

The Fisher Market Game: Equilibrium and Welfare

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Abstract

The *Fisher market* model is one of the most fundamental resource allocation models in economics. In a Fisher market, the prices and allocations of goods are determined according to the preferences and budgets of buyers to clear the market.

In a *Fisher market game*, however, buyers are *strategic* and report their preferences over goods; the market-clearing prices and allocations are then determined based on their reported preferences rather than their real preferences. We show that the Fisher market game always has a pure Nash equilibrium, for buyers with linear, Leontief, and Cobb-Douglas utility functions, which are three representative classes of utility functions in the important Constant Elasticity of Substitution (CES) family. Furthermore, to quantify the social efficiency, we prove Price of Anarchy bounds for the game when the utility functions of buyers fall into these three classes respectively.

Introduction

The *Fisher market* (Brainard and Scarf 2000) is one of the most fundamental models within mathematical economics studied in an extensive body of literature alongside the Arrow-Debreu model. The basic setting is that of a set of buyers aiming to purchase multiple goods in a way that maximizes their utility subject to budget constraints. The outcome where supply equals demand is known as a *market equilibrium*, and has the property that the buyers spend their entire budgets, all goods are sold, and the bundle purchased by each buyer maximizes his utility (given his budget and the equilibrium prices of the goods). While the general equilibrium theory is essentially a non-algorithmic theory (Nisan et al. editors 2007), Fisher markets enjoy desirable computational properties. In particular, a market equilibrium is guaranteed to exist under mild conditions (Arrow and Debreu 1954), and it can be computed *efficiently* when the utility functions belong to the important class of *Constant Elasticity of Substitution (CES)* functions (Solow 1956; Arrow et al. 1961). The most prominent types of functions in this class are the Linear, Leontief, and Cobb-Douglas utility functions.

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Due to their attractive computational properties, Fisher markets represent an appealing mechanism for allocating resources in multiagent systems. However, the classical model implicitly assumes that the true preferences of the buyers are known. If preferences are private to the market – which often is the case in multiagent systems – the buyers might manipulate their reports to influence the computed market equilibrium and hence obtain a better allocation. This strategic behavior naturally motivates the study of the Fisher market as a game, where the market equilibrium is computed based on the reports rather than the true preferences; clearly, the outcome may be different from the one intended.

The Fisher market game was first studied by Adsul et al. (2010) for buyers with linear utility functions. The authors showed the existence of pure Nash equilibria under mild assumptions and provided necessary conditions for a strategy profile to be a pure Nash equilibrium. Chen, Deng, and Zhang (2011) and Chen et al. (2012) characterized the extent to which a buyer can improve his utility by deviating from being truthful for CES utility functions. More recently, Babaioff et al. (2013) examined strategic behavior in settings where markets are used as auction mechanisms. Unlike Fisher markets, where money has no intrinsic value to buyers, the buyers in the markets studied by Babaioff et al. have *quasi-linear* utilities.

In this paper, we study the Fisher market game for buyers with linear, Leontief, and Cobb-Douglas utility functions respectively. We explore two research questions. First, while a Fisher market almost always has a market equilibrium, does a Fisher market game always have a game-theoretic equilibrium? Second, we are interested in quantifying the social welfare loss due to the strategic behavior of buyers in the Fisher market game. We analyze the *Price of Anarchy (PoA)* (Koutsoupias and Papadimitriou 1999; Roughgarden and Tardos 2002), which is defined as the ratio between the social welfare at the worst game-theoretic equilibrium and the social welfare at the optimal “centralized” allocation, for the game.

Our contributions are as follows. When buyers have Leontief and Cobb-Douglas utility functions, we show that the Fisher market game always have a pure Nash equilibrium under mild conditions. Together with the results of Adsul et al. (2010), which identified a particular pure Nash equilibrium of the Fisher market game for buyers with linear utility

ties, these results prove the existence of a pure Nash equilibrium in the Fisher market game for all the three typical CES utility functions. We then prove asymptotic PoA bounds for the Fisher market game for linear, Leontief, and Cobb-Douglas utilities. For Leontief and Cobb-Douglas functions, we obtain tight PoA bounds of $\Theta(1/n)$ and $\Theta(1/\sqrt{n})$ respectively, where n is the number of buyers in the game. For linear utility functions, the PoA is upper bounded by $O(1/\sqrt{n})$ and lower bounded by $\Omega(1/n)$. The tight PoA bound for linear utility functions is left as an open question.

The Model

A Fisher market \mathcal{M} consists of a set $N = \{1, \dots, n\}$ of buyers (agents) and a set $M = \{1, \dots, m\}$ of divisible goods (items). Every buyer i has:

- an initial budget $B_i > 0$, which can be viewed as some currency that can be used to acquire goods but has no intrinsic value to the buyer, and
- a utility function $u_i : [0, 1]^m \rightarrow \mathbf{R}$ that maps a quantity vector of the m items to a real value. $u_i(\mathbf{x}_i)$ represents the buyer's utility when receiving \mathbf{x}_i amount of the items.

Without loss of generality, the supply of each good is assumed to be one unit, and the total budget of all buyers is normalized to one, i.e. $\sum_{i=1}^n B_i = 1$.¹

Utility Functions

In this paper, we consider buyers with linear, Leontief, or Cobb-Douglas utility functions. These utility functions are the most widely used classes in the general *Constant Elasticity of Substitution (CES)* utility function family. Utility functions in the CES family take the form of

$$u_i(\mathbf{x}_i) = \left(\sum_{j=1}^m a_{ij} \cdot x_{ij}^\rho \right)^{\frac{1}{\rho}}$$

where ρ parameterizes the family, and $-\infty < \rho \leq 1$, $\rho \neq 0$. The Leontief, Cobb-Douglas, and linear utility functions are respectively obtained when ρ approaches $-\infty$, approaches 0, and equals 1:

$$\text{Leontief: } u_i(\mathbf{x}_i) = \min_{j \in [m]} \left\{ \frac{x_{ij}}{a_{ij}} \right\}.$$

$$\text{Cobb-Douglas: } u_i(\mathbf{x}_i) = \prod_{j \in [m]} x_{ij}^{a_{ij}}.$$

$$\text{Linear: } u_i(\mathbf{x}_i) = \sum_{j \in [m]} a_{ij} x_{ij}.$$

The Leontief function captures utility of items that are *perfect complements*, e.g. left and right shoes; the Linear function captures utility of items that are *perfect substitutes*, e.g. Pepsi and Coca-Cola. The Cobb-Douglas function expresses a perfect balance between complements and substitutes.

The a_{ij} is a parameter of the utility functions. It quantifies how receiving more item j affects buyer i 's utility, while the exact effect depends on the specific class of utility functions.

¹These are standard assumptions that are often made for convenience in analyzing the Fisher market model. They do not affect our results in this paper.

We hence call $\mathbf{a}_i = (a_{ij})_{j \in [m]}$ a valuation vector. In this paper, we will only consider buyers having utility functions from the *same* class (linear, Leontief, or Cobb-Douglas) but with possibly different valuation vectors. Once the class of utility functions is fixed, a buyer's utility function can be completely described by its valuation vector \mathbf{a}_i .

Market Equilibrium

Each buyer in the market wants to spend its entire budget to acquire a bundle of items that maximizes its utility. A market outcome is defined as a tuple $\langle \mathbf{p}, \mathbf{x} \rangle$, where \mathbf{p} is a vector of prices for the m items and $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an allocation of the m items, with p_j denoting the price of item j and x_{ij} representing the amount of item j received by buyer i . A market outcome that maximizes the utility of each buyer subject to its budget constraint and clears the market is called a *market equilibrium* (Nisan et al. editors 2007). Formally, $\langle \mathbf{p}, \mathbf{x} \rangle$ is a market equilibrium if and only if:

- For all $i \in N$, \mathbf{x}_i maximizes buyer i 's utility given prices \mathbf{p} and budget B_i .
- Each item j either is completely sold or has price 0, i.e. $(\sum_{i=1}^n x_{ij} - 1)p_j = 0, \forall j \in [m]$.
- All budgets get spent, i.e. $\sum_{j=1}^m p_j \cdot x_{ij} = B_i, \forall i \in [n]$.

A market equilibrium is guaranteed to exist if each item is desired by at least one buyer and each buyer desires at least one item (Maxfield 1997). For buyers with utility functions from the same class in the CES family (i.e. for some fixed ρ), the equilibrium allocation can be captured by the celebrated Eisenberg-Gale convex program (1959), one of the few algorithmic results in general equilibrium theory:

$$\begin{aligned} \max \quad & \sum_{i=1}^n B_i \cdot \log(u_i) \\ \text{s.t.} \quad & u_i = \left(\sum_{j=1}^m a_{ij} \cdot x_{ij}^\rho \right)^{\frac{1}{\rho}}, \quad \forall i \in [n] \\ & \sum_{i=1}^n x_{ij} \leq 1, \quad \forall j \in [m] \\ & x_{ij} \geq 0, \quad \forall i \in [n], j \in [m] \end{aligned}$$

For some values of ρ , for example $\rho = 1$, the objective function of this convex program is not strictly concave, which means that there may be multiple market equilibria.

The Fisher Market Game

When the Fisher market is used as a mechanism for allocating resources among self-interested agents, it induces a game. In this game, each agent first reports its preference to the center and the center then determines a market equilibrium according to the budgets of the agents and their reported preferences. An agent's utility in the game is his utility of the allocated items with respect to his true preference.

In the same spirit as Adsul et. al (2010), we define the Fisher market game as a game with complete information among all agents. The definition is for agents with CES utility functions with a fixed ρ . Hence, an agents's utility function can be described by its valuation vector \mathbf{a}_i .

Definition 1 (Fisher Market Game). Given a set of items $M = \{1, \dots, m\}$ and a set of agents $N = \{1, \dots, n\}$, where each agent i has budget B_i and valuation vector \mathbf{a}_i , the Fisher Market Game is such that:

- The pure strategy space of each agent i is the set of all possible valuation vectors that i may report: $S_i = \{\mathbf{s}_i \mid \mathbf{s}_i \in \mathbb{R}_{\geq 0}^m\}$. We refer to a strategy \mathbf{s}_i as a report.
- Given a strategy profile $\mathbf{s} = (\mathbf{s}_i)_{i=1}^n$, the outcome of the game is any fixed market equilibrium of the Fisher market given by $\langle B_i, \mathbf{s}_i \rangle_i$, after removing all items j such that $\sum_{i \in N} s_{ij} = 0$.
- Let $\mathbf{x}(\mathbf{s}) = (\mathbf{x}_1(\mathbf{s}), \dots, \mathbf{x}_1(\mathbf{s}))$ denote the market allocation for strategy profile \mathbf{s} . For all $i \in N$, agent i 's utility at \mathbf{s} is $u_i(\mathbf{x}_i(\mathbf{s}))$, written as $u_i(\mathbf{s})$ for shorthand.

The above definition does not allow agents to strategize on their budgets, since Chen, Deng, and Zhang (2011) proved that agents can never gain by misreporting their budgets.

Existence of Pure Nash Equilibria

We study the existence of pure Nash equilibria for the main classes of CES utility functions and begin with Cobb-Douglas valuations.

According to the standard definition, a strategy profile is a pure Nash equilibrium if no agent can increase its utility by deviating to some other strategy. In the Fisher market game, since the outcome of the game might be one of several market equilibria, we define a *pure Nash equilibrium of the Fisher market game* to be a strategy profile where for any deviation of any agent i , agent i 's payoff does not increase, for any market equilibrium of the resulting strategy profile.

Cobb-Douglas Utilities

The main result of this section is that the Fisher market game with Cobb-Douglas utilities has pure Nash equilibria for a large class of valuations that captures most scenarios of interest. That is, existence is guaranteed when the game is *strongly competitive* (i.e. for each item $j \in [m]$, there exists more than one agent with non-zero valuation for it) and the preferences are *unit-sum* (i.e. $\sum_j a_{ij} = 1, \forall i \in [n]$). The unit-sum preferences assumption is a common normalization originating in social choice theory and widely used in both economics and computer science to model situations where agents' preferences are not measurable in monetary terms and hence all agents have equal weights in the system.

Strong competitiveness is required in order for pure Nash equilibrium to exist, since if there is an item desired by a single agent, that agent has an incentive to assign less and less value on this item and can still receive it entirely. The very same condition is used by Adsul *et. al* (2010), Feldman *et al* (2009), and Zhang (2005).

Theorem 1. *The Fisher market game with Cobb-Douglas utilities has a pure Nash equilibrium under unit-sum valuations whenever the game is strongly competitive.*

Recall that given an allocation $\mathbf{x} = (x_{ij})$, where x_{ij} is the amount received by agent i from good j , the utilities are:

$$u_i(\mathbf{x}) = \prod_{j \in [m]: a_{ij} \neq 0} x_{ij}^{a_{ij}}, \forall i \in [n],$$

where a_{ij} is the value of agent i for item j and B_i its budget.

Moreover, the market equilibrium and market prices are unique and have the following succinct form (Eaves 1985):

$$p_j = \sum_{i=1}^n a_{ij} B_i \quad \text{and} \quad x_{ij} = \frac{a_{ij} B_i}{\sum_{k=1}^n a_{kj} B_k}.$$

The game has discontinuous payoffs; thus standard methods for proving existence of pure Nash equilibria do not apply. Our main tool is a non-trivial result due to Reny (1999) on the existence of Nash equilibria in general games with discontinuous payoffs. We set up the required machinery starting with the following definitions.

Definition 2. Agent i can secure a payoff of α at strategy $(\mathbf{s}_i, \mathbf{s}_{-i}) \in S$ if there is $\bar{\mathbf{s}}_i \in S_i$ such that $u_i(\bar{\mathbf{s}}_i, \mathbf{s}'_{-i}) \geq \alpha$ for all \mathbf{s}'_{-i} close enough to \mathbf{s}_{-i} . [i.e. if there exists $\varepsilon > 0$ such that for any \mathbf{s}'_{-i} with $|\mathbf{s}'_{-i} - \mathbf{s}_{-i}| < \varepsilon$ then $u_i(\bar{\mathbf{s}}_i, \mathbf{s}'_{-i}) \geq \alpha$].

In other words, agent i can secure the payoff α if the agent has a strategy guaranteeing at least α not only at the strategy profile $(\bar{\mathbf{s}}_i, \mathbf{s}_{-i})$, but also at all profiles where agent i plays $\bar{\mathbf{s}}_i$ but the other agents slightly deviate from \mathbf{s}_{-i} .

Definition 3. A pair $(\mathbf{s}, \mathbf{u}) \in S \times \mathbb{R}^n$ is in the closure of the graph of the vector payoff function if $\mathbf{u} \in \mathbb{R}^n$ is the limit of the vector of agent payoffs for some sequence of strategies $(\mathbf{s}^k)_{k \geq 1}$ converging to \mathbf{s} . That is, if $\mathbf{u} = \lim_k (u_1(\mathbf{s}^k), \dots, u_n(\mathbf{s}^k))$ for some $\mathbf{s}^k \rightarrow \mathbf{s}$.

Definition 4. A game $G = (S_i, u_i)_{i=1}^n$ is better-reply secure if whenever $(\mathbf{s}^*, \mathbf{u}^*)$ is in the closure of the graph of its vector payoff function and \mathbf{s}^* is not a Nash equilibrium, some agent i can secure a payoff strictly above u_i^* at \mathbf{s}^* .

Theorem 2. (Reny 1999) *If the strategy profile of each agent i , S_i , is a non-empty, compact, convex subset of a metric space, the utility function of each agent i , $u_i(\mathbf{s}_1, \dots, \mathbf{s}_n)$ is quasi-concave in the agent's own strategy, \mathbf{s}_i , and the game $G = (S_i, u_i)_{i=1}^n$ is better-reply secure, then G has at least one pure Nash equilibrium.*

We show that the Fisher market game with Cobb-Douglas utilities and unit-sum valuations is better-reply secure.

Lemma 1. *The Fisher market game with Cobb-Douglas utilities and unit-sum valuations is better-reply secure.*

Proof. Since all games with continuous payoffs are better-reply secure, it is sufficient to check the property at the points where the utility functions are discontinuous (Reny 2006). In the Fisher market game with Cobb-Douglas utilities, the discontinuity occurs when there exists an item j such that all agents assign a value of zero towards that item. That is, the utility functions are discontinuous at the points in the set

$$\mathcal{D} = \{\mathbf{s} \in S \mid \exists j \in [m] \text{ such that } s_{ij} = 0, \forall i \in N\}$$

Let $(\mathbf{s}^*, \mathbf{u}^*)$ be in the closure of the graph of the vector payoff function, where $\mathbf{s}^* \in \mathcal{D}$. Then $\mathbf{u}^* = \lim_{K \rightarrow \infty} (u_1(\mathbf{s}^K), \dots, u_n(\mathbf{s}^K))$ for some sequence of strategies $\mathbf{s}^K \rightarrow \mathbf{s}^*$. For each sequence term \mathbf{s}^K , let s_{ij}^K be the report of agent i for item j and S_j^K the sum of reported values for item j . Let

J be the set of items that no agent declares as valuable (i.e. with strictly positive value) in \mathbf{s}^* :

$$J = \{j \in [m] \mid s_{ij}^* = 0 \text{ and } a_{ij} \neq 0, \forall i \in N\}$$

Using an average argument, there exist an agent i , item l and index $N_0 \in \mathbb{N}$ such that $\frac{s_{il}^K}{S_i^K} \leq \frac{1}{2}$, for all $K \geq N_0$. That is, agent i gets at most 50% of item l in every term of the sequence $(\mathbf{s}^K)_{K \geq 1}$ (except possibly for the first $N_0 - 1$ terms). Let agent i and item l be fixed for the remainder of the proof. Let $S_j^* = \sum_{k=1}^n s_{kj}^*$ be the sum of values of the agents for item j at the strategy profile \mathbf{s}^* , $S_j^K = \sum_{k=1}^n s_{kj}^K$ the sum of values for item j at the strategy profile \mathbf{s}^K , and $L_i = \{j \in [m] \mid s_{ij}^* > 0\}$ the set of items that agent i declares as valuable in the limit.

For every item $k \notin J$, we have that $\lim_{K \rightarrow \infty} \frac{s_{ik}^K}{S_k^K} = \frac{s_{ik}^*}{S_k^*}$. Then the utility of agent i in the limit of the sequence of strategies $(\mathbf{s}^K)_{K \geq 1}$, can be rewritten as follows:

$$\begin{aligned} u_i^* &= \lim_{K \rightarrow \infty} \left(\prod_{j \in J} \left(\frac{s_{ij}^K}{S_j^K} \right)^{\alpha_{ij}} \cdot \prod_{j \in L_i} \left(\frac{s_{ij}^K}{S_j^K} \right)^{\alpha_{ij}} \right) \\ &= \prod_{j \in J} \lim_{K \rightarrow \infty} \left(\frac{s_{ij}^K}{S_j^K} \right)^{\alpha_{ij}} \cdot \prod_{j \in L_i} \left(\frac{s_{ij}^*}{S_j^*} \right)^{\alpha_{ij}} \\ &\leq \left(\frac{1}{2} \right)^{\alpha_{il}} \cdot \prod_{j \in L_i} \left(\frac{s_{ij}^*}{S_j^*} \right)^{\alpha_{ij}} \end{aligned}$$

We illustrate the case $u_i^* > 0$. If $u_i^* = 0$, the analysis is simpler; agent i can easily secure a strictly positive payoff by declaring a small valuation on the items in J , for every ε -perturbation of the other agents' strategies around \mathbf{s}_{-i}^* .

Define the constants $\alpha = \sum_{j \in L_i: a_{ij} \neq 0} a_{ij}$ and $\gamma = 2^{\frac{\alpha_{il}}{\alpha}}$, where $\gamma > 1$. Let $\delta > 0$ be fixed such that $\delta < \frac{(\gamma-1) \cdot S_j^*}{\gamma \cdot S_j^* - s_{ij}^*}$, for all $j \in L_i$. Consider a new strategy profile, \mathbf{s}'_i , for agent i , such that

$$s'_{ij} = \begin{cases} (1 - \delta) s_{ij}^* & \text{if } j \in L_i \\ \left(\frac{\delta}{|J|} \right) \cdot \left(\sum_{k \in L_i} s_{ik}^* \right) & \text{if } j \in J \\ s_{ij}^* (= 0) & \text{otherwise} \end{cases}$$

Agent i 's utility when playing \mathbf{s}'_i against strategies \mathbf{s}_{-i}^* is:

$$\begin{aligned} u_i(\mathbf{s}'_i, \mathbf{s}_{-i}^*) &= \prod_{j \in J} \left(\frac{s'_{ij}}{s'_{ij}} \right)^{\alpha_{ij}} \cdot \prod_{j \in L_i} \left(\frac{(1 - \delta) s_{ij}^*}{S_j^* - \delta \cdot s_{ij}^*} \right)^{\alpha_{ij}} \\ &= \prod_{j \in L_i} \left(\frac{(1 - \delta) s_{ij}^*}{S_j^* - \delta \cdot s_{ij}^*} \right)^{\alpha_{ij}} \end{aligned}$$

Then for each $j \in L_i$, the following inequality holds:

$$\gamma^{\alpha_{ij}} \cdot \left(\frac{(1 - \delta) \cdot s_{ij}^*}{S_j^* - \delta \cdot s_{ij}^*} \right) > \frac{s_{ij}^*}{S_j^*} \quad (1)$$

By taking the product of Inequality (1) over all items $j \in L_i$, we obtain that $u_i(\mathbf{s}'_i, \mathbf{s}_{-i}^*) > u_i^*$. The utility of agent i

is continuous at $(\mathbf{s}'_i, \mathbf{s}_{-i}^*)$, and so for small changes in the strategies of the other agents around \mathbf{s}_{-i}^* , agent i still gets a better payoff than at u_i^* . That is, there exists $\varepsilon > 0$ such that for all feasible strategies \mathbf{s}'_{-i} of the other agents, where $\|\mathbf{s}'_{-i} - \mathbf{s}_{-i}^*\| < \varepsilon$, it is still the case that $u_i(\mathbf{s}'_i, \mathbf{s}'_{-i}) > u_i^*$. It follows that the game is better-reply secure. \square

It can be easily seen that the utility function of each agent is quasi-concave in the agent's own strategy.

Proof of Theorem 1. The strategy set of each agent in the Fisher market game with unit-sum, Cobb-Douglas utilities is non-empty, compact, and convex. Moreover, the utilities are quasi-concave in the agents' own strategies; by Lemma 1, the game is also better-reply secure. Thus the conditions of Reny's theorem are met, and so a pure Nash equilibrium is guaranteed to exist.

Leontief Utilities

For the class of Leontief functions, we prove the existence of a pure Nash equilibrium by directly constructing a set of equilibrium strategies. Namely, the uniform strategy profile is a Nash equilibrium regardless of the true valuations; moreover, the statement holds even for games that fail to be strongly competitive. The high level explanation is that Leontief utilities exhibit perfect complementarity; thus reporting a smaller valuation of an item that no other agent desires does not result in an increased utility for the deviator (since utility is taken as a minimum over the allocation/valuation ratios).

We start by analyzing two-agent markets and then extend the characterization to markets with multiple agents.

Theorem 3. *Given a Fisher market game for two agents with Leontief utilities, the uniform tuple of strategies is a pure Nash equilibrium, and the agents' utilities are $\frac{B_1}{\max_j \{a_{1j}\}}$ and $\frac{B_2}{\max_j \{a_{2j}\}}$, respectively.*

In order to prove Theorem 3, we build upon a result due to Chen, Deng, and Zhang (2011), that describes the best response strategies in two-agent markets. First, define for each agent $i \in [n]$ the following terms: $a_i^{\max} = \max_{j \in [m]} \{a_{ij}\}$, $a_i^{\min} = \min_{j \in [m]} \{a_{ij}\}$, $s_i^{\max} = \max_{j \in [m]} \{s_{ij}\}$, $s_i^{\min} = \min_{j \in [m]} \{s_{ij}\}$.

Then given any two-agent market and an arbitrary fixed strategy \mathbf{s}_2 of agent 2, the best response strategy of agent 1 is: $\mathbf{s}_1 = (s_{1j})_{j \in [m]}$, where $s_{1j} = 1 - s_{2j} \cdot \frac{B_2}{s_2^{\max}}$. In addition, given fixed strategies $(\mathbf{s}_1, \mathbf{s}_2)$, the market equilibrium allocation is unique and the utility of agent 1 is:

$$u_1(\mathbf{s}_1, \mathbf{s}_2) = \min_{j \in [m]} \left\{ \frac{1 - s_{2j} \cdot \frac{B_2}{s_2^{\max}}}{a_{1j}} \right\}$$

Agent 2's allocation is given by: $x_{2j} = s_{2j} \cdot \frac{B_2}{s_2^{\max}}$, $\forall j \in [m]$, and its utility is the minimum possible (as evaluated using its strategy, \mathbf{s}_2); that is, $u'_2(\mathbf{s}_1, \mathbf{s}_2) = \frac{B_2}{s_2^{\max}}$.

Recall that the prices p_j and budgets B_i satisfy the identity: $\sum_{j \in [m]} p_j = \sum_{i \in [n]} B_i = 1$, and so the utility of agent i (as evaluated using the agent's strategy \mathbf{s}_i) is:

$$u'_i = \frac{B_i}{\sum_{j \in [m]} p_j s_{ij}}, \text{ where } \frac{B_i}{s_i^{\max}} \leq u'_i \leq \frac{B_i}{s_i^{\min}}$$

At a high level, by using \mathbf{s}_1 , agent 1 forces agent 2 to get the minimum possible utility (as evaluated with the reported preferences \mathbf{s}_i); this translates to the worst possible allocation for agent 2, while agent 1 gets all the remaining items.

In order for a pair of strategies $(\mathbf{s}_1, \mathbf{s}_2)$ to be a pure Nash equilibrium, the utility of each agent i (evaluated using its report) satisfies: $u'_i(\mathbf{s}_1, \mathbf{s}_2) = \frac{B_i}{s_i^{m\alpha x}}$. Otherwise, some agent could increase its allocation by using the above best response strategy (which would decrease the other agent's allocation). Theorem 3 follows Lemmas 2 and 3.

Lemma 2. *For every pair of strategies $(\mathbf{s}_1, \mathbf{s}_2)$ that is a pure Nash equilibrium of the Fisher market game with two agents and Leontief preferences, the utility of each agent i , as evaluated using its true preferences, satisfies the inequality: $u_i(\mathbf{s}_1, \mathbf{s}_2) \leq \frac{B_i}{a_i^{m\alpha x}}$.*

Lemma 3. *The uniform strategy guarantees agent 1 a payoff of $\frac{B_1}{a_1^{m\alpha x}}$, regardless of agent 2's strategy.*

Note that by using its true preferences, agent i receives: $u_i = B_i / \left(\sum_{j \in [m]} p_j \cdot a_{ij} \right)$, where $B_i / a_i^{m\alpha x} \leq u_i \leq B_i / a_i^{min}$. Thus in any pure Nash equilibrium, the agents fare worse compared to truthful play.

Next we generalize Theorem 3 to any number of agents. Note that the best response strategy of Chen, Deng, and Zhang (2011) does not apply to our game directly. However, we observe that the uniform strategy remains a pure Nash equilibrium regardless of the number of agents.

Theorem 4. *Given a Fisher market game with Leontief preferences, the uniform strategy is a Nash equilibrium for any number of agents, with utilities $u_i = \frac{B_i}{a_i^{m\alpha x}}$, for all $i \in [n]$.*

Proof. Let i be any agent and \mathbf{s}_{-i} an arbitrary fixed strategy of the other agents. From the objective function of the Eisenberg-Gale convex program, it can be observed that all the other agents can be seen as equivalent to a (combined) single agent. Thus the market equilibrium allocation can be computed by reducing the game to two agents, i and $-i$. By Theorem 3, agent i has no incentive to deviate from the uniform strategy; thus the uniform strategy is also a pure Nash equilibrium of the n -agent game. It can be verified that the utilities are $u_i = \frac{B_i}{a_i^{m\alpha x}}$, for each $i \in [n]$. \square

Linear Utilities

Finally, the existence of pure Nash equilibria for linear utilities was established by Adsul *et. al* (2010) for strongly competitive games in a more restricted model than ours. In their model they require that the outcome of the game on a given set of reports \mathbf{s} , is the market equilibrium that maximizes the product of every agent's utility according to their true valuations, i.e. if there is a market equilibrium \mathcal{E} that every agent prefers to every other market equilibrium, then the outcome of the game is \mathcal{E} . This is well-defined because the true valuations are known to all agents but implicitly assumes that the market equilibrium is selected by the agents themselves.

Theorem 5. [Adsul *et. al* (2010)] *Given any Fisher market game with linear utilities, there exists a (symmetric) pure*

Nash equilibrium in which the payoffs are identical to those obtained when agents play truthfully.

Price of Anarchy Bounds

Having examined the existence of pure Nash equilibria in the Fisher market game, we proceed to study its *Price of Anarchy (PoA)* (Nisan *et al.* editors 2007) and give asymptotic bounds for the three main classes of CES utility functions. The (pure) PoA is defined as the ratio between the welfare of the worst-case pure Nash equilibrium and the maximum welfare (taken over all strategy profiles). Note that a PoA of $O(\alpha)$ implies that the equilibrium efficiency cannot be asymptotically better (i.e. higher) than the optimal welfare by a factor of α . Similarly, a PoA of $\Omega(\alpha)$ implies that the equilibrium welfare is at least as high as α times the optimal welfare.

First, note that in all three cases, if the valuations can be completely arbitrary, then the PoA can also be arbitrarily bad. Following the recent literature (Zhang 2005), we study the normalized game, in which each agent's utility is proportional to its budget if it owns all of the items. More formally, for each agent $i \in [n]$, we have that $u_i(\mathbf{0}) = 0$ and $B_i / u_i(\mathbf{1}) = \delta$, where $\mathbf{0}$ and $\mathbf{1}$ are the all 0 and all 1 vectors. We will refer to this assumption as the δ -normalization. In addition, we focus on unit-sum valuations; such normalizations are commonly encountered in other resource allocation and social choice settings.

Linear Utilities

For linear utilities, we begin with the following upper bound.

Theorem 6. *The Fisher market game with linear utilities has a Price of Anarchy of $O(1/\sqrt{n})$.*

Proof. Consider an instance with $n = m^2 + m$ agents and m items, where the budget of each agent i is $B_i = 1$.

For every agent $i \in \{1, \dots, m\}$, define its valuation vector as $\mathbf{a}_i = (0, \dots, 0, 1, 0, \dots, 0)$, that is, the vector in which the i^{th} coordinate is set to 1 and all other entries are zero.

For every agent $i \in \{m+1, \dots, m^2+m\}$, define its valuation vector as $\mathbf{a}_i = \left(\frac{1}{m}, \dots, \frac{1}{m}\right)$.

By checking the KKT conditions of the Eisenberg-Gale convex program, we obtain the market equilibrium:

- The prices are: $p_j = m + 1, \forall j \in [m]$
- The allocations are: $x_{ii} = \frac{1}{m+1}, \forall i \in \{1, \dots, m\}, x_{ij} = \frac{1}{m(m+1)}, \forall i \in \{m+1, \dots, m^2+m\}, \forall j \in [m]$, and $x_{ij} = 0$ everywhere else.

Moreover, for any truthful reporting, any market equilibrium gives the same utility to every agent. Thus, regardless of the chosen allocation, the social welfare under truthfulness is $\frac{2m}{m+1}$. By Theorem 5, there exists a Nash equilibrium in which the social welfare is the same as that of the truthful strategy profile, and so there exists a Nash equilibrium with a social welfare of $\frac{2m}{m+1}$. The optimal social welfare is at least $m - 1$; given by the strategies $\mathbf{s}_i = (0, 0, \dots, 1, \dots, 0)$ for players $i = 1, \dots, m - 1$ and $\mathbf{s}_i = (0, 0, \dots, 1)$ for players

$i = m, \dots, m+m^2$. The PoA is $\frac{2m}{(m-1)(m+1)}$; asymptotically, the bound is $O(1/\sqrt{n})$. \square

We also establish the following lower bound.

Theorem 7. *The Fisher market game with linear utilities has a Price of Anarchy of $\Omega(1/n)$.*

Proof. By δ -normalization, we have: $B_i/(\sum_{j=1}^m a_{ij}) = \delta$. Given unit-sum preferences, we get $B_i = B_k, \forall i \neq k$.

By only assigning value to its most preferred item, each agent i can guarantee a payoff bounded as follows:

$$u_i \geq B_i \cdot \frac{a_i^{max}}{\sum_{k=1}^n B_k} = \frac{a_i^{max}}{n}$$

i.e., in the worst case, all agents prefer the same item, so the price of the item is $\sum_{k=1}^n B_k$ and each agent gets a fraction of $B_i/(\sum_{k=1}^n B_k)$. The optimal social welfare is $W^* \leq \sum_{i=1}^n a_i^{max}$, and so the PoA is:

$$PoA = \frac{\sum_{i=1}^n u_i}{W^*} \geq \frac{\frac{1}{n} \sum_{i=1}^n a_i^{max}}{\sum_{i=1}^n a_i^{max}} = \frac{1}{n}. \quad \square$$

Cobb-Douglas Utilities

Recall that under Cobb-Douglas utilities with unit-sum valuations, the Fisher market game allocates to each agent i exactly a $x_{ij} = \frac{a_{ij} B_i}{\sum_{k=1}^n a_{kj} B_k}$ fraction of every item j . This allocation coincides with that of the *proportional-share allocation mechanism*, studied by Feldman et al. (2009) and Zhang (2005).

In the proportional-share mechanism, each agent i has a budget B_i that it can freely distribute over the m items. The report of each agent is an m -dimensional vector $\mathbf{s}_i = (s_{i1}, \dots, s_{im})$, with the property that $\sum_{j=1}^m s_{ij} = B_i$. Given any instance of the agents' reports, $\mathbf{s} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$, the price of each item j is set to $p_j = \sum_{k=1}^n s_{kj}$, and agent i receives $x_{ij} = \frac{s_{ij}}{p_j}$ units of item j . If all agents report zero for some item, then that item is kept by the center.

Theorem 8. *The Fisher market game with Cobb-Douglas utilities has a Price of Anarchy of $\Theta(1/\sqrt{n})$.*

Proof. For the lower bound, we use the technique employed by Zhang (2005) to show that the linear proportional-share mechanism has a PoA of $\Omega(1/\sqrt{n})$.

The key observation is that the technique used in their proof does not require a specific form of the utility functions; thus the lower bound holds more generally for any concave, non-decreasing utility function. In their proof, the optimal allocation is not necessarily a feasible allocation of the Fisher market game, but its value is actually at least as large as the optimal value of the game and hence the lower bound still holds. As a result, the Fisher market game with Cobb-Douglas utilities has a PoA of $\Omega(1/\sqrt{n})$.

To see the upper bound, consider the same instance that we constructed in Theorem 6 to prove the upper bound on the PoA for the Fisher market game for linear utilities (with $n = m^2 + m$ agents and m items). With a simple check, it can be seen that reporting truthfully is a Nash equilibrium.

Moreover, under truthfulness, the social welfare is $\frac{2m}{m+1}$, while the optimal social welfare is at least $m - 1$, given by the same strategy profile used in proving the upper bound for linear utility functions. Thus the PoA is $\frac{2m}{(m-1)(m+1)}$, and we get the asymptotic bound of $O(1/\sqrt{n})$. \square

Leontief Utilities

Finally, for Leontief utilities, we give the next tight bound.

Theorem 9. *The Fisher market game with Leontief utilities has a Price of Anarchy of $\Theta(1/n)$.*

Proof. The upper bound proof is omitted. For the lower bound, by reporting truthfully, agent i can guarantee

$$u_i = \frac{B_i}{(\sum_{j=1}^m p_j a_{ij})} \geq \frac{B_i}{(\sum_{j=1}^m p_j a_i^{max})} = \frac{B_i}{(\sum_{k=1}^n B_k a_i^{max})}$$

The optimal welfare is $W^* \leq \sum_{i=1}^n u_i(\mathbf{1}) = \sum_{i=1}^n \frac{1}{a_i^{max}}$, and so the PoA is:

$$\sum_{i=1}^n \frac{u_i}{W^*} \geq \frac{B_i/(\sum_{k=1}^n B_k a_i^{max})}{\sum_{i=1}^n (1/a_i^{max})}$$

The δ -normalization implies: $a_i^{max} B_i = \delta$, and so:

$$PoA = \sum_{i=1}^n \frac{B_i^2}{(\sum_{i=1}^n B_i)^2}$$

By the Cauchy-Schwarz inequality, we have:

$$PoA = \frac{1}{n} \cdot \frac{(\sum_{i=1}^n 1^2)(\sum_{i=1}^n B_i^2)}{(\sum_{i=1}^n B_i)^2} \geq \frac{1}{n} \cdot \frac{(\sum_{i=1}^n 1 \cdot B_i)^2}{(\sum_{i=1}^n B_i)^2} = \frac{1}{n}. \quad \square$$

Table 1: Summary: Welfare of the Fisher Market Game. Lower bound (*) is due to Zhang (2005)

	Cobb-Douglas	Leontief	Linear
UB	$O(1/\sqrt{n})$	$O(1/n)$	$O(1/\sqrt{n})$
LB	$\Omega(1/\sqrt{n})$ (*)	$\Omega(1/n)$	$\Omega(1/n)$

Future Work

It remains an interesting open question whether a pure Nash equilibrium exists for CES functions with any value of ρ . The main challenge to this question is that there is no explicit formula for the allocation, instead, the allocation rule is a part of the feasible solution to a convex program. This challenge also carries over into proving a tight lower bound on the PoA of the Fisher market game with linear utility functions. For the latter case, an additional difficulty is that the market equilibrium may not be unique and hence a proof should take all market equilibria into account.

For the linear case, the results of Adsumt et al (2010) assume a somewhat artificial choice of market equilibrium. In real markets, however, the choice of the market equilibrium is upon the market designer, who does not know the true valuations and hence cannot select the ‘‘universally preferred’’ allocation. It remains open to find a natural choice of market equilibrium for the linear case that could be handled by the market designer.

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