Incentivizing High-Quality Content from Heterogeneous Users: On the Existence of Nash Equilibrium

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Abstract

We study the existence of pure Nash equilibrium (PNE) for the mechanisms used in Internet services (e.g., online reviews and question-answering websites) to incentivize users to generate high-quality content. Most existing work assumes that users are homogeneous and have the same ability. However, real-world users are heterogeneous and their abilities can be very different from each other due to their diversity in background, culture, and profession. In this work, we consider the following setting: (1) the users are heterogeneous and each of them has a private type indicating the best quality of the content he/she can generate; (2) all the users share a fixed total reward. With this setting, we study the existence of pure Nash equilibrium of several mechanisms composed by different allocation rules, action spaces, and information availability. We prove the existence of PNE for some mechanisms and the non-existence for some other mechanisms. We also discuss how to find a PNE (if exists) through either a constructive way or a search algorithm.

Introduction

More and more Internet websites rely on users' contributions to collect high-quality content, including knowledge-sharing services (e.g., Yahoo! Answers and Quora), online product commenting and rating services (e.g., Yelp, mobile app stores), and e-commerce websites (e.g., Amazon.com). For simplicity, we call websites that rely on User-Generated Content *UGC websites*. To attract more users and incentivize them to contribute high-quality content, those sites usually give high-quality contributors some rewards, in terms of either virtual value or monetary return. To collect more rewards, some users might strategically interact with those websites. Therefore, to maximize the quality of the content generated from users, a UGC website needs to carefully design its mechanism and analyze users' behaviors. We call the mechanisms used by those UGC sites *UGC mechanisms*.

Recently, a lot of effort has been placed to the design and analysis of UGC mechanisms (Ghosh and Hummel 2011; Ghosh and McAfee 2011; Easley and Ghosh 2013). For example, (Ghosh and McAfee 2011) designs a simple voting rule under a sequential and simultaneous model, in which both the quality of contributions and the number of contributors are endogenously determined. (Ghosh and Hummel 2011) studies the rank-based allocation mechanism and shows that the mechanism always incentivizes higher quality equilibria than the proportional allocation rule. (Ghosh and Kleinberg 2013) models the online education forums with two parameters which represent the frequency of checking forums by teachers and students separately. A brief survey about UGC mechanisms can be found in (Ghosh 2012).

Most of those works assume that users are homogeneous, i.e., they are of the same ability while contributing to the sites. However, in the real world, users' abilities can be very different from each other due to their diversity in background, culture, and profession. For example, an experienced photographer can write a high-quality comment to a photo, which is very difficult for a non-experienced user. Thus, in this work, we study the game-theoretic problem raised in Internet services with heterogeneous users. We introduce the concept of "type" for the problem, which denotes the ability of a user: the larger the type of a user is, the better content he/she can contribute to the site. We further assume that each user needs to afford a cost to participate in the game and contribute content. The cost reflects the effort of content generation, e.g., time spent on writing a review and the payment for mobile network usage. In our work, we assume the costs are bounded. We believe this assumption is more reasonable and practical than the unbounded-cost assumption used in (Easley and Ghosh 2013; Ghosh and McAfee 2011).

We study two allocation rules: one is the top K allocation rule (Jain, Chen, and Parkes 2009; Easley and Ghosh 2013) in which users with the highest K qualities will share the reward equally; the other is the proportional allocation rule (Ghosh and McAfee 2011; Jain, Chen, and Parkes 2009; Nisan et al. 2007; Chen 2009), in which all the participants (who make non-zero contribution) will share the reward proportionally to their contributed qualities.[1] We consider two action spaces: one is the binary action space, in which each user can only choose to participate in or not; the other is the

[1]See (Kelly 1997; Kelly, Maulloo, and Tan 1998; Cachon and Lariviere 1999; Sandholm and Lai 2010; Stoica et al. 1996; Bachrach, Syrgkanis, and Vojnović 2013) for more details about the proportional allocation rule.

continuous action space, in which each user can choose the quality of his/her contributed content. Besides, we study the problem in both the full-information setting and the partial-information setting.

We study the existence of pure Nash equilibrium for several UGC mechanisms (with different allocation rules and action spaces). Our main results can be summarized as follows.

1. For the *full-information setting*, we prove the existence of PNE for the mechanism with the proportional allocation rule and the continuous action space. The key of the proof is to construct a perturbed game, prove the existence of PNE for the perturbed game, and then prove that the PNE of the perturbed game will converge to the equilibrium of the original game. We then design an efficient algorithm to find a PNE for the mechanism. We also prove that the PNE exists for the mechanism with the top K allocation rule and the binary action space, as well as the mechanism with the proportional allocation rule and the binary action space. However, it does not exist for the mechanism with the top K allocation rule and the continuous action space.
2. For the *partial-information setting*, we prove the existence of a symmetric PNE for the mechanism with the top K allocation rule and the continuous action space. The key of the proof is to construct a simple but (maybe) infeasible symmetric strategy and then convert it to a feasible symmetric equilibrium strategy by repeated calibration. Our proof also provides a method to construct a symmetric PNE. For the binary action space, we prove the existence of equilibrium for the mechanisms with both the top K allocation rule and the proportional allocation rule.

Model

In this section, we describe the model for analyzing the incentives created by various UGC mechanisms, when contributors are strategic agents with heterogeneous abilities.

There is a set of N strategic users in a UGC website, and user i has a private type $q_i \in [0,1]$, which indicates the best quality of the content he/she can contribute to the site. Without loss of generality, we number the users according to the descending order of their types, i.e., $q_1 \geq q_2 \ldots \geq q_N$. Let x_i ($0 \leq x_i \leq q_i$) denote user i's action, which indicates the quality of the content he/she actually contributes to the site, and c_i as the corresponding cost he/she needs to afford. In this work, we consider the linear cost for simplicity: $c_i = c\frac{x_i}{q_i}$, where c is a constant denoting the upper bound of the cost.[2]

We study two action spaces. The first is a binary action space: each user can only choose either not to contribute or to contribute some content with quality q_i (i.e., $x_i \in \{0, q_i\}$). The second is a continuous action space: the quality x_i that user i contributes to the site is a real value between 0 and q_i (i.e., $x_i \in [0, q_i]$). We say that a user does not participate in the game if $x_i = 0$, and a user participates in the game if $x_i > 0$.

[2]If each user has a personalized cost upper bound C_i, we can absorb C_i into the private type q_i by means of scaling: $q_i = \frac{q_i c}{C_i}$.

The site has a fixed total reward R to be allocated to the contributors, depending on their contributions. We study two allocation rules: the top-K allocation (Jain, Chen, and Parkes 2009) and the proportional allocation (Ghosh and McAfee 2011). The former allocates $\frac{R}{K}$ to each of the users who contribute the K largest qualities. Note that if $N < K$, each user can still get $\frac{R}{K}$ reward only. The latter allocates the reward to all users proportionally to their contributions: the reward r_i allocated to user i is $\frac{x_i}{\sum_j x_j}R$ if $x_i > 0$ and 0 otherwise.

While analyzing the model, we consider two settings: the full-information setting and the partial-information setting. In the full-information setting, the types $\{q_i\}_{i\in[N]}$ are deterministic and known to all the users. In the partial-information setting, the type of each user is assumed to be drawn from a publicly known distribution F, which has a continuous first order derivative. Each user only knows his/her own type q_i.

With the above notations, the utility of user i can be written as $u_i(x_i, x_{-i}) = r_i(x_i, x_{-i}) - c_i(x_i)$, where $r(x_i, x_{-i})$ is the reward of user i given his/her own strategy x_i and the strategies x_{-i} of other players. We assume that all the users are rational and target at the maximization of their (expected) utilities.

Full-information Setting

In this section, we study the mechanisms in the full-information setting. In this setting, the type q_i of any user is known to all the users. This setting corresponds to the real-world scenarios where the users are familiar with each other. For example, considering a professional mathematical question posted in Yahoo! Answer, there will be only a few users in the community who can answer the question and they know each other quite well.

Combining the different choices of the allocation rule and the action space, there are four mechanisms of our interest. Due to space restrictions, we will provide detailed elaborations on the mechanism with the proportional allocation rule and the continuous action space only, and directly list the results of the other three mechanisms.

$\mathcal{M}_1$: Proportional Allocation, Continuous Action Space

We first prove the existence of PNE for the mechanism $\mathcal{M}_1$ and then present an algorithm to search its PNE.

Theorem 1 *For the full-information setting, there exists a PNE for the mechanism $\mathcal{M}_1$.*

Proof. Consider an action profile $\{x_i\}_{i\in[N]}$. Denote $x_{-i} = \sum_{j\neq i} x_j$. If $\sum_{i=1}^N x_i > 0$, the utility of user i is

$$u_i(x_i, x_{-i}) = R\frac{x_i}{x_i + x_{-i}} - \frac{x_i}{q_i}c, \tag{1}$$

constrained by $0 \leq x_i \leq q_i$.

The first order derivative of u_i w.r.t x_i is

$$\frac{\partial u_i}{\partial x_i} = R\frac{x_{-i}}{(x_i + x_{-i})^2} - \frac{c}{q_i}. \tag{2}$$

By setting the above derivative to zero, we get the best-response strategy of user i:

$$x_i(q_i, x_{-i}) = \sqrt{\frac{Rq_i x_{-i}}{c}} - x_{-i}. \quad (3)$$

We have the following observations for this best response strategy:

1. Clearly $x_i = 0$ is not the best response when $x_{-i} = 0$ because user i would not be rewarded with $x_i = 0$. Actually there is no best response for user i when $x_{-i} = 0$: if he/she gets a positive utility by contributing $\delta > 0$, he/she will profitably deviate by contributing $\frac{\delta}{2}$. Therefore zero-contribution ($x_i = 0, \forall i$) is not an equilibrium strategy, but it is a fixed point of Eqn. (3).
2. x_i calculated from Eqn. (3) can be smaller than zero or larger than q_i, which is not a feasible action. If $x_i(q_i, x_{-i}) < 0$, it means that Eqn. (2) will be smaller than zero when $x_i > 0$, so it is better to make zero contribution. If $x_i(q_i, x_{-i}) \geq q_i$, Eqn. (2) will be larger than 0, so $u_i(x_i, x_{-i})$ increases w.r.t. x_i, and it is better to contribute q_i.

Based on the above observations, we consider a perturbed game (Feldman, Lai, and Li 2009), in which the action of each user is lower bounded by a small quantity $\epsilon > 0$ and the best-response strategy of i is given as below.

$$x_i^*(q_i, x_{-i}, \epsilon) = \begin{cases} \epsilon & \text{if } x_i(q_i, x_{-i}) \leq \epsilon \\ x_i(q_i, x_{-i}) & \text{if } \epsilon < x_i(q_i, x_{-i}) < q_i \\ q_i & \text{if } x_i(q_i, x_{-i}) \geq q_i \end{cases} \quad (4)$$

where $x_i(\cdot, \cdot)$ is defined in Eqn. (3). In the remaining part of the proof, we show that (1) there exists a PNE for the perturbed game; (2) as $\epsilon \to 0$, no PNE of the perturbed game will converge to a zero-contribution point and (3) by setting $\epsilon \to 0$, we can get the PNE for the original game.

Denote the space $[0, q_1] \times [0, q_2] \times \cdots \times [0, q_N]$ as X, which is convex and compact. Define a mapping f from X to itself, in which for any fixed ϵ, $\forall i \in [N]$, $f_i(\boldsymbol{x}, \epsilon) = x_i^*(q_i, x_{-i}, \epsilon)$. It is easy to verify that f is a continuous mapping. According to Brouwer fixed-point theorem (Border 1989), f has at least one fixed point in X, which is the equilibrium of the perturbed game. Let $\boldsymbol{x}^\epsilon$ denote one of the fixed points. Since X is compact, we could always find a series of $\{\epsilon_n\} \to 0$ whose corresponding $\boldsymbol{x}^{\epsilon_n}$ converge (we denote the limit point as $\boldsymbol{x}^0$).

Next we show $\boldsymbol{x}^0$ is not a zero-contribution point. Otherwise, $\sum_{i=1}^N x_i^{\epsilon_n}$ could be infinitely close to zero as $n \to \infty$. Furthermore, $x_1^{\epsilon_n}$ and $x_2^{\epsilon_n}$ are strictly less than q_1 and q_2 respectively. We set $\sum_{i=3}^N x_i^{\epsilon_n} = \delta_n$ and $Q = \frac{c}{q_1} + \frac{c}{q_2}$. By Eqn. (3) we obtain:

$$\begin{aligned} \sqrt{\frac{Rq_1(\delta_n + x_2^{\epsilon_n})}{c}} - (\delta_n + x_2^{\epsilon_n}) &= x_1^{\epsilon_n} \\ \sqrt{\frac{Rq_2(\delta_n + x_1^{\epsilon_n})}{c}} - (\delta_n + x_1^{\epsilon_n}) &= x_2^{\epsilon_n}. \end{aligned} \quad (5)$$

From the above two equations, we can obtain

$$x_1^{\epsilon_n} + x_2^{\epsilon_n} = \frac{1}{2}\left(\frac{R}{Q} + \sqrt{\left(\frac{R}{Q}\right)^2 + 4\delta_n \frac{R}{Q}}\right) - \delta_n, \quad (6)$$

which will not converge to zero as $n \to \infty$. This contradicts with the assumption that $\sum_{i=1}^N x_i^{\epsilon_n}$ could be infinitely close to zero as $n \to \infty$.

Finally we prove $\boldsymbol{x}^0$ is the equilibrium strategy of the original game by contradiction. There are three possible alternative cases: user j would like to deviate i) from $x_j^0 = 0$ to $x_j^{'} > 0$; ii) from $x_j^0 \in (0, q_j)$ to $x_j^{'} \in [0, q_j] \backslash \{x_j^0\}$; iii) from $x_j^0 = q_j$ to $x_j^{'} < q_j$. We only discuss the first case and the discussions on the other two cases are just similar. Suppose user j could profitably deviate by contributing no less than $\delta (> 0)$. Then we obtain

$$\sqrt{\frac{Rq_j x_{-j}^0}{c}} - x_{-j}^0 \geq \delta. \quad (7)$$

Since $x_j^0 = 0$, given any sufficiently small positive ϵ, we have $\exists N^{'}$, $\forall n > N^{'}$, $x_j^{\epsilon_n} < \epsilon$, i.e.,

$$\sqrt{\frac{Rq_j x_{-j}^{\epsilon_n}}{c}} - x_{-j}^{\epsilon_n} < \epsilon. \quad (8)$$

$\boldsymbol{x}^{\epsilon_n} \to \boldsymbol{x}^0$ implies $x_{-j}^{\epsilon_n} \to x_{-j}^0$. However, as $\epsilon \to 0$, we can verify that $x_{-j}^{\epsilon_n}$ will not converge to x_{-j}^0 by Eqn. (7) and Eqn. (8) (details can be found in the extended version of this paper (Xia et al. 2014)). This contradicts with $\boldsymbol{x}^{\epsilon_n} \to \boldsymbol{x}^0$.

Thus, $\boldsymbol{x}^0$ must be the equilibrium strategy of the original game. ■

Given the existence of PNE, we can leverage the following three lemmas about the properties of an equilibrium profile to find a PNE strategy. Due to space restrictions, we leave the proofs of them to the extended version of this paper (Xia et al. 2014).

Lemma 2 *In an equilibrium profile, a user with a larger type will contribute no less than a user with a smaller type. That is, if $q_i < q_j$, we have $x_i \leq x_j$ in an equilibrium.*

Lemma 3 *Consider two users with $q_i \leq q_j$. If $x_i = q_i$ in an equilibrium, then we also have $x_j = q_j$ in the same equilibrium.*

Given Mechanism $\mathcal{M}_1$, in which N users compete for the reward R, we can induce a *local game* with n users, i.e., only the first n users compete for the reward R. As shown in the following lemma, an equilibrium of the induced local game can be extended to an equilibrium of the original game (i.e., Mechanism $\mathcal{M}_1$ with N users) under certain conditions.

Lemma 4 *If $\{x_i\}_{i \in [n]}$ is an equilibrium of the induced local game with the first n users and $\sum_{i=1}^n x_i \geq \frac{Rq_{n+1}}{c}$, then $\{y_i\}_{i \in [N]}$, where $y_i = x_i, \forall 1 \leq i \leq n$ and $y_i = 0, \forall n+1 \leq i \leq N$, is an equilibrium of the original game.*

Based on the above lemmas, we propose Algorithm 1, which judges whether a PNE of an induced game can be extended to a PNE of the original game, and returns such a PNE if it exists.

Algorithm 1 Algorithm to find a PNE of the original game from an induced local game

Input: $\boldsymbol{q} = (q_1, q_2, \cdots q_n)$ where $q_1 \geq q_2 \geq \cdots \geq q_n$;

Output: $\boldsymbol{x}^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)})$

1: Calculate the y_{ni} $\forall i \in [n]$ with Eqn. (9). If none of y_{ni} is smaller than zero or larger than q_i, $\boldsymbol{x}^{(n)} \leftarrow \boldsymbol{y}_n$; verify whether it can be extended to a PNE of the original game by Lemma 4; if so, return $\boldsymbol{x}^{(n)}$.
2: **for** $m \leftarrow 1 : n$ **do**
3: Calculate x_{nim} $\forall i \in \{m+1, \cdots, n\}$ with Eqn. (13). $x_i^{(n)} \leftarrow q_i$ $\forall i \in \{1, \cdots, m\}$, $x_i^{(n)} \leftarrow x_{nim}$ $\forall i \in \{m+1, \cdots, n\}$ if they are all feasible;
4: **if** $\boldsymbol{x}^{(n)}$ is a local PNE (verified by Lemma 5) **then**
5: Verify whether $\boldsymbol{x}^{(n)}$ can be extended to a PNE of the original game and return it if so;
6: **end if**
7: **end for**

$$y_{ni} = \frac{R(n-1)}{c\sum_{k=1}^{n}\frac{1}{q_k}}\left[1 - \frac{n-1}{q_i\sum_{k=1}^{n}\frac{1}{q_k}}\right] \tag{9}$$

$$Q_m = \sum_{i=1}^{m} q_i \tag{10}$$

$$A_{nm} = \sum_{i=m+1}^{n} \frac{c}{Rq_i} \qquad \forall n \leq N, n \geq m+1 \tag{11}$$

$$s_{nm} = \frac{(n-m-1) + \sqrt{(n-m-1)^2 + 4Q_m A_{nm}}}{2A_{nm}} \tag{12}$$

$$x_{nim} = s_{nm} - \frac{c}{Rq_i} s_{nm}^2 \tag{13}$$

Lemma 5 *In Algorithm 1, $\boldsymbol{x}^{(n)}$ is a PNE of the induced local game with n users if the following condition holds:*

- *If $m < n$,*

$$\frac{Rq_1}{2c}\left(1 - \sqrt{1 - \frac{4c}{R}}\right) \leq \sum_{k=1}^{n} x_i^{(n)} \leq \frac{Rq_m}{2c}\left(1 + \sqrt{1 - \frac{4c}{R}}\right);$$

- *If $m = n$,*

$$R \geq \frac{cQ_m^2}{q_i(Q_m - q_i)},\ \forall i \in \{1, m\}.$$

Theorem 6 *Recursively calling Algorithm 1 from $n = 2$ to N, one obtains a PNE of an induced local game, which can be extended to a PNE of the original game.*

Proof sketch. Accordingly to Theorem 1, the induced local game has a PNE.

For $n \leq N$, if the algorithm stops at Step 1, then $\boldsymbol{x}^{(n)}$ can be extended to a PNE of the original game. Otherwise, given the top m users contributing their types, Eqn. (13) could be seen as the strategies that users $m+1, \cdots, n$ do not want to deviate if they are feasible. If the top m users do not want to deviate either, we find a PNE of the induced local game. Lemma 2 and 3 describe the structure of all the PNEs, and therefore Algorithm 1 will traverse all the local PNEs of the induced local game. Furthermore, we note that a PNE of the original game is also a PNE of some induced local game. So the PNE of the original game could certainly be found by Algorithm 1. ■

Other Three Mechanisms

Due to space limitation, we list the results of the other three mechanisms under the full-information setting in this subsection. The detailed analysis can be found in (Xia et al. 2014).

- Top-K allocation rule, binary action space: (1) If $R < Kc$, zero contribution (i.e., no participation) is the unique equilibrium. (2) If $R = Kc$, the contribution of any K users can be an equilibrium. (3) If $R > Kc$, the contribution from the first K users is the unique equilibrium.
- Top-K allocation rule, continuous action space: No equilibrium exists.
- Proportional allocation rule, binary action space: If $R < c$, zero contribution ($x_i = 0, \forall i$) is the unique equilibrium; otherwise, there exist multiple equilibria.

Partial-information Setting

In this section, we investigate the existence of PNE for the UGC mechanisms under the partial-information setting. We discuss three mechanisms here and leave the fourth one to the future work due to its complication.

$\mathcal{M}_2$: Top-K Allocation, Continuous Action Space

We first give a general description to a symmetric equilibrium strategy, then prove the existence of PNE when the distribution F is uniform, finally generalize the result to any distribution.

Let us consider a symmetric increasing strategy $\beta(\cdot)$: each user i with quality q_i will contribute to the site with quality $\beta(q_i)$.

Lemma 7 *If $\beta(\cdot)$ is an equilibrium strategy, then $q_i > 0 \Rightarrow \beta(q_i) > 0$.*

Proof. It is easy to see that for a given equilibrium strategy $\beta(\cdot)$, if $q_i < q_j$ and $\beta(q_i) > 0$, we have $\beta(q_j) > 0$. Otherwise, user j can contribute $x_j = \beta(q_i) + \delta$, where δ is positive and sufficiently close to zero, to get a positive utility, which contradicts with that $\beta(\cdot)$ is an equilibrium strategy.

Suppose that there exists some $q > 0$ such that $\beta(q) = 0$. As a result, we have $\beta(x) = 0, \forall x \in [0, q]$. For a user whose type falls in $[0, q]$, if he/she contributes ϵ, his/her expected utility is

$$\frac{R}{K}\sum_{n=N-K}^{N-1}\binom{N-1}{n}F(q)^n(1-F(q))^{N-n-1} - \frac{\epsilon}{q_i}c \tag{14}$$

We can always find an ϵ that is small enough so as to ensure the above expected utility to be positive. Therefore $\beta(q_i)$ is not an equilibrium, which leads to a contradiction. Thus, there does not exist a $q > 0$ such that $\beta(q) = 0$. ■

Let function $T(x)$ denote the probability that a user with type x wins the game (e.g., he/she is one of the top K contributors) given all the users adopt the same increasing strategy $\beta(\cdot)$. We have

$$T(x) = \sum_{j=0}^{K-1} \binom{N-1}{j} F(x)^{N-1-j}(1-F(x))^j. \quad (15)$$

Suppose that users $j \neq i$ follow the increasing symmetric equilibrium strategy $\beta(\cdot)$. If user i pretends that his/her type is x and contributes $\beta(x)$, then his/her expected utility is

$$u_i(x; q_i) = \frac{R}{K}T(x) - \frac{\beta(x)}{q_i}c. \quad (16)$$

The first order derivative of u_i is[3]

$$\frac{\partial u_i(x;q_i)}{\partial x} = \frac{R}{K}\frac{\partial T(x)}{\partial x} - \frac{c\beta^{'}(x)}{q_i}. \quad (17)$$

If $\beta(q_i)$ is an equilibrium strategy for user i, his/her expected utility should be maximized at $x = q_i$. That is, we should have

$$\frac{\partial u_i(x;q_i)}{\partial x}|_{x=q_i} = 0. \quad (18)$$

Note that $\beta(0) = 0$. Solving the above equation, we get

$$\beta(x) = \frac{R(N-1)}{cK}\binom{N-2}{K-1}\int_0^x tF(t)^{N-K-1}(1-F(t))^{K-1}\mathrm{d}F(t) \quad (19)$$

Then we have the following results.

Lemma 8 *If $\beta(x) \leq x, \forall x \in [0,1]$, then the function $\beta(\cdot)$ in the above equation is an increasing equilibrium strategy.*

However, it is possible that $\beta(x)$ expressed by Eqn. (19) is larger than x. For example, if F is the uniform distribution over $[0,1]$, $\beta(x)$ can be written as below.

$$\beta(x) = \frac{R}{cK}(N-1)\binom{N-2}{K-1}\sum_{k=0}^{K-1}(-1)^{K-k-1}\binom{K-1}{k}\frac{x^{N-k}}{N-k} \quad (20)$$

Then we have

$$\beta(1) = \frac{R}{cK}\frac{N-K}{N}, \quad (21)$$

which might be larger than 1.

If $\beta(x) > x$ for some $x \in [0,1]$, $\beta(x)$ will not be an equilibrium strategy anymore. We need to calibrate $\beta(x)$. For ease of description, we first illustrate how to make calibration when F is the uniform distribution, and then extend the result to any distribution.

With some derivations, one can get that the equation $\beta(x) = x$ has at most two solutions in the region $(0,1]$ when the distribution F is uniform. If there exist two solutions (denoted as x_1 and x_2 ($x_1 < x_2$)), there will be an $x_p(> x_1)$ that satisfies $\beta^{'}(x_p) = 1$. Then we have:

[3]The computation of $\frac{\partial T(x)}{\partial x}$ could be found in (Xia et al. 2014).

Theorem 9 *If F is the uniform distribution over $[0,1]$ and $\beta(x) = x$ has two solutions in $(0,1]$, the following $\beta^*(\cdot)$ function is an equilibrium, where x_1 and x_p are defined above.*

$$\beta^*(x) = \begin{cases} \beta(x) & x \in [0, x_1] \\ x & x \in (x_1, x_p] \\ \beta(x) - \beta(x_p) + x_p & x \in (x_p, 1] \end{cases} \quad (22)$$

Proof. First, suppose $x \in [0, x_1]$. Since $\beta(x) \leq x$, we have that $\beta^*(x) = \beta(x)$ is the best response of type x.

Second, it is clear that the first order derivative $\beta^{'}(x)$ is larger than 1 for any $x \in (x_1, x_p)$. Suppose that all other users follow strategy $\beta^*(\cdot)$ except user i with $q_i \in (x_1, x_p)$, and suppose he/she pretends to have a type x.

- If $x \in (x_1, x_p)$ and $x \leq q_i$, we have

  $$u_i(x; q_i) = \frac{R}{K}T(x) - \frac{x}{q_i}c,$$

  and

  $$\frac{\partial u_i(x;q_i)}{\partial x} > \frac{R}{K}\frac{\partial T(x)}{\partial x} - \frac{\beta^{'}(x)}{q_i}c \geq \frac{R}{K}\frac{\partial T(x)}{\partial x} - \frac{\beta^{'}(x)}{x}c = 0.$$

  Therefore, the larger x is, the larger utility he/she will get. However, since the contributed quality is upper bounded by his/her type q_i, the best choice for the user is to take the action $x_i = q_i$.
- If $x \in [0, x_1]$, we have

  $$\frac{\partial u_i(x;q_i)}{\partial x} = \frac{R}{K}\frac{\partial T(x)}{\partial x} - \frac{\beta^{'}(x)}{q_i}c > \frac{R}{K}\frac{\partial T(x)}{\partial x} - \frac{\beta^{'}(x)}{x}c = 0.$$

  So the user should pretend to have a type x_1, which is, however, still worse than revealing the true type q_i.

Thus, for any x in $(x_1, x_p]$, the best response is $\beta^*(x) = x$.

Third, for any x in $(x_p, 1]$, we have $\beta^{'}(x) \leq 1$. Integrating $\beta^{'}(x)$ from x_p to x and using $\beta(x_p) = x_p$, we get

$$\beta^*(x) - x_p = \beta(x) - \beta(x_p).$$

It is easy to verify that $\beta^*(x) \leq x$ for any x in $(x_p, 1]$. Therefore, we have that $\beta^*(x) = \beta(x) + x_p - \beta(x_p)$ is the best response for any x in $(x_p, 1]$.

Thus, the theorem is proved. ■

Figure 1 shows an equilibrium strategy for $N = 11, K = 5, c = 1, R = 8$.

Next we generalize the above results. For a general distribution F over $[0,1]$, we first initialize $\beta^*(x) = \beta(x), \forall x \in [0,1]$ and then calibrate $\beta^*(x)$ as follows.

1. Check whether $\beta^*(x) > x$ starting from $x = 0$ to $x = 1$.
2. Suppose $[x_1, x_2]$ is the first interval that $\beta^*(x) > x$, and x_p is the point in this interval satisfying $\beta^{'*}(x_p) = 1$. Let o denote the value of $\beta^*(x)$ at x_p (i.e., $o = \beta^*(x_p)$), and then calibrate $\beta^*(x) = x, \forall x \in [x_1, x_p]$ and $\beta^*(x) = \beta^*(x) - o + x_p, \forall x \in (x_p, 1]$.
3. Continue to check whether $\beta^*(x) > x$ starting from $x = x_p$ to $x = 1$. If there is still some interval with $\beta^*(x) > x$, we calibrate $\beta^*(x)$ as shown in the previous step.

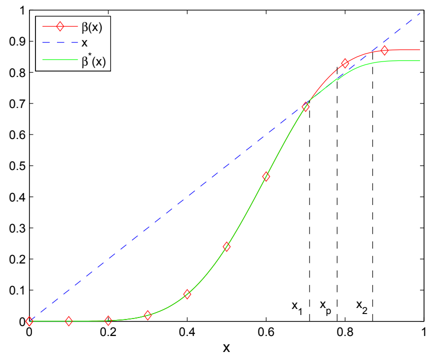

Figure 1: An example equilibrium strategy

4. We repeat the checking and calibrating procedure until $\beta^*(x) \le x, \forall x \in [0, 1]$.

After the calibration process, we obtain an equilibrium strategy $\beta^*(x)$ from $\beta(x)$, which is shown in Eqn. (19), for any distribution F. Therefore we have the following theorem.

Theorem 10 *$\mathcal{M}_2$ has at least one symmetric PNE.*

$\mathcal{M}_3$: Proportional Allocation, Binary Action Space

Now we study the existence of PNE for the mechanism with the proportional allocation rule and the binary action space in the partial-information setting.

For user i, let us consider such a symmetric cut-off equilibrium strategy (Fudenberg and Tirole 1991):

$$\beta_i(x) = \begin{cases} q_i & \text{if } x \ge x^* \\ 0 & \text{if } x < x^*, \end{cases} \tag{23}$$

where x^* is a threshold parameter. Suppose that users $j \neq i$ follow the above strategy. Then the expected utility of user i can be written as follows if he/she participates in the game $(x_i = q_i)$.

$$u_i(q_i; x^*) = \sum_{k=0}^{N-1} \binom{N-1}{k} F(x^*)^{N-1-k} (1 - F(x^*))^k u_i(q_i, k; x^*) - c \overset{def}{=} y(q_i, x^*) - c, \tag{24}$$

In the above equation, $u_i(q_i, k; x^*)$ is the expected utility of user i given another k users participating in the game whose qualities are larger than x^*. This can be written as

$$R \int_{x^*}^{1} \ldots \int_{x^*}^{1} \frac{q_i}{q_i + t_1 + t_2 + \ldots + t_k} \mathrm{d}F(t_1|x^*) \ldots \mathrm{d}F(t_k|x^*),$$

where $F(t|x^*)$ represents the conditional probability distribution given one's type is larger than x^*. In Eqn. (24), when $k = 0$, $u_i(q_i, k; x^*) = R$, which means that user i gets all the reward R since no other user participates in the game ($k = 0$ means $x_j = 0, \forall j \neq i$).

If Eqn. (23) is an equilibrium strategy, the best response of user i is also to follow the strategy given that all the other users follow the strategy. That is,

$$u_i(q_i; x^*) = \begin{cases} > 0 & \text{if } q_i \ge x^* \\ = 0 & \text{if } q_i = x^* \\ < 0 & \text{if } x < x^* \end{cases} \tag{25}$$

It is not difficult to get that

- $y(q_i, x^*)$ increases w.r.t. q_i,
- $y(0, 0) - c = -c < 0$ and $y(1, 1) - c = R - c > 0$.

We further assume that $F(\cdot|\cdot)$ is a continuous function; consequently, $y(t, t)$ is continuous. Therefore, there exists an x^* satisfying the three conditions in Eqn. (25); in turn, this x^* makes Eqn. (23) a (symmetric) equilibrium strategy. Thus we have the following theorem.

Theorem 11 *$\mathcal{M}_3$ has at least one PNE if $R > c$.*

$\mathcal{M}_4$: Top-K Allocation, Binary Action Space

We can construct a symmetric equilibrium using a similar method to $\mathcal{M}_3$, which is also a cut-off strategy for everyone:

Theorem 12 *Denote the unique root of the following equation as x^*.*

$$\sum_{n=N-K}^{N-1} \binom{N-1}{n} F(x)^n (1 - F(x))^{N-1-n} = \frac{cK}{R} \tag{26}$$

$\forall i \in [N]$, we have

$$\beta(q_i) = \begin{cases} q_i & \text{if } q_i \ge x^* \\ 0 & \text{if } q_i < x^* \end{cases} \tag{27}$$

is an equilibrium strategy of $\mathcal{M}_4$.

Conclusions and Future work

In this paper, we have studied the UGC mechanisms under a new framework: users are heterogeneous and the best qualities users can contribute are different from each other. Under this framework, we have considered several mechanisms involving two allocation rules, two action spaces, and two information settings. We proved the existence of multiple PNE for some mechanisms, the existence and uniqueness of PNE for some mechanisms, and the non-existence of PNE for some other mechanisms.

As for the future work, there are quite a few interesting points worth investigating. First, the study on the mechanism with the proportional allocation rule and the continuous action space is missing for the partial-information setting in this paper. We plan to investigate on this setting in the future. Second, we plan to conduct the efficiency analysis for those mechanisms whose equilibria are proven to exist. Third, we plan to study the mixed Nash equilibrium for the UGC mechanisms. Fourth, we will investigate more general cost functions (e.g., concave functions). Fifth, we will make comparisons between different UCG mechanisms and identify the best one for practical use.

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References

Bachrach, Y.; Syrgkanis, V.; and Vojnović, M. 2013. Incentives and efficiency in uncertain collaborative environments. In *Web and Internet Economics*. Springer. 26–39.

Border, K. 1989. *Fixed point theorems with applications to economics and game theory*. New York, NY, USA: Cambridge University Press.

Cachon, G. P., and Lariviere, M. A. 1999. An equilibrium analysis of linear, proportional and uniform allocation of scarce capacity. *IIE Transactions* 31(9):835–849.

Chen, D. Y.-C. 2009. *Essays on mobile advertising and commerce*. Ph.D. Dissertation, Harvard University Cambridge, Massachusetts.

Easley, D., and Ghosh, A. 2013. Incentives, gamification, and game theory: an economic approach to badge design. In *Proceedings of the fourteenth ACM conference on Electronic commerce*, EC '13, 359–376.

Feldman, M.; Lai, K.; and Li, Z. 2009. The proportional-share allocation market for computational resources. *IEEE Trans. Parallel Distrib. Syst.* 20(8):1075–1088.

Fudenberg, D., and Tirole, J. 1991. *Game theory*. MIT press.

Ghosh, A., and Hummel, P. 2011. A game-theoretic analysis of rank-order mechanisms for user-generated content. In *Proceedings of the 12th ACM conference on Electronic commerce*, EC '11, 189–198.

Ghosh, A., and Kleinberg, J. 2013. Incentivizing participation in online forums for education. In *Proceedings of the fourteenth ACM conference on Electronic commerce*, EC '13, 525–542.

Ghosh, A., and McAfee, P. 2011. Incentivizing high-quality user-generated content. In *Proceedings of the 20th international conference on World wide web*, WWW '11, 137–146.

Ghosh, A. 2012. Social computing and user-generated content: a game-theoretic approach. *SIGecom Exch.* 11(2):16–21.

Jain, S.; Chen, Y.; and Parkes, D. C. 2009. Designing incentives for online question and answer forums. In *Proceedings of the 10th ACM conference on Electronic commerce*, EC '09, 129–138.

Kelly, F.; Maulloo, A.; and Tan, D. 1998. Rate control in communication networks: shadow prices, proportional fairness and stability. In *Journal of the Operational Research Society*.

Kelly, F. 1997. Charging and rate control for elastic traffic. *European transactions on Telecommunications* 8(1):33–37.

Nisan, N.; Roughgarden, T.; Tardos, E.; and Vazirani, V. V. 2007. *Algorithmic Game Theory*. New York, NY, USA: Cambridge University Press.

Sandholm, T., and Lai, K. 2010. Dynamic proportional share scheduling in hadoop. In *Job scheduling strategies for parallel processing*, 110–131. Springer.

Stoica, I.; Abdel-Wahab, H.; Jeffay, K.; Baruah, S.; Gehrke, J.; and Plaxton, C. 1996. A proportional share resource allocation algorithm for real-time, time-shared systems. In *Real-Time Systems Symposium, 1996., 17th IEEE*, 288–299.

Xia, Y.; Qin, T.; Yu, N.; and Liu, T.-Y. 2014. Incentivizing high-quality content from heterogeneous users: On the existence of nash equilibrium. *arXiv preprint arXiv:1404.5155*.