Two Case Studies for Trading Multiple Indivisible Goods with Indifferences

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Abstract

Individual rationality, Pareto efficiency, and strategy-proofness are crucial properties of decision making functions, or *mechanisms*, in social choice literatures. In this paper we investigate mechanisms for exchange models where each agent is initially endowed with a set of goods and may have indifferences on distinct bundles of goods, and monetary transfers are not allowed. Sönmez (1999) showed that in such models, those three properties are not compatible in general. The impossibility, however, only holds under an assumption on preference domains. The main purpose of this paper is to discuss the compatibility of those three properties when the assumption does not hold. We first establish a preference domain called *top-only preferences*, which violates the assumption, and develop a class of exchange mechanisms that satisfy all those properties. Each mechanism in the class utilizes one instance of the mechanisms introduced by Saban and Sethuraman (2013). We also find a class of preference domains called m*-chotomous preferences*, where the assumption fails and these properties are incompatible.

1 Introduction

The *housing market problem* (Shapley and Scarf 1974) is a fundamental exchange model where each agent is initially endowed with an indivisible good, say a house, and monetary transfers are not allowed. When agents' preferences over the goods are strict, Gale's *top-trading-cycles* (TTC) algorithm returns a unique core. Furthermore, an exchange mechanism based on the TTC algorithm is the only one that satisfies three desirable properties; individual rationality, Pareto efficiency, and strategy-proofness (Ma 1994).

Considering the multiple endowment of agents is a natural extension of the housing market problem. Since agents can have complicated preferences for the situations, designing Pareto efficient exchange mechanisms is not straightforward in the model. Pápai (2003) proposed a class of exchange mechanisms that satisfy individual rationality and strategy-proofness along with a weaker notion of efficiency. Konishi, Quint, and Wako (2001) dealt with a model where there is more than one type of goods, say houses and cars, and showed that the core may be empty in their model.

Introducing indifferences in agents' preferences is another extension. In realistic exchange models, agents' preferences over goods are not always strict, that is, some agent may be indifferent about getting one of two distinct bundles of goods. Concerning the existence of such indifferences, much progress has been made in the past half decade (Alcalde-Unzu and Molis 2011; Jaramillo and Manjunath 2012; Aziz and de Keijzer 2012; Plaxton 2013). Particularly, Saban and Sethuraman (2013) provided a class of exchange mechanisms that satisfy all three properties.

In this paper, we investigate an exchange model where each agent initially owns multiple indivisible goods and agents' preferences contain indifferences; i.e., we consider both of these two extensions together. Our main objective is to discuss the compatibility of individual rationality, Pareto efficiency, and strategy-proofness for our model. To the best of our knowledge, there has been very little research dealing with this model, even though it can reflect many realistic trading situations.

Sönmez (1999) is one notable work whose situation closely resembles ours. His elegant characterization of the relationships between preference structures and the core implies that, when at least one agent has more than one good, the above three properties are not compatible in general. This impossibility result, however, crucially depends on an assumption that for every agent, any bundle distinct from its initial endowment is either strictly better or strictly worse than the initial endowment under her preference. Therefore, a broad class of preference domains remains where the assumption does not hold, and thus it has not been clarified whether these three properties are compatible.

Our results complement the findings of Sönmez (1999). We establish a domain of agents' preferences called *top-only preferences* that actually violates the assumption on preferences. We show that under the domain, these three properties are compatible, i.e., there is an exchange mechanism that is individually rational, Pareto efficient, and strategy-proof. Indeed, we propose a class of such exchange mechanisms based on Saban and Sethuraman (2013), and to guarantee Pareto efficiency we need to implement an appropriate demand update scheme. We also find a class of preference domains called m*-chotomous preferences*, under which the three properties cannot be simultaneously satisfied by any exchange mechanism.

2 Preliminaries

Let N be the set of n agents and K be the set of indivisible goods in a market. Each agent $i \in N$ is initially endowed with subset $w_i \subseteq K$ of indivisible goods, or an *endowment*. Assume that goods are heterogeneous, i.e., $w_i \cap w_j = \varnothing$ for any pair $i, j(\neq i) \in N$, and $\bigcup_{i \in N} w_i = K$. Let $w = (w_i)_{i \in N}$ denote an *endowment distribution* to N.

Let $\mathcal{X}_N$ denote the set of all *feasible assignments* of goods K to agents N. Assignment $x \in \mathcal{X}_N$ is feasible if it is a distribution of K to N, i.e., $x_i \cap x_j = \varnothing$ for any $i, j(\neq i) \in N$ and $\bigcup_{i \in N} x_i = K$, where x_i is a bundle assigned to agent i under x. Note that any endowment distribution w is regarded as an assignment, and the set of all possible endowment distributions coincides with $\mathcal{X}_N$.

Each agent i also has *preference relation* R_i over the set of all possible subsets of K. We assume R_i is complete, transitive, and reflexive. For any pair of two subsets $L, L' \subseteq K$, $L R_i L'$ means that L is weakly preferred to L' under preference R_i. Let P_i and I_i respectively indicate the strict and indifferent components of R_i, so that for any pair $L, L' \subseteq K$, $L R_i L'$ means either $L P_i L'$ or $L I_i L'$. Let $\mathscr{R}$ indicate the set of all possible preferences. Let $R = (R_i)_{i \in N} \in \mathscr{R}^n$ denote a *preference profile* of agents N, and $R_{-i} = (R_j)_{j \neq i} \in \mathscr{R}^{n-1}$ denote a preference profile of agents excluding i.

Now we formally define *exchange mechanisms*. Exchange mechanism $\varphi : \mathcal{X}_N \times \mathscr{R}^n \to \mathcal{X}_N$ assigns feasible assignment $\varphi(w, R)$ to each pair (w, R) of endowment distribution w and preference profile R, where $\varphi_i(w, R)$ denotes the bundle assigned to agent i by exchange mechanism φ under assignment $\varphi(w, R)$.

Definition 1 (Individual Rationality). *Under endowment distribution $w \in \mathcal{X}_N$, assignment $x \in \mathcal{X}_N$ is individually rational if $x_i R_i w_i$ holds for any $i \in N$. Exchange mechanism φ is* individually rational (IR) *if $\forall w \in \mathcal{X}_N$, $\forall R \in \mathscr{R}^n$, $\varphi(w, R)$ is individually rational under w.*

In other words, for every agent, participating the exchange mechanism is better than not participating as long as she reports her preference truthfully. Under an IR exchange mechanism, every agent has a natural incentive to participate.

Definition 2 (Pareto Efficiency). *Assignment $y \in \mathcal{X}_N$ Pareto dominates another $x \in \mathcal{X}_N$ at preference profile R if $\forall i \in N$, $y_i R_i x_i$, and $\exists j \in N$, $y_j P_j x_j$. Exchange rule φ is* Pareto efficient (PE) *if $\forall w \in \mathcal{X}_N$, $\forall R \in \mathscr{R}^n$, any $y \in \mathcal{X}_N$ does not Pareto dominate $\varphi(w, R)$ at R.*

A Pareto efficient exchange mechanism is "optimal" in the sense that it is impossible to make any agent better off without making at least one agent worse off. When assignment $y \in \mathcal{X}_N$ Pareto dominates another assignment $x \in \mathcal{X}_N$ at R, we represent the relation as $y \rightharpoonup_R x$.

Definition 3 (Strategy-proofness). *Exchange rule φ is* strategy-proof (SP) *if $\forall w \in \mathcal{X}_N$, $\forall R \in \mathscr{R}^n$, $\forall i \in N$, $\forall R'_i \in \mathscr{R}$, it holds that $\varphi_i(w, R) R_i \varphi_i(w, (R'_i, R_{-i}))$.*

Under an SP exchange mechanism, for every agent, reporting her true preference is a dominant strategy. Furthermore, from the revelation principle, focusing on SP exchange mechanisms is without loss of generality if we are only interested in markets with dominant strategy equilibria.

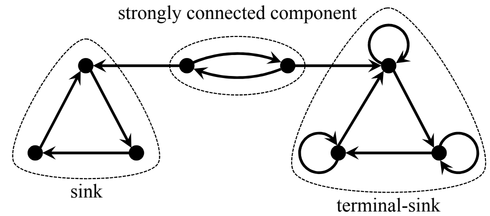

Figure 1: Terminal-sink in directed graph

3 Related Works

Housing Market Problem: The housing market problem (Shapley and Scarf 1974) is represented as setting $|K| = n$ and $|w_i| = 1$ for any $i \in N$. In housing market problem, agents' preferences are usually assumed to be *strict*, such that for any two distinct bundles, $L, L'(\neq L) \subseteq K$, and any preference R_i, either $L P_i L'$ or $L' P_i L$ holds. It has been known that Gale's *top-trading-cycles* algorithm (TTC) always returns the unique core assignment. Furthermore, Ma (1994) showed that TTC is the only exchange mechanism that is simultaneously IR, PE, and SP for the housing market problem.

Impossibility for Multiple Goods: Sönmez (1999) extended the housing market problem to more general situations where each agent's initial endowment is not restricted to a single good, which is called *indivisible goods exchange*. In the paper he introduced the following assumption on agents' preferences:

Assumption 1 (Assumption A (Sönmez 1999)). *Given $w \in \mathcal{X}_N$, for any $i \in N$, any $R_i \in \mathscr{R}$, and any $x \in \mathcal{X}_N$, $x_i I_i w_i$ if and only if $x_i = w_i$.*

In other words, an agent is indifferent between an assignment and her initial endowment if and only if she keeps her initial endowment. For example, the assumption holds when all preferences are *strict*, e.g., the housing market problem. Sönmez showed that under Assumption 1 (and another richness assumption), there exists no exchange mechanism that is IR, PE, and SP when $|K| > n$ holds, i.e., when at least one agent is endowed with more than one good. However, the existence of such mechanisms satisfying IR, PE, and SP has not been clarified under preference domains where Assumption 1 does not hold.

Preferences with Indifferences: In the last half decade, many researches have dealt with a modified housing market problem where agents' preferences can have indifferences. Their main research interest is to construct exchange mechanisms that satisfy all three properties. Saban and Sethuraman (2013) proposed a class of exchange mechanisms that satisfy IR and PE. Their class contains many previously proposed mechanisms, such as top-cycles-rule (Jaramillo and Manjunath 2012) and top trading absorbing sets (Alcalde-Unzu and Molis 2011). Furthermore, they show a condition called *local invariance*, which is a necessary and sufficient condition for those mechanisms to be SP.

4 Proposed Mechanisms

Our objective in this paper is to discuss preference structures under which the above three properties can be simultaneously satisfied, instead of developing new exchange mechanisms. However, the most natural and direct way to show the compatibility is to develop such mechanisms. Therefore, in this section we propose a new class of exchange mechanisms for our exchange model with multiple goods. The basic idea is natural; each agent $i \in N$ is replaced by $|w_i|$ atomic agents $\{i_k | k \in w_i\}$, each of which is endowed with good $k \in w_i$ and agent i's preference R_i. Now that the augmented market is a housing market problem with indifferences, run the IR, PE, and SP exchange mechanism proposed by Saban and Sethuraman (2013). Note that, since agents (not atomic agents) may have more than one good in this model, each agent's demand must be appropriately updated during the algorithm.

We next introduce some terms for directed graphs. A directed graph is *strongly connected* if there is a path from each node in the graph to every other node. The *strongly connected components* of a directed graph are its maximal strongly connected subgraphs. Strongly connected component S is a *sink* if all the neighbors of each node in S are also in S. A sink is a *terminal-sink* if all the nodes in it have an edge pointing to themselves. In Fig. 1, all the components circled by dotted lines are strongly connected, and the rightmost one is the only terminal-sink.

We also briefly describe three functions TTC-GRAPH, TERMINAL-SINK, and CYCLE. Given a pair of a set N' of agents and a profile $T = (T_i)_{i \in N} \in (2^K)^n$ of bundles, TTC-GRAPH returns directed graph G constructed from N' and T. Formally, for every remaining atomic agent $i_k \in N'$ and for each atomic agent $i'_{k'}$ currently assigned one good from T_i, add an edge from node i_k to node $i'_{k'}$ to graph G. For a given G, TERMINAL-SINK returns a set of nodes that are in a terminal-sink in G. For a given G, CYCLE returns a set of all edges that are in a cycle in G.

Now we are ready to describe our proposed mechanisms, whose formal description is given in Algorithm 1. At lines 2–8, each agent i is replaced by $|w_i|$ atomic agents $\{i_k | k \in w_i\}$, where N' indicates the set of all atomic agents. Lines 9–25 basically correspond to the procedure of the mechanisms proposed by Saban and Sethuraman (2013), except for lines 16–18. At line 9, the loop terminates if $N' = \varnothing$, and go to line 26. Otherwise, at line 10, directed graph G is constructed by the TTC-GRAPH function for given N' and T. At lines 13–14, for each terminal-sink found in the graph, each atomic agent i_k is removed from the market, with the good ω_{i_k} currently assigned to it. Each atomic agent that is still in the market updates its demand by the UPDATE-TOPS function, and directed graph G is re-drawn at lines 16–19, and re-run TERMINAL-SINK. At line 21, for each node (atomic agent), the F-RULE chooses only one outgoing edge in some manner. At lines 22–24, each atomic agent i_k that is included in a cycle is assigned good $\omega'_{i'_{k'}}$ to which she is currently pointing, and go to line 9. At line 26, each agent $i \in N$ is given set of goods $\phi_i = \bigcup_{k \in w_i} \omega_{i_k}$ that were removed from the market with its atomic agents.

Algorithm 1 New Trading Family

Input: N, K, w, R
Output: $\varphi = (\varphi_i)_{i \in N}$
1: $N' \leftarrow \varnothing$, $K' \leftarrow K$
2: **for** $i \in N$ **do**
3:     $\phi_i \leftarrow \varnothing$
4:     $T_i \leftarrow$ UPDATE-TOPS(R_i, ϕ_i, K')
5:     **for** $k \in w_i$ **do**
6:         $N' \leftarrow N' \cup \{i_k\}$, $\omega_{i_k} \leftarrow k$
7:     **end for**
8: **end for**
9: **while** $N' \neq \varnothing$ **do**
10:     $G \leftarrow$ TTC-GRAPH(N', T)
11:     **while** TERMINAL-SINK$(G) \neq \varnothing$ **do**
12:         **for** $i_k \in$ TERMINAL-SINK(G) **do**
13:             $\phi_i \leftarrow \phi_i \cup \{\omega_{i_k}\}$
14:             $N' \leftarrow N' \setminus \{i_k\}$, $K' \leftarrow K' \setminus \{\omega_{i_k}\}$
15:         **end for**
16:         **for** $i \in N$ **do**
17:             $T_i \leftarrow$ UPDATE-TOPS(R_i, ϕ_i, K')
18:         **end for**
19:         $G \leftarrow$ TTC-GRAPH(N', T)
20:     **end while**
21:     $G \leftarrow F$-RULE(G), $\omega' \leftarrow \omega$
22:     **for** $(i_k, i'_{k'}) \in$ CYCLE(G) **do**
23:         $\omega_{i_k} \leftarrow \omega'_{i'_{k'}}$
24:     **end for**
25: **end while**
26: **return** $\varphi = (\phi_i)_{i \in N}$

Now we introduce the UPDATE-TOPS function, which is one key feature of our proposed mechanisms. When an agent has more than one good as her endowment, her demand varies during the algorithm based on the bundles she has obtained so far; "I still want good a since I've never gotten anything" and "I no longer want a because I just got another b." To guarantee Pareto efficiency, UPDATE-TOPS(R_i, ϕ_i, K') prescribes the set of goods agent i *currently* wants among remaining goods K', under her preference R_i and the bundle ϕ_i she has gotten so far. Its formal description, which depends on the structure of preference domains, will be given in the following sections.

F-RULE is a "tie-breaking" scheme that chooses only one good from given set T_i. As previously discussed (Saban and Sethuraman 2013), F-RULE must be determined carefully so that the algorithm eventually terminates and strategy-proofness holds. To make the description in this paper self-contained, we decided to use an F-RULE originally used in the *top trading absorbing sets* (TTAS) mechanism (Alcalde-Unzu and Molis 2011).

> *A priority ordering over K is given. For each atomic agent i_k, choose good $g \in T_i$ with the highest priority that has not been assigned yet to i_k. If all goods in T_i have been assigned to it at least m times, choose the good with highest priority among those that have not been assigned to her $m + 1$ times yet.*

5 Top-Only Preferences

In this section we establish a preference domain called the *top-only preference domain*, where our proposed mechanisms are IR, PE, and SP.

Definition 4 (Top-Only Preferences). *The* top-only preference domain $\mathscr{R}_T$ *is the set of all possible preferences* R_i *satisfying that there is strict ordering* $\succ_i$ *of* K *s.t. for any pair* $L, L' \subseteq K$, *(i)* $L P_i L'$ *iff either* $t(\succ_i, L) \succ_i t(\succ_i, L')$ *holds or* $t(\succ_i, L) = t(\succ_i, L')$ *and* $|L| > |L'|$ *hold, and (ii)* $L I_i L'$ *iff* $t(\succ_i, L) = t(\succ_i, L')$ *and* $|L| = |L'|$ *hold, where* $t(\succ_i, L)$ *denote the most preferred good in L under* $\succ_i$.

Intuitively, when agent i gets $L \subseteq K$, her utility is solely determined by the most preferred good among L, i.e., $t(\succ_i, L)$. However, every other good also gives her a fixed amount of utility. For example, consider a situation where you have three tickets for different movies, all of which are scheduled to be shown from 6pm to 9pm on the same night. Since you can just use one, you choose the most interesting movie. At the cinema's box office, you get a refund for the other two tickets, each of which will give you a fixed amount of money.

The UPDATE-TOPS function of our proposed mechanisms for the top-only preference domain is formalized as follows:

$$\text{UPDATE-TOPS}(R_i, \phi_i, K') = \begin{cases} \{t(\succ_i, K')\} & \text{if } \phi_i = \varnothing, \\ K' & \text{otherwise.} \end{cases}$$

At each round of the algorithm, it returns (i) the most preferred good among all the remaining ones until the agent receives something, and (ii) all remaining goods after she receives something. Actually, each agent's demand differs before and after she first receives a good. To achieve Pareto efficiency, UPDATE-TOPS considers such a dynamic change in agents' preferences.

Our proposed mechanisms are obviously IR under the top-only preference domain, since (i) each agent weakly prefers the good she first gets to any good she initially owns, and (ii) the number of goods given to each agent is the same as that of her initial endowment. In what follows we focus on Pareto efficiency and strategy-proofness.

Example

Now we demonstrate the behavior of our proposed mechanisms under the top-only preference domain. Consider $N = \{1, 2\}$, $K = \{a, b, c\}$, $w = (\{a, b\}, \{c\})$, and preferences $R_1, R_2 \in \mathscr{R}_T^2$ whose corresponding strict orderings $\succ_1, \succ_2$ satisfy $a \succ_1 b \succ_1 c$ and $b \succ_2 c \succ_2 a$, respectively. For the F-RULE described in the previous section, a choice of priority ordering does not matter for this example.

Figure 2 (a) shows trading graph G drawn for the first time at line 10. There exists a terminal-sink that only contains good a, which is given to identity 1_a and removed from the market at lines 13-14. At lines 16-18, the UPDATE-TOPS function changes T_1 so that at line 19, its atomic agent 1_b points to both b and c. Here, the F-RULE chooses good c for node 1_b at line 21 (Fig. 2 (b1)), since good c has not been assigned to 1_b, and b has already been assigned.

At line 23, goods b and c are traded (but not yet removed from the market), and the algorighm goes to line 9. In the next round, identity 2_c points to only good b that is currently assigned to itself, i.e., there is a self-loop, so b is given to identity 2_c and removed from the market. The only remaining good c, which is currently assigned to 1_b, is then removed in the same round. As a result, agents 1 and 2 get $\{a, c\}$ and $\{b\}$, respectively, which is Pareto efficient under (R_1, R_2).

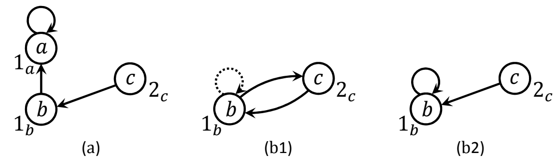

Figure 2: Behavior of our proposed mechanism

Note that without the UPDATE-TOPS function, the final assignment would not be Pareto efficient, i.e., just splitting each agent into atomic agents is not adequate. Actually, in the above example, after the removal of identity 1_a with good a, identity 1_b only points to itself (Fig. 2 (b2)) and is also removed in the same round with good b. The final assignment is then $(\{a, b\}, \{c\})$, which is Pareto dominated by $(\{a, c\}, \{b\})$ (note that $\{a, b\}$ and $\{a, c\}$ are indifferent for agent 1 under top-only preference R_1).

Pareto Efficiency

In this section we show that our proposed mechanisms are PE under the top-only preference domain. We first prove two lemmas that will be used for the main theorem.

Lemma 1. *For any* $R \in \mathscr{R}_T^n$, *if there exists* $y, z \in \mathcal{X}_N$ *s.t.* $y \rightharpoonup_R z$, *then there exists* $x \in \mathcal{X}_N$ *s.t.* $x \rightharpoonup_R z$ *and* $|x_i| = |z_i|$ *for any* $i \in N$.

Proof Sketch. Since the most preferred good almost solely determines each agent's utility under the preference domain, for each agent $i \in N$, it must *not* be the case that $t(\succ_i, z_i) \succ_i t(\succ_i, y_i)$, and for some agent $i' \in N$ it *must* be the case that $t(\succ_{i'}, y_{i'}) \succ_{i'} t(\succ_{i'}, z_{i'})$. Therefore, balancing the number of goods for each agent at assignment y by re-allocating the non-top goods of the agents constructs assignment x, which also Pareto dominates z. □

Lemma 2. *Let* φ *denote a proposed mechanism. For any* $w \in \mathcal{X}_N$, *any* $R \in \mathscr{R}_T^n$, *and any* $i \in N$, *let* $D_i := \{g \in K | g \succ_i t(\succ_i, \varphi_i(w, R))\}$. *Then, for any* $i \in N$ *and any* $g \in D_i$, *there exists* $i' \neq i$ *s.t.* $g = t(\succ_{i'}, \varphi_{i'}(w, R))$.

Proof Sketch. From the description of Algorithm 1, such good g is removed only when it is included in a terminal-sink. When an atomic agent of an agent i' is removed from the market, none of its other atomic agents can be removed until all other agents receive at least one good. Therefore, if g is not the top good of any agent i', then it must be removed *after* all agents get their first goods, including agent i. However, agent i prefers g to her current top good, and thus it cannot be assigned to such i' who has already received her top good. □

Theorem 1. *Any proposed mechanism is PE under the top-only preference domain.*

Proof. We assume for the sake of contradiction that our proposed mechanism φ is not PE under the top-only preference domain. More precisely, $\exists N$, $\exists K$, $\exists w \in \mathcal{X}_N$, $\exists R \in \mathscr{R}_T^n$, $\exists x \in \mathcal{X}_N$ s.t. $x \rightharpoonup_R \varphi(w,R)$. By applying Lemma 1, we can assume w.l.o.g. that $|x_i| = |\varphi_i(w,R)|$ for any $i \in N$. Therefore, from the definition of top-only preferences, there must be at least one agent $i \in N$ s.t. the top good is improved, i.e., $t(\succ_i, x_i) \succ_i t(\succ_i, \varphi_i(w,R))$. To complete the proof, we derive a contradiction by showing that no agent's top good is improved.

We first observe that for each agent who gets (i.e., one of her atomic agent is removed with) a good at round 1, the top good is never improved. This is because every atomic agent points to the most preferred good among K at round 1.

Assume that for each round s s.t. $1 \leq s \leq r$, every agent i who first gets a good at round s never satisfies $t(\succ_i, x_i) \succ_i t(\succ_i, \varphi_i(w,R))$ on assignment x. Since $x \rightharpoonup_R \varphi(w,R)$, the assumption implies $t(\succ_i, x_i) = t(\succ_i, \varphi_i(w,R))$ for any such i, i.e., the top goods are the same between x_i and $\varphi_i(w,R)$.

Consider round $s = r+1$. If there is agent i who first gets a good at round $r+1$ and satisfies $t(\succ_i, x_i) \succ_i t(\succ_i, \varphi_i(w,R))$, good $t(\succ_i, x_i)$ must be removed at some round *before* $r+1$ from the definition of the mechanism. Here, from Lemma 2, good $t(\succ_i, x_i)$ is the top good of another agent i' at assignment $\varphi(w,R)$. That is, there is agent i' who first gets a good at a round before $r+1$ and loses good $t(\succ_{i'}, \varphi_{i'}(w,R)) = t(\succ_i, x_i)$ when the assignment is changed to x, i.e., $t(\succ_{i'}, \varphi_{i'}(w,R)) \neq t(\succ_{i'}, x_{i'})$. This derives a contradiction. □

Strategy-proofness

We next show that under the top-only preference domain, our proposed mechanisms are SP.

Theorem 2. *Any proposed mechanism is SP under the top-only preference domain.*

Proof Sketch. By definition, the mechanism gives each agent the same number of goods as her initial endowment. Therefore, for each agent, improving her top good is the only way to get higher utility by a misreport.

However, for any agent and any misreport, all goods removed from the market before round r in which she first get a good in the truth-telling case are still removed from the market in the exact same way as in misreport cases. This is because, for any misreport of any agent, only the difference in the TTC-GRAPH at each round is the outgoing edges from her atomic agents, and then every originally constructed cycle is still constructed even if these outgoing edges are changed. □

Theorems 1 and 2 show that there exists at least one preference domain, namely the top-only preference domain, outside the discussions of Sönmez (1999), where the impossibility never carries over. In the next section we demonstrate the opposite example; there also exists at least one preference domain where the impossibility does carry over.

6 m-chotomous Preferences

In this section we define a different class of preference domains, called *m-chotomous preferences*, as another example of preference domains that do not satisfy Assumption 1.

Definition 5 (m-chotomous preference domain)**.** *For given integer $m \in \{1, \dots, |K|\}$, the m-*chotomous preference do*main $\mathscr{R}_m$ is the set of all possible preferences R_i satisfying that there is partition $(A_1, A_2, \dots, A_m)$ of K s.t. for any pair $L, L' \subseteq K$, (i) LP_iL' iff $\exists q \in \{1, \dots, m\}$ s.t. $|A_q \cap L| > |A_q \cap L'|$ and $\forall q' \leq q-1$, $|A_{q'} \cap L| = |A_{q'} \cap L'|$, and (ii) LI_iL' iff $\forall q \in \{1, \dots, m\}$, $|A_q \cap L| = |A_q \cap L'|$.*

The m-chotomous preference domain can be considered a modification of the well known additively separable preference domain. The main difference of m-chotomous preferences from additively separable ones is that, even if you get arbitrarily many goods from one subset A_q, you strictly prefer just getting only one good from more preferred subset A_{q-1}. For any integer $m \in \{1, \dots, |K|-1\}$, the m-chotomous preference domain violates Assumption 1, while the $|K|$-chotomous preference domain is a strict preference domain and thus Assumption 1 holds.

These preferences seem quite realistic in situations where there is (or at least there seems to be) some "domination relation" between goods, and each agent can utilize several goods at one time. For example, when a university is hiring lecturers for different levels of mathematics courses, it prefers those who can teach Ph.D. students to those who can only teach undergraduates, since the former can also teach undergraduates. For this example, we have a 2-chotomous (or a dichotomous) preference over possible subsets of lecturers.

Impossibility Result

We first show that under the m-chotomous preference domain for any integer $m \in \{3, \dots, |K|-1\}$, there exists no exchange mechanism that satisfies IR, PE, and SP The result show that, even without Assumption 1, the compatibility of those three properties is not trivial.

Theorem 3. *For any integer $m \in \{3, \dots, |K|-1\}$, there exists no exchange mechanism that is IR, PE, and SP under the m-chotomous preference domain.*

Proof. We assume $K = \{a,b,c,d\}$ and $m = 3$, i.e., the trichotomous preference. Assume $N = \{1,2\}$, $(w_1, w_2) = (\{b,c,d\}, \{a\})$, and $R_1, R_2 \in \mathscr{R}_3$ have the following partitions of K, respectively:

$$\begin{array}{lcl} R_1 & : & (\{a,b\}, \{c\}, \{d\}) \\ R_2 & : & (\{b\}, \{a,c\}, \{d\}) \end{array}$$

Note that for any K s.t. $|K| > 4$ and any $m > 3$, we can modify this example and derive the non-existence of possible assignments in the same form as the proof below, by assigning each additional good in $K \setminus \{a,b,c,d\}$ to different agents $3, 4, \dots$ who solely prefers their initial endowment. Thus, focusing on this example is without loss of generality.

In that case, the followings are the only possible assignments under their truth-tellings for an IR and PE exchange mechanism; i) $(\{a,b\}, \{c,d\})$, ii) $(\{a,b,d\}, \{c\})$, and iii)

$(\{a,c,d\},\{b\})$. When i) or ii) occurs, agent 2 reports R_2' with partition $(\{b\},\{a\},\{c,d\})$ and changes the assignment to $(\{a,c,d\},\{b\})$, which is the only possible assignment under preference profile (R_1,R_2'). Therefore, the mechanism violates strategy-proofness.

When iii) occurs, agent 1 reports R_1' with partition $(\{b\},\{a\},\{c,d\})$ and changes the assignment to either $(\{a,b\},\{c,d\})$ or $(\{a,b,d\},\{c\})$, which are the only possible assignments under preference profile (R_1',R_2). Therefore, the mechanism violates strategy-proofness. □

When $m=|K|$, the preferences are strict, and thus the impossibility shown by Sönmez can be applied. On the other hand, when $m=1$, the "no-trade" mechanism trivially satisfies all the properties, since the initial endowment itself is already Pareto efficient. When $m=2$, clarifying whether they are SP in the dichotomous preference domain remains future work.

Pareto Efficiency

Although the impossibility was shown in the previous subsection, there might be situations in which we have to design a trading mechanism for m-chotomous preferences. Here we show that our proposed mechanisms are PE under the m-chotomous preference domain for any integer $m \le |K|$. We first formally describe the UPDATE-TOPS function of our proposed mechanisms for m-chotomous preferences:

$$\text{UPDATE-TOPS}(R_i,\phi_i,K') = \{g \in K' | \forall h \in K', \{g\} R_i \{h\}\}$$

In other words, at each round of the algorithm, it returns a set of goods that is comprised of the highest ranked among all goods that remain in the round. Note that the output does not depend on ϕ_i, i.e., the set of goods given to agent i so far during the mechanism.

We are now ready to show the Pareto efficiency of our mechanisms under the m-chotomous preference domain for any integer $m \le |K|$. We first prove a lemma, which is quite similar to Lemma 1, and thus the proof is omitted for space limitations.

Lemma 3. *For any integer $m \le |K|$, and any $R \in \mathscr{R}_m^n$, if there exists $y,z \in \mathcal{X}_N$ s.t. $y \rightharpoonup_R z$, then there exists $x \in \mathcal{X}_N$ s.t. $x \rightharpoonup_R z$ and $|x_i| = |z_i|$ for any $i \in N$.*

Theorem 4. *For any integer $m \le |K|$, any proposed mechanism is PE under m-chotomous preference domain.*

Proof. Let φ be any proposed mechanism. For the sake of contradiction, we assume that there exist N, K, $m \le |K|$, $w \in \mathcal{X}_N$, $R \in \mathscr{R}_m^n$, and $x \in \mathcal{X}_N$ s.t. $x \rightharpoonup_R \varphi(w,R)$. From Lemma 3, we assume without loss of generality that $|x_i| = |\varphi_i(w,R)|$ for any $i \in N$.

Now let $M \subseteq N$ be the set of agents who receive different bundles under these assignments x and $\varphi(w,R)$. Since x Pareto dominates $\varphi(w,R)$, at least one such agent exists. Then for each $i \in M$, let us define two bundles of goods $\pi_i := \varphi_i(w,R) \setminus x_i$ and $\tau_i := x_i \setminus \varphi_i(w,R)$. In other words, π_i is the set of goods that $i \in M$ loses after a change of assignment from $\varphi(w,R)$ to x, and τ_i is the set of goods that she acquires after the change. We can easily observe that $\pi_i \cap \tau_i = \varnothing$, $|\pi_i| = |\tau_i|$, and $\tau_i R_i \pi_i$ hold for any $i \in N$. Also, $\tau_j P_j \pi_j$ must hold for some $j \in M$; otherwise x never Pareto dominates $\varphi(w,R)$.

Under the m-chotomous preference domain, for any such $i \in M$, there is good $t_i^* \in \tau_i$ s.t. $\forall p_i \in \pi_i$, $\{t_i^*\} R_i \{p_i\}$; otherwise $\pi_i P_i \tau_i$ holds and x never Pareto dominates $\varphi(w,R)$. Since such good t_i^* is assigned to different agent $j (\neq i) \in M$ under original assignment $\varphi(w,R)$, i.e., $\forall i \in M$, $\exists j \neq i$ s.t. $t_i^* \in \pi_j$, we can construct a finite "cycle" of length $l \le |M|$ by a subset of agents $M' \subseteq M$ with $|M'| = l$ so that for any $i' \in M'$, $\{t_{i'}^*\} R_{i'} \{t_{(i' \bmod l)+1}^*\}$. There are only two cases; (i) $\exists i' \in M'$, $\{t_{i'}^*\} P_{i'} \{t_{(i' \bmod l)+1}^*\}$, and (ii) $\forall i' \in M'$, $\{t_{i'}^*\} I_{i'} \{t_{(i' \bmod l)+1}^*\}$.

If (i) occurs, then assignment $\varphi(w,R)$ is simply Pareto improved by a trading cycle of single goods, i.e., a trade by the constructed cycle. However, from the definition of UPDATE-TOPS for m-chotomous preference domains, proposed mechanism φ is just applying TTAS (or another mechanism in the class introduced by (Saban and Sethuraman 2013)) for the augmented market. Thus, the existence of such an improvement violates the fact that the TTAS mechanism is Pareto efficient for the augmented market.

If (ii) occurs, the trade of single goods by the cycle never changes any agent's utility from original assignment $\varphi(w,R)$. Therefore, we define φ' as a new assignment after the trade by the cycle. Obviously it holds that $x \rightharpoonup_R \varphi'$. Here we define set of agents $\tilde{M} \subseteq N$ who receives different bundles under x and φ'. Obviously $\tilde{M} \subseteq M$, and we apply the same argument above iteratively. This iteration must terminate in a finite step by holding the condition (i), since the number of goods is finite. □

7 Conclusion

In this paper we discussed the compatibility of individual rationality, Pareto efficiency, and strategy-proofness. We first showed that under the top-only preference domain, our proposed mechanisms satisfy all these properties. We also revealed that they are incompatible under many m-chotomous preference domains. All these domains, except for the $|K|$-chotomous preference domain, violate Assumption 1, and thus those complement Sönmez's findings.

We strongly believe that these results will initiate research for comprehensively understanding the (in)compatibility of these three properties in markets without monetary transfers. As future work, we will completely characterize the class of preference domains under which the existence of exchange mechanisms satisfying the three properties is guaranteed. For this direction, we must also investigate the richness of preference domains; actually the problem is quite easy when there is only one preference. Even if we identify some preference domains that are not in that class, a complexity approach may help us obtain somewhat positive results, e.g., proving that finding the best manipulation is NP-hard (Teo, Sethuraman, and Tan 2001; Pini et al. 2011).

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