team members.

Figure 3: Winning rate in the real Computer Go system.



(a) Size of the set of actions that receive a nonzero probability. (b) $p_{i,0}$ as the size of the board grows.

Figure 5: Verifying the assumptions in the real system.

a slope of 0.010 for the diverse team (adjusted R^2: 0.808, $p = 0.0036$) and 0.005 for the uniform team (adjusted R^2: 0.5695, $p = 0.0305$). Therefore, the diverse team improves its winning rate faster than the uniform team. To check if this is a significant difference, we evaluate the interaction term in a linear regression with multiple variables. We find that the influence of board size is higher on Diverse than on Uniform with $p = 0.0797$ (estimated coefficient of "size of the board * group type": -10.321, adjusted R^2: 0.7437). Moreover, on the 9x9 board Diverse is worse than Uniform ($p = 0.0663$), while on the 21x21 board Diverse is better with high statistical significance ($p = 1.941 \times 10^{-5}$). We also analyze the performance of the teams subtracted by the average strength of their members (Figure 3 (b)), in order to calculate the increase in winning rate achieved by "teamwork" and compensate fluctuations on the winning rate of the agents as we change the board size. Again, the diverse team improves faster than the uniform team. A linear regression results in a slope of 0.0104 for Diverse (adjusted R^2: 0.5549, $p = 0.0546$) and 0.0043 for Uniform (adjusted R^2: 0.1283, $p = 0.258$).

We also evaluate the performance of teams of 4 agents (Diverse 4 and Uniform 4). For Diverse 4, we removed FuegoΔ and FuegoΘ from the Diverse team. As can be seen in Figure 4, the impact of adding more agents is higher for the diverse team in a larger board size(21x21). In the 9x9 board, the difference between Diverse 4 and Diverse 6 is only 4.4%, while in 21x21, it is 14%. Moreover, we can see a higher impact of adding agents for the diverse team, than for the uniform team. These results would be expected according to Theorem 2.

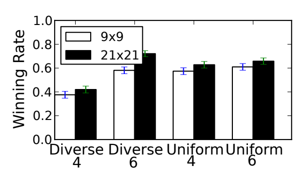


Figure 4: Winning rates for 4 and 6 agents teams.

As can be seen, the prediction of our theory holds: the diverse team improves significantly as we increase the action space. The improvement is enough to make it change from playing worse than the uniform team on 9x9 to playing better than the uniform team with statistical significance on the 21x21 board. Furthermore, we show a higher impact of adding more agents when the size of the board is larger.

Analysis

To test the assumptions of our model, we estimate a pdf for each one of the agents. For each board size, and for each one of 1000 games from our experiments, we randomly choose a board state between the first and the last movement. We make Fuego evaluate the chosen board, but we give it a time limit 50x higher than the default one. Therefore, we use this much stronger version of Fuego to approximate the true ranking of all actions. For each board size, we run all agents in each board sample and check in which position of the approximated true ranking they play. This allow us to build a histogram for each agent and board size combination. Some examples can be seen in the appendix.

We study how the pdfs of the agents change as we increase the action space. Our hypothesis is that weaker agents will have a behavior closer to *ST* agents, while stronger agents to *NST* agents. In Figure 5(a) we show how many actions receive a probability higher than 0. As can be seen, Fuego does not behave exactly like an *NST* agent. However, it does have a slower growth rate than the other agents. A linear regression gives the following slopes: 13.08, 19.82, 19.05, 15.82, 15.69, 16.03 for Fuego, Gnugo, Pachi, Mogo, FuegoΔ and FuegoΘ, respectively (R^2: 0.95, 0.98, 0.94, 0.98, 0.98, 0.98, respectively). It is clear, therefore, that the probability mass of weak agents is distributed into bigger sets of actions as we increase the action space; and even though the strongest agent does not behave in the idealized way, it does have a slower growth rate.

We also verify how the probability of playing the best action ($p_{i,0}$) changes for each one of the agents as the number of actions increase. Figure 5(b) shows that even though $p_{i,0}$ decreases for all agents, it does not decrease much (on average, they decreased about 25% from 9x9 to 21x21).

Conclusion

We present a new model to analyze diversity in teams. It allows us to show that the performance of diverse teams increase as the size of the action space gets larger, and also that diverse teams converge faster than uniform teams. Besides, in large action spaces the performance of a diverse team converges exponentially fast to the optimal one as the number of agents increases. Experimental results with real Computer Go agents match the predictions of our theory.

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