

# Item Bidding for Combinatorial Public Projects

Evangelos Markakis and Orestis Telelis

Department of Informatics,  
Athens University of Economics and Business, Greece.  
{markakis,telelis}@gmail.com

## Abstract

We present and analyze a mechanism for the Combinatorial Public Project Problem (CPPP). The problem asks to select  $k$  out of  $m$  available items, so as to maximize the social welfare for autonomous agents with combinatorial preferences (valuation functions) over subsets of items. The CPPP constitutes an abstract model for decision making by autonomous agents and has been shown to present severe computational hardness, in the design of truthful approximation mechanisms. We study a non-truthful mechanism that is, however, practically relevant to multi-agent environments, by virtue of its natural simplicity. It employs an Item Bidding interface, wherein every agent issues a separate bid for the inclusion of each distinct item in the outcome; the  $k$  items with the highest sums of bids are chosen and agents are charged according to a VCG-based payment rule. For fairly expressive classes of the agents' valuation functions, we establish existence of socially optimal pure Nash and strong equilibria, that are resilient to coordinated deviations of subsets of agents. Subsequently we derive tight worst-case bounds on the approximation of the optimum social welfare achieved in equilibrium. We show that the mechanism's performance improves with the number of agents that can coordinate, and reaches half of the optimum welfare at strong equilibrium.

## Introduction

Public project problems model situations where a central authority (e.g., a government or municipality) aims at carrying out a project in the common interest of all members of a community, such as a bridge or a new road (Mas-Colell, Whinston, and Green 1995; Moore 2006). Several variations have been considered in the literature, motivated by different applications, see e.g., (Moulin 1988)[Chapters 6-8]. Our focus is on the *Combinatorial Public Project Problem* (CPPP), which was introduced in (Papadimitriou, Schapira, and Singer 2008) as a prototypical model for decision making by autonomous strategic agents with *combinatorial preferences*. In the CPPP, an authority aims at combining at most  $k$  components from a given set of  $m$  distinct items, to build a composite service or facility, in favor of  $n$  strategic

agents. Each agent values different subsets of items according to a *private valuation function*, defined over all subsets of  $m$  items. The problem amounts to devising a *mechanism*, through which the authority will determine an *efficient outcome* – a welfare maximizing subset of items, along with the payments that the agents should issue for the outcome.

We analyze the performance of a mechanism for the CPPP, with respect to the approximation of the optimum social welfare that it achieves in equilibrium. The mechanism employs a simple *Item Bidding* interface for eliciting the agents' preferences and a natural rule for outcome determination. Under the item bidding interface, each agent issues a separate *bid* for the inclusion of each distinct item in the outcome. In effect, each agent “compresses” his combinatorial valuation function into an *additive* bid vector. The mechanism then selects the  $k$  items with the highest sums of bids. The agents' payments are determined by an adaptation of the familiar Vickrey-Clarke-Groves (VCG) pricing scheme (Vickrey 1961; Clarke 1971; Groves 1973). To alleviate the agents' strategic behavior, *Mechanism Design* has traditionally advocated the implementation of *truthful* reporting of preferences in a *dominant strategies* equilibrium. Our item bidding mechanism is *not* truthful in general; nevertheless, its study is motivated by several practical and theoretical considerations.

Item bidding mechanisms have received considerable attention recently in the context of combinatorial auctions (Christodoulou, Kovács, and Schapira 2008; Syrgkanis and Tardos 2013; de Keijzer et al. 2013), as a simple and practical means, that is already deployed successfully in real-world online markets. In contrast, most known truthful mechanisms for agents with combinatorial preferences consist of complex algorithmic schemes for determining the outcome and payments; their complexity hinders their practical deployment and discourages the agents from participating in the underlying strategic game. Concerning the CPPP in particular, a series of works (Papadimitriou, Schapira, and Singer 2008; Schapira and Singer 2008; Buchfuhrer, Schapira, and Singer 2010) have established severe computational inapproximability results for tractable *truthful* mechanisms. On the other hand, for quite expressive classes of the agents' valuation functions, the optimization problem underlying the CPPP is long known to be **NP**-hard, yet approximable within a constant factor, by the greedy al-

gorithm of (Nemhauser, Woolsey, and Fischer 1978). It is compelling to examine whether other than truthful mechanism models exist, with comparably favorable performance.

Our approach follows (Lucier et al. 2013), where item bidding and the simple outcome determination rule were paired with a similarly natural “pay-as-bid” rule, to yield a “first-price” type of mechanism for the CPPP. The authors observed that their mechanism’s performance depends largely on how well agents can coordinate their bidding decisions. To this end, they proved favorable inefficiency bounds for *strong equilibria*, that are resilient to coordinated joint deviations of subsets of agents. We quantify this observation in a detailed manner, by analyzing the inefficiency of  $\ell$ -strong equilibria of our mechanism, to show that its performance improves gracefully with the “allowed” maximum size,  $\ell$ , of agent subsets that can coordinate.

**Contribution** We describe a simple deterministic item bidding mechanism for the CPPP, and analyze its performance at equilibrium. For the fairly general class of *fractionally subadditive* valuation functions (also termed **XOS**), we prove existence of *socially optimal* pure Nash equilibria. We also show that our item bidding mechanism admits strong equilibria for at least a smaller – yet expressive – class of *capped-Linear* valuation functions (**cL**). These results signify the importance of employing the VCG-based pricing scheme; in contrast, the “pay-as-bid” rule used in (Lucier et al. 2013), prevents existence of Nash equilibria in general (Maskin and Riley 2000), even for a singleton “project”.

Subsequently, we quantify the mechanism’s performance at  $\ell$ -strong equilibrium, by deriving bounds on the  $\ell$ -strong Price of Anarchy (Andelman, Feldman, and Mansour 2009). In doing so, we make a standard *no-overbidding* assumption as, e.g., in (Christodoulou, Kovács, and Schapira 2008), detailed in the next section. For agents with **XOS** valuation functions we prove an upper bound of  $1 + \lceil n/\ell \rceil$ ; we give a lower bound of  $\max\{2, n/\ell\}$  even when agents have **cL** valuation functions. For a more restricted class of valuation functions – termed Unit Demand (**UD**), we show an improved upper bound of  $1 + \min\{\lceil n/\ell \rceil, (1 + \lceil n/(k \cdot \ell) \rceil)\}$ . This bound is also attained in the worst case, even when agents have *uniform UD* valuation functions; we prove a lower bound of  $\max\{2, n/(k \cdot \ell)\}$  in this case. In effect, our results suggest that the mechanism recovers half of the optimum social welfare at strong equilibrium. This along with its simplicity makes it a natural choice for multi-agent environments where coordination among agents is possible.

Due to space limitations, some of the proofs are deferred to the full version of our work.

## Related Work

Public project problems have been studied within the AI community mainly in the context of truthful *redistribution mechanisms* (Cavallo 2006; Guo et al. 2013; Naroditskiy et al. 2012; Guo et al. 2011). The CPPP was introduced in (Papadimitriou, Schapira, and Singer 2008); the authors proved communication and computational complexity lower bounds of  $\Omega(\sqrt{m})$  on the problem’s approximability by de-

terministic *truthful* mechanisms, when agents have submodular valuation functions. These results established seminally the increase in hardness of the underlying optimization problem, incurred by the requirement for truthfulness. Schapira and Singer (2008) proved non-constant lower bounds for more general valuation functions and devised a simple truthful  $O(\sqrt{m})$ -approximation mechanism for agents with subadditive valuation functions. A detailed study of the problem’s complexity for subclasses of subadditive valuation functions appeared in (Buchfuhrer, Schapira, and Singer 2010). The only known constant-approximation mechanism for the CPPP with submodular valuation functions is randomized and *truthful-in-expectation* (Dughmi 2011).

The very recent work of (Lucier et al. 2013) is the most related to ours. The authors studied arguably the simplest possible item bidding mechanism, using a “pay-as-bid” pricing rule. They established a tight inefficiency ratio of  $\Theta(\log n)$  for this mechanism at strong equilibrium, for arbitrary valuation functions of the agents. For a restricted class of valuation functions, they showed that a sequential first-price mechanism fully optimizes the social welfare.

A significant volume of recent research concerns the study of *item bidding* mechanisms for combinatorial auctions. This line of research was initiated by (Christodoulou, Kovács, and Schapira 2008), and followed by (Bhawalkar and Roughgarden 2011; Hassidim et al. 2011; Feldman et al. 2013). (de Keijzer et al. 2013) studied the performance of Multi-Unit item bidding auctions. (Syrkkanis and Tardos 2013) developed a unified framework for the analysis of item bidding auctions, in the incomplete information model.

## Definitions and Preliminaries

We consider a set  $[m] = \{1, \dots, m\}$  of *items* and a set  $[n] = \{1, \dots, n\}$  of *agents*. Each agent has *private* combinatorial preferences over  $2^{[m]}$ , expressed by a non-decreasing valuation function  $v_i : 2^{[m]} \mapsto \mathbb{R}^+$ . Given  $k \in \mathbb{Z}^+$  and in light of the agents’ private preferences, the aim is to choose  $X \subseteq [m]$ ,  $|X| = k$ , maximizing the *Social Welfare*,  $SW(X) = \sum_i v_i(X)$ .

We study an *Item Bidding* mechanism, wherein each agent  $i \in [n]$  issues a *separate* bid  $b_i(j)$  for each item  $j \in [m]$ , within a bid vector  $\mathbf{b}_i = (b_i(1), \dots, b_i(m))$ . As usual, we use  $\mathbf{b}_{-i}$  to denote the profile  $(\mathbf{b}_1, \dots, \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, \dots, \mathbf{b}_n)$ . Given a *bidding profile*  $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ , the mechanism determines an outcome  $\mathbb{X}^k(\mathbf{b}) \subseteq [m]$ , consisting of the  $k$  items with *the highest sums of bids*. It also determines charges  $\mathbf{p}(\mathbf{b}) = (p_1(\mathbf{b}), \dots, p_n(\mathbf{b}))$  to the agents. For this purpose we use a direct adaptation of the VCG payment rule that, for a profile  $\mathbf{b}$ , determines  $p_i(\mathbf{b})$  as:

$$p_i(\mathbf{b}) = \sum_{j \in \mathbb{X}^k(\mathbf{b}_{-i})} b_{i'}(j) - \sum_{j \in \mathbb{X}^k(\mathbf{b})} b_{i'}(j) \quad (1)$$

The *utility* of agent  $i$  under profile  $\mathbf{b}$  is:

$$u_i(\mathbf{b}) = v_i(\mathbb{X}^k(\mathbf{b})) - p_i(\mathbf{b})$$

**Additional Notation** To facilitate succinctness in our technical presentation, we define for any  $I \subseteq [n]$ ,  $X \subseteq [m]$ :

$$V_I(X) \equiv \sum_{i \in I} v_i(X) \text{ and: } V_{-I}(X) \equiv \sum_{i \in [n] \setminus I} v_i(X)$$

In effect,  $SW(X) = V_{[n]}(X)$ , but we will use  $SW(X)$  in this case. Moreover, for an *item-bidding* profile  $\mathbf{b}$ , define:

$$S_I^k(\mathbf{b}) \equiv \sum_{i \in I} \sum_{j \in \mathbb{X}^k(\mathbf{b})} b_i(j) \text{ and: } S_{-I}^k(\mathbf{b}) \equiv S_{[n] \setminus I}^k(\mathbf{b})$$

For simplicity, we write  $S^k(\mathbf{b})$  instead of  $S_{[n]}^k(\mathbf{b})$  and  $S_{-i}^k(\mathbf{b})$  instead of  $S_{[n] \setminus \{i\}}^k(\mathbf{b})$ . To illustrate the use of this notation, we can rewrite (1) as:

$$p_i(\mathbf{b}) = S^k(\mathbf{b}_{-i}) - S_{-i}^k(\mathbf{b}) \quad (2)$$

**Example** We provide an example to clarify how the mechanism works. Assume  $m = 4$ ,  $k = 2$  and  $n = 3$  agents, bidding as described in Figure 1. We have  $\mathbb{X}^k(\mathbf{b}) = \{1, 3\}$ . If agent 1 does not participate, the outcome becomes  $\mathbb{X}^k(\mathbf{b}_{-1}) = \{3, 4\}$ . Similarly we can see that  $\mathbb{X}^k(\mathbf{b}_{-2}) = \{1, 3\}$  (agent 2 does not affect the outcome) and  $\mathbb{X}^k(\mathbf{b}_{-3}) = \{1, 2\}$ . Hence, the payment for agent 1 will be:  $p_1(\mathbf{b}) = S^k(\mathbf{b}_{-1}) - S_{-1}^k(\mathbf{b}) = 21 - 20 = 1$ . Clearly the payment of agent 2 is 0 since  $\mathbb{X}^k(\mathbf{b}) = \mathbb{X}^k(\mathbf{b}_{-2})$ . And finally, for agent 3 we have  $p_3(\mathbf{b}) = S^k(\mathbf{b}_{-3}) - S_{-3}^k(\mathbf{b}) = 17 - 12 = 5$ .

1	2	3	4
5	8	0	0
3	1	4	2
4	0	9	6

Figure 1: A bidding profile with 3 agents and 4 items.

**Valuation Functions** We evaluate the mechanism’s performance for agents with valuation functions belonging to the rich class of *Fractionally Subadditive* functions (Feige 2009). This class was also defined syntactically in the seminal work of (Lehmann, Lehmann, and Nisan 2006), under the name **XOS**. The **XOS** class is a strict superset of the widely studied (in Mechanism Design and in optimization at large) class of *Submodular* functions (**SM**), which express *decreasing marginal value* of each additional item to an agent. We also consider the class of *Unit Demand* functions (**UD**) and special versions of the **SM** and **UD** classes – referred to as *capped-Linear* (**cL**) and *uniform UD* (**uUD**) – for which we show that they are worst-case for the mechanism’s performance.

**Definition 1.** A function  $v : 2^{[m]} \mapsto \mathbb{R}^+$  belongs to the class

- **XOS**, if there exists a family of  $r$  additive functions  $\{\alpha_t : [m] \mapsto \mathbb{R}^+ \mid t = 1, \dots, r\}$ , such that  $v(X) = \max_{t=1, \dots, r} \alpha_t(X)$ , for every  $X \subseteq [m]$ .
- **SM**, if for any two sets  $X \subseteq Y \subseteq [m]$ , and for any  $j \in [m]$ :  $v(X \cup \{j\}) - v(X) \geq v(Y \cup \{j\}) - v(Y)$ .
- **UD**, if  $v(X) = \max\{v(\{j\}) \mid j \in X\}$ , for every  $X \subseteq [m]$ .

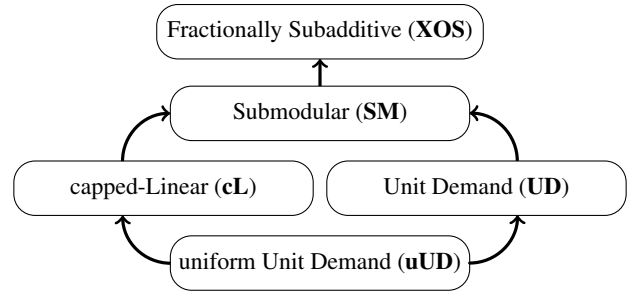


Figure 2: “Contained in” relations among the considered combinatorial valuation function classes.

- **cL**, if there exist  $S \subseteq [m]$ ,  $c \in \{1, \dots, m\}$  and  $\tau > 0$  such that  $v(X) = \tau \cdot \min\{|S \cap X|, c\}$ , for every  $X \subseteq [m]$ .
- **uUD**, if  $v$  belongs to **UD** and there exists  $\nu > 0$  such that, for every  $j \in [m]$ ,  $v(\{j\}) \in \{\nu, 0\}$ ; equivalently, if  $v$  belongs to **cL**, with  $c = 1$ .

It is easy to verify that if  $v$  is **cL** or **uUD**, then it is *symmetric* over the referenced subset  $S$ , but *not* necessarily over  $[m]$ . That is, for every  $X \subseteq [m]$ :

$$v(X \cup \{j\}) = v(X \cup \{j'\}), \text{ for any two } j, j' \in S \setminus X$$

The containment relations among the considered function classes are depicted in Figure 2. By the results of (Buchfuhrer, Schapira, and Singer 2010) [cf. THEOREM 2.1], the CPPP remains **NP**-hard even for **uUD** valuation functions. Moreover, no tractable truthful  $o(\sqrt{m})$ -approximation mechanism is known for this class; a computational hardness result by (Buchfuhrer, Schapira, and Singer 2010) [cf. THEOREM 2.2] suggests an  $\Omega(\sqrt{m})$  approximation lower bound for a wide category of truthful mechanisms.

**Solution Concepts** We study the Item Bidding mechanism’s performance at  $\ell$ –Strong Equilibrium, a solution concept due to (Aumann 1959), which specifies pure Nash equilibria that are resilient to *coordinated* deviations of subsets of at most  $\ell$  agents, for any  $\ell = 1, \dots, n$ . Formally:

**Definition 2.** A bidding profile  $\mathbf{b}$  is an  $\ell$ –Strong Equilibrium if and only if, for every subset  $I$  of at most  $\ell \geq 1$  agents and for every joint deviation  $\mathbf{b}'_I$  of  $I$ , there exists at least one agent  $i \in I$  such that  $u_i(\mathbf{b}'_I, \mathbf{b}_{-I}) \leq u_i(\mathbf{b})$ .

For  $\ell = 1$  this definition coincides with that of a pure Nash equilibrium. We use the term *strong equilibrium* when  $\ell = n$ . To quantify the mechanism’s inefficiency, we will derive upper and lower bounds on the  $\ell$ –Strong Price of Anarchy (Andelman, Feldman, and Mansour 2009), that is, *the worst case ratio of the optimum social welfare over the welfare achieved at  $\ell$ –strong equilibrium*.

Following previous works on item-bidding mechanisms, we make a standard *no-overbidding assumption*. That is, we assume that for any subset of items, the sum of bids submitted by an agent for this subset does not exceed his value for it:  $\sum_{j \in X} b_i(j) \leq v_i(X)$  for any  $X \subseteq [m]$ . This assumption is justified by the fact that overbidding strategies are weakly dominated in general; they can be made strictly

dominated by seamless modifications of the mechanism, see, e.g. (Christodoulou, Kovács, and Schapira 2008).

## Pure Nash and Strong Equilibria

In this section we examine existence of stable configurations under the item bidding mechanism. In particular, we show that socially optimal pure Nash equilibria do exist, for agents with the most general type of preferences among the ones we consider, i.e., **XOS** valuation functions. Subsequently, we examine existence of  $\ell$ -strong equilibria.

**Theorem 1.** *The Item Bidding mechanism with VCG-based Pricing for the CPPP admits socially optimal pure Nash equilibria, when agents have **XOS** valuation functions.*

This state of affairs ceases to hold, when we allow coordinated deviations of subsets of agents and consider the associated stability notion of *strong* equilibria. Consider an instance of the CPPP with  $n = 3$  agents,  $m = 2$  items and  $k = 1$ . Define  $v_1(1) = v_2(1) = v_3(1) = 1$  and  $v_1(2) = 0$ ,  $v_2(2) = v_3(2) = 1 + \epsilon$ . The socially optimal outcome is  $X^* = \{1\}$ , with social welfare of 3. As long as agent 1 does not overbid, he may be bidding at  $b_1^*(1) \leq 1$  in support of  $X^*$ ; agents 2 and 3 can bid  $\mathbf{b}_2 = (0, 1 + \epsilon)$  and  $\mathbf{b}_3 = (0, 1 + \epsilon)$ , to force  $\{2\}$  as an outcome, at zero price for each of them. Then  $(\mathbf{b}_1^*, \mathbf{b}_2, \mathbf{b}_3)$  is a strong equilibrium.

We show however, that socially optimal strong equilibria do exist in item bidding mechanisms, when agents have **cL** valuation functions.

**Theorem 2.** *The Item Bidding mechanism with VCG-based Pricing for the CPPP admits socially optimal strong equilibria, when agents have **cL** valuation functions.*

The **cL** and **uUD** classes prove to be general enough, to provide essentially tight worst-case lower bounds for the inefficiency upper bounds that we develop for **XOS** and **UD** valuation functions in the next section. Although we do not concern ourselves with *how* agents converge to equilibrium, we have recently verified that an iterative best response procedure – similar to the one studied in (Christodoulou, Kovács, and Schapira 2008) – converges in finite time. We defer the details to the full version of our work.

## Inefficiency Bounds for Strong Equilibria

We quantify the inefficiency of Item Bidding with VCG-based Pricing at  $\ell$ -strong equilibrium. We develop upper bounds on the  $\ell$ -strong Price of Anarchy of the mechanism, for agents with **XOS** and **UD** valuation functions. We also provide matching lower bounds, for agents with valuation functions in the classes **cL** and **uUD** respectively. First we state a Lemma that we use in our proofs of upper bounds.

**Lemma 1.** *Let  $\mathbf{b}$  denote an  $\ell$ -strong equilibrium of the Item Bidding mechanism for the CPPP, with VCG-based Pricing. For every subset  $I$  of  $\rho \leq \ell$  agents and for every joint non-overbidding deviation  $\mathbf{b}'_I$  of  $I$ , there exists a sequence  $I = I_1 \supset I_2 \supset \dots \supset I_\rho \supset I_{\rho+1} = \emptyset$  of subsets of agents, satisfying  $|I_q \setminus I_{q+1}| = 1$  for  $q = 1, \dots, \rho$ , such that:*

$$1. \mathcal{S}^k(\mathbf{b}'_{I_{q+1}}, \mathbf{b}_{-I_q}) + V_{I_q \setminus I_{q+1}}(\mathbb{X}^k(\mathbf{b})) \geq \mathcal{S}^k(\mathbf{b}'_{I_q}, \mathbf{b}_{-I_q})$$

$$2. \mathcal{S}^k(\mathbf{b}'_{I_{q+1}}, \mathbf{b}_{-I_{q+1}}) + V_{I \setminus I_{q+1}}(\mathbb{X}^k(\mathbf{b})) \geq \mathcal{S}^k(\mathbf{b}'_I, \mathbf{b}_{-I})$$

where  $I_q \setminus I_{q+1} = \{q\}$ , for  $q = 1, \dots, \rho$ , without loss of generality.

## Fractionally Subadditive Agents

**Theorem 3.** *In  $\ell$ -Strong Equilibrium, the Item Bidding mechanism for the CPPP with VCG-based Pricing recovers at most  $1 + \lceil \frac{n}{\ell} \rceil$  times less welfare than the socially optimal outcome, when agents have **XOS** valuation functions.*

*Proof.* Let  $\mathbf{b}$  denote an  $\ell$ -strong equilibrium and  $I \subseteq [n]$  be any subset of agents of size  $\ell$ . Let  $X^*$  denote the socially optimal outcome. We will consider a particular joint deviation strategy  $\mathbf{b}'_I$ , so as to apply Lemma 1. For every  $i \in I$ , consider the representation of  $v_i$  by a set of additive valuation functions,  $\{\alpha_{i,1}, \dots, \alpha_{i,r_i}\}$ ; let  $\alpha_i$  denote one particular additive function from this set, satisfying  $v_i(X^*) = \sum_{j \in X^*} \alpha_i(\{j\})$ . Then, define the following strategy for each  $i \in I$ :  $b'_i(j) = \alpha_i(\{j\})$ , for every  $j \in X^*$  and  $b'_i(j) = 0$  otherwise (i.e., if  $j \notin X^*$ ).

Let us apply inequality 2 of Lemma 1, for  $q = \ell$ :

$$\mathcal{S}^k(\mathbf{b}'_{I_{\ell+1}}, \mathbf{b}_{-I_{\ell+1}}) + V_{I \setminus I_{\ell+1}}(\mathbb{X}^k(\mathbf{b})) \geq \mathcal{S}^k(\mathbf{b}'_I, \mathbf{b}_{-I})$$

Because  $I_{\ell+1} = \emptyset$ , the first term of the left-hand side equals  $\mathcal{S}^k(\mathbf{b})$ ; moreover, by our no-overbidding assumption we have  $\mathcal{S}^k(\mathbf{b}) \leq SW(\mathbb{X}^k(\mathbf{b}))$ , thus:

$$SW(\mathbb{X}^k(\mathbf{b})) + V_I(\mathbb{X}^k(\mathbf{b})) \geq \mathcal{S}^k(\mathbf{b}'_I, \mathbf{b}_{-I})$$

Next, we observe that for every item  $j' \in X^* \setminus \mathbb{X}^k(\mathbf{b}'_I, \mathbf{b}_{-I})$ , there exists an item  $j \in \mathbb{X}^k(\mathbf{b}'_I, \mathbf{b}_{-I}) \setminus X^*$ , with a higher sum of bids; that is, because  $b'_i(j) = 0$  for all  $i \in I$ ,  $\sum_{i \notin I} b_i(j) \geq \sum_{i \in I} b'_i(j')$ . Thus, we derive:

$$\begin{aligned} SW(\mathbb{X}^k(\mathbf{b})) + V_I(\mathbb{X}^k(\mathbf{b})) &\geq \sum_{i \in I} \sum_{j \in X^*} b'_i(j) \\ &= \sum_{i \in I} \sum_{j \in X^*} \alpha_i(j) = V_I(X^*) \end{aligned}$$

Summing the latter inequality over at most  $\lceil n/\ell \rceil$  disjoint subsets  $I$  of agents, yields the stated result.  $\square$

Next, we give a matching lower bound on the inefficiency of  $\ell$ -Strong Equilibria in item bidding mechanisms for the CPPP, for the class of **cL** valuation functions.

**Theorem 4.** *The Item Bidding mechanism with VCG-based Pricing for the CPPP recovers in  $\ell$ -Strong Equilibrium at least  $\max\{2, \frac{n}{\ell}\}$  times less welfare than the socially optimal outcome in the worst-case, even when agents have **cL** valuation functions.*

*Proof.* Given integers  $k, n$ , and  $\ell \leq n$ , consider  $m = 2k$  items, that we partition in two sets,  $J = \{j_1, j_2, \dots, j_k\}$ ,  $J' = \{j'_1, j'_2, \dots, j'_k\}$ . Consider a subset  $L \subset [n]$ , containing  $r \equiv \min\{\ell, \lceil n/2 \rceil\}$  agents. For every agent  $i \in L$ , let  $v_i$  be a **cL** function, with parameters  $\tau_i = 1$ ,  $S_i = J \cup J'$  and

$c_i = m$ . For every agent  $i \in [n] \setminus L$ , let  $v_i$  be a  $\mathbf{cL}$  function, with parameters  $\tau_i = 1$ ,  $S_i = J$  and  $c_i = m$ .

The socially optimal outcome is  $J$ , with social welfare equal to  $n \cdot k$ . Consider the bidding configuration  $\mathbf{b}$ , wherein for every agent  $i \in L$ , we have  $b_i(j) = 1$  for every item  $j' \in J'$ , and  $b_i(j) = 0$  for  $j \in J$ . Also,  $b_i(j) = 0$ , for every  $i \in [n] \setminus L$  and  $j \in J \cup J'$ . Then,  $\mathbb{X}^k(\mathbf{b}) = J'$  and  $SW(\mathbf{b}) = r \cdot k$ . We claim that  $\mathbf{b}$  is a  $\ell$ -Strong Equilibrium. First, we notice that every agent  $i \in L$  obtains the largest utility possible under  $\mathbf{b}$ , equal to  $k$  – because the payments equal 0. Thus, no such agent may take part in any subset of jointly deviating agents. We therefore have to argue only about subsets among the remaining agents in  $[n] \setminus L$ , who receive *zero* utility under  $\mathbf{b}$ .

Consider first the case that  $\ell \leq \lceil n/2 \rceil$ . Then  $r = \ell$ . Note that any subset of agents of size strictly less than  $\ell$  cannot change the current outcome. This is due to the assumption that agents do not bid above their real value and by the fact that the currently selected items under  $\mathbf{b}$  have score equal to  $\ell$ . Consider now a subset  $T$ , with  $|T| = \ell$ . The members of  $T$  can manage to at most match exactly the total bid on items in  $J$  with those in  $\mathbb{X}^k(\mathbf{b}) = J'$ . Suppose that if there are such ties in a potential deviation by  $T$ , the tie-breaking rule favors a number of  $s$  items from  $J$ , which are selected in the new outcome, combined with  $k - s$  items from  $J'$ . But in that case, it is straightforward to verify from (2) that the payment for each  $i \in T$  is exactly  $s$ . Thus, the deviation will not raise the utility of any member of  $T$  beyond 0.

Next, let  $\ell > \lceil n/2 \rceil$ . Then,  $r = \lceil n/2 \rceil$  and, obviously, the only possibility for a deviation that can change the outcome is when  $n$  is even, and all the agents of  $[n] \setminus L$  deviate jointly within a subset of  $n/2$  agents. But in that case we can again apply exactly the same argument as above and argue that the utility of each  $i \in [n] \setminus L$  cannot rise beyond 0.  $\square$

Notice that the most efficient  $\ell$ -strong equilibrium is always socially optimal, as was shown in Theorem 2. In contrast, the *least* efficient one can be at least a factor  $\max\{2, \frac{n}{\ell}\}$  less efficient than the social optimum.

## Unit Demand Agents

We present an improved upper bound for agents with **UD** valuation functions. Our final upper bound on the mechanism's inefficiency for this class of valuation functions will emerge from a consolidation of this improved upper bound with the one that we derived in the previous subsection. Subsequently, we give a matching worst-case lower bound for agents with **uUD** valuation functions.

**Theorem 5.** *In  $\ell$ -Strong Equilibrium, the Item Bidding mechanism for the CPPP with VCG-based Pricing recovers at most  $2 + \frac{n}{\ell \cdot k}$  times less welfare, than the socially optimal outcome, when agents have **UD** valuation functions.*

*Proof.* Let  $\mathbf{b}$  denote a  $\ell$ -strong equilibrium and  $X^*$  be the socially optimal outcome. Fix *any* single item  $j \in X^*$  and define  $N_j = \{i \in [n] \mid v_i(j) = \max_{r \in X^*} v_i(\{r\})\}$  and  $n_j = |N_j|$ . If  $j \in X^* \cap \mathbb{X}^k(\mathbf{b})$ , then  $V_{N_j}(\mathbb{X}^k(\mathbf{b})) \geq$

$V_{N_j}(X^*)$  and, after summing over all  $j \in X^* \cap \mathbb{X}^k(\mathbf{b})$ :

$$\sum_{j \in X^* \cap \mathbb{X}^k(\mathbf{b})} V_{N_j}(\mathbb{X}^k(\mathbf{b})) \geq \sum_{j \in X^* \cap \mathbb{X}^k(\mathbf{b})} V_{N_j}(X^*) \quad (3)$$

For the sequel, assume that  $j \in X^* \setminus \mathbb{X}^k(\mathbf{b})$ . Consider any subset of agents  $I \subseteq N_j$ , with  $|I| = \rho = \min\{n_j, \ell\}$ . For each agent  $i \in I$ , we define a bidding strategy  $\mathbf{b}'_i$  with:  $b'_i(j) = v_i(j)$  and  $b'_i(r) = 0$ , for all  $r \neq j$ . Lemma 1 now holds for  $I$  and  $\mathbf{b}'_I$ . Before proceeding, consider the succession of coalitions  $I_1, \dots, I_{\rho+1}$  specified in the Lemma:

$$t = \min \left\{ s = 0, \dots, \rho \mid j \notin \mathbb{X}^k(\mathbf{b}'_{I_{s+1}}, \mathbf{b}_{-I_{s+1}}) \right\} \quad (4)$$

Notice that, because  $j \in X^* \setminus \mathbb{X}^k(\mathbf{b})$ ,  $t$  is well defined; indeed,  $j \notin \mathbb{X}^k(\mathbf{b}) = \mathbb{X}^k(\mathbf{b}'_{\emptyset}, \mathbf{b}_{-\emptyset}) = \mathbb{X}^k(\mathbf{b}'_{I_{\rho+1}}, \mathbf{b}_{-I_{\rho+1}})$ , thus,  $t \leq \rho$ . Moreover, if  $j \notin \mathbb{X}^k(\mathbf{b}'_{I_t}, \mathbf{b}_{-I_t}) = \mathbb{X}^k(\mathbf{b}'_{I_1}, \mathbf{b}_{-I_1})$ , we have  $t = 0$ . We will omit a separate treatment of this latter case; for the sequel we assume  $t \geq 1$ .

We analyze the first term of the left-hand side of inequality 1 of Lemma 1, for when  $q = t$ . We have  $j \notin \mathbb{X}^k(\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_{t+1}})$ ; then, either  $j \notin \mathbb{X}^k(\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_t})$ , or  $j \in \mathbb{X}^k(\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_t})$ . We handle here only one of these cases; they both lead to the same result. Consider the case  $j \in \mathbb{X}^k(\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_t})$ . Then, we have:

$$\begin{aligned} \mathcal{S}^k(\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_t}) &= \mathcal{S}^k(\mathbf{b}'_{I_t}, \mathbf{b}_{-I_t}) - b'_t(j) \\ &= \mathcal{S}^k(\mathbf{b}'_{I_t}, \mathbf{b}_{-I_t}) - \sum_{i \in I_t} b'_i(j) + \sum_{i \in I_{t+1}} b'_i(j) \\ &\leq \mathcal{S}^k(\mathbf{b}'_{I_t}, \mathbf{b}_{-I_t}) - \sum_{i \in I_t} b'_i(j) + \frac{1}{k} SW(\mathbb{X}^k(\mathbf{b})) \end{aligned} \quad (5)$$

To justify the last inequality, define  $\mathbf{c} = (\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_{t+1}})$ . Then, because  $j \notin \mathbb{X}^k(\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_{t+1}})$ , we have:

$$\begin{aligned} \sum_{i \in I_{t+1}} b'_i(j) &\leq \sum_{i \in [n]} c_i(j) \leq \min \left\{ \sum_{i \in [n]} c_i(j') \mid j' \in \mathbb{X}^k(\mathbf{c}) \right\} \\ &\leq \frac{1}{k} \mathcal{S}^k(\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_{t+1}}) \leq \frac{1}{k} SW(\mathbb{X}^k(\mathbf{b}'_{I_{t+1}}, \mathbf{b}_{-I_{t+1}})) \end{aligned}$$

where the latter is at most  $\frac{1}{k} SW(\mathbb{X}^k(\mathbf{b}))$ .

Using (5) in inequality 1 of Lemma 1 (with  $q = t$ ) gives:

$$\frac{1}{k} \cdot SW(\mathbb{X}^k(\mathbf{b})) + v_t(\mathbb{X}^k(\mathbf{b})) \geq \sum_{i \in I_t} b'_i(j) = V_{I_t}(X^*) \quad (6)$$

Using inequality 2 of Lemma 1, for  $q = t - 1$ , we obtain:

$$\begin{aligned} V_{I \setminus I_t}(\mathbb{X}^k(\mathbf{b})) &\geq \mathcal{S}^k(\mathbf{b}'_I, \mathbf{b}_{-I}) - \mathcal{S}^k(\mathbf{b}'_{I_t}, \mathbf{b}_{-I_t}) \\ &= \mathcal{S}^k_{I \setminus I_t}(\mathbf{b}'_I, \mathbf{b}_{-I}) = V_{I \setminus I_t}(X^*) \end{aligned} \quad (7)$$

The previous-to-last equality is due to  $\mathbb{X}^k(\mathbf{b}'_I, \mathbf{b}_{-I}) = \mathbb{X}^k(\mathbf{b}'_{I_t}, \mathbf{b}_{-I_t})$ , by definition of  $t$  in (4). Now we sum (6)

and (7) to obtain (also by using  $V_{I \setminus I_{t+1}}(\mathbb{X}^k(\mathbf{b})) \leq V_I(\mathbb{X}^k(\mathbf{b}))$ ):

$$\frac{1}{k} \cdot SW(\mathbb{X}^k(\mathbf{b})) + V_I(\mathbb{X}^k(\mathbf{b})) \geq V_I(X^*) \quad (8)$$

We sum up (8) over at most  $\lceil n_j/\ell \rceil$  disjoint subsets of agents per  $j \in X^* \setminus \mathbb{X}^k(\mathbf{b})$ , and over all  $j \in X^* \setminus \mathbb{X}^k(\mathbf{b})$ , to obtain:

$$\begin{aligned} & \sum_{j \in X^* \setminus \mathbb{X}^k(\mathbf{b})} \left[ \frac{1}{k} \cdot \lceil \frac{n_j}{\ell} \rceil \cdot SW(\mathbb{X}^k(\mathbf{b})) + V_{N_j}(\mathbb{X}^k(\mathbf{b})) \right] \\ & \geq \sum_{j \in X^* \setminus \mathbb{X}^k(\mathbf{b})} V_{N_j}(X^*) \end{aligned} \quad (9)$$

The final result emerges by combination of (3) with (9); the latter is clearly worst-case, so that we may assume  $X^* \cap \mathbb{X}^k(\mathbf{b}) = \emptyset$ . Then, from (9) we deduce:

$$\begin{aligned} SW(X^*) & \leq \left( 1 + \frac{1}{k} \sum_{j \in X^*} \lceil n_j/\ell \rceil \right) \cdot SW(\mathbb{X}^k(\mathbf{b})) \\ & \leq \left( 2 + n/(\ell \cdot k) \right) \cdot SW(\mathbb{X}^k(\mathbf{b})) \end{aligned}$$

which proves the theorem.  $\square$

Observe that the upper bound obtained by Theorem 5 can be worse than  $1 + \lceil n/\ell \rceil$ , that we obtained in Theorem 4 for  $\mathbf{XOS} \supset \mathbf{UD}$  valuation functions. For example, when  $\ell = n$ , the two results yield bounds  $2 + 1/k$  and 2 respectively. It can be verified that the bound of Theorem 5 is the better one, when  $\ell < \frac{n(k-1)}{k}$  for any  $k \geq 2$ , thus, definitely when  $\ell < \frac{n}{2}$ . By consolidating the two bounds we obtain:

**Corollary 1.** *In  $\ell$ -Strong Equilibrium, the Item Bidding mechanism for the CPPP with VCG-based Pricing recovers at most  $1 + \min\{\lceil \frac{n}{\ell} \rceil, (1 + \frac{n}{\ell \cdot k})\}$  times less welfare than the socially optimal outcome, when agents have  $\mathbf{UD}$  valuation functions.*

We prove an almost matching lower bound on the mechanism's performance, even for  $\mathbf{uUD}$  valuation functions.

**Theorem 6.** *For any  $k = o(n)$ , the Item Bidding mechanism for the CPPP with VCG-based Pricing recovers in  $\ell$ -Strong Equilibrium at least  $\max\{2, \frac{n}{k \cdot \ell}\}$  times (asymptotically in  $n$ ) less welfare than the socially optimal outcome, when agents have  $\mathbf{uUD}$  valuation functions.*

*Proof.* Given  $k, \ell$ , consider a CPPP instance with  $m = 2k$  items and  $n = \omega(k)$  agents. We index the items by  $j_1, j_2, \dots, j_k, j_1^*, j_2^*, \dots, j_k^*$ . We partition the  $n$  agents into  $k$  subsets,  $A_1, \dots, A_k$ , each containing at least  $\lfloor n/k \rfloor$  agents; each subset also contains at most 1 of the remaining (at most)  $k-1$  agents, so that all  $n$  agents are distributed among these  $k$  subsets. Thus,  $\lceil n/k \rceil \geq |A_t| \geq \lfloor n/k \rfloor$ . Next we make another  $k$  subsets of agents,  $B_1, \dots, B_k$ , where each  $B_t$  contains  $r = \min\{\ell, \lceil (\max_t |A_t|)/2 \rceil\}$  distinct agents from  $A_t$ , for  $t = 1, \dots, k$ . For  $t = 1, \dots, k$ , we define the  $\mathbf{uUD}$  valuation functions of agents in  $B_t, A_t$ , as follows:

$$\forall i \in B_t: \quad v_i(\{j\}) = \begin{cases} 1 & \text{if } j = j_t^* \text{ or } j = j_t \\ 0 & \text{otherwise} \end{cases}$$

$$\forall i \in A_t \setminus B_t: \quad v_i(\{j\}) = \begin{cases} 1 & \text{if } j = j_t^* \\ 0 & \text{otherwise} \end{cases}$$

The socially optimal outcome is  $\{j_1^*, j_2^*, \dots, j_k^*\}$  with total value of  $n$ . Consider a bidding configuration  $\mathbf{b}$ , with:

$$b_i(j) = \begin{cases} 1 & \text{if } j = j_t \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in B_t$$

$$b_i(j) = 0 \quad \forall i \in A_t \setminus B_t \quad \forall j \in \{j_t, j_t^* \mid t = 1, \dots, k\}$$

Then,  $\mathbb{X}^k(\mathbf{b}) = \{j_1, j_2, \dots, j_k\}$  and:

$$\begin{aligned} SW(\mathbf{b}) & = \sum_{t=1}^k |B_t| \leq \sum_{t=1}^k \min\left\{\ell, \left(1 + \frac{|A_t|}{2}\right)\right\} \\ & \leq \sum_{t=1}^k \min\left\{\ell, \frac{n+3k}{2k}\right\} \leq \min\left\{k \cdot \ell, \frac{n+3k}{2}\right\} \end{aligned}$$

Then, notice that  $n/SW(\mathbf{b}) \geq \max\left\{\frac{n}{k \cdot \ell}, \frac{2n}{n+3k}\right\}$ , which asymptotically in  $n$  yields the result, for  $k = o(n)$ .

We show that  $\mathbf{b}$  is a  $\ell$ -strong equilibrium. Observe that, under  $\mathbf{b}$ , all agents in  $\cup_{t=1}^k B_t$  receive their maximum utility possible (equal to 1); thus, none of these agents could be part of a jointly deviating subset. For the remaining agents in each  $A_t \setminus B_t$ , notice that  $|B_t| \geq \min\{\ell, \lceil (\max_t |A_t|)/2 \rceil\} \geq \min\{\ell, |A_t \setminus B_t|\}$ , by construction. Thus, no subset of at most  $\min\{\ell, |A_t \setminus B_t|\}$  agents from  $A_t \setminus B_t$  may concentrate a bid larger than  $|B_t|$  on item  $j_t^*$  (recall that agents from  $B_t$  do not have incentive to participate in any deviating subset). If such a subset manages to insert  $j_t^*$  in the outcome due to a favorable tie-breaking rule, all its members gain a value of 1 at a price of 1, thus not increasing their utility beyond 0.  $\square$

## Conclusions and Future Work

We presented and analyzed a simple Item Bidding mechanism for the CPPP. We have derived tight upper and lower bounds on the Price of Anarchy for  $\ell$ -strong equilibria both for the general class of  $\mathbf{XOS}$  valuation functions but also for other fairly expressive subclasses. Our analysis exhibits increasingly favorable performance as the number of agents that are allowed to coordinate increases. We believe that the performance of item bidding mechanisms along with their simple and natural interface makes them appealing for practical deployment in multi-agent environments.

Several interesting open problems remain. It would be interesting to investigate existence of strong equilibria for classes beyond the ones that we established their existence for; experimental evidence allows us to conjecture that they do exist for  $\mathbf{UD}$  and, possibly,  $\mathbf{XOS}$  valuation functions. Computation of strong equilibria and related convergent procedures constitute compelling issues for future research, as well. Considering even richer valuation function classes, such as *Subadditive* ones, would broaden our understanding of the mechanism's performance. Another direction is to enrich the family of item bidding mechanisms, by examining different payment or outcome determination rules.

## Acknowledgements

This research has been co-financed by the European Union (European Social Fund – ESF) and the Greek State, through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF), within (i) the framework of Action “Supporting Postdoctoral Researchers”: Research Project DDCOD, PE6-213 (Action’s Beneficiary: General Secretariat for Research and Technology) and (ii) the Research Funding Program: THALES (Project AGT). Investing in knowledge society through the European Social Fund. This work is also supported by an internal research funding program of the Athens University of Economics and Business.

## References

- Andelman, N.; Feldman, M.; and Mansour, Y. 2009. Strong price of anarchy. *Games and Economic Behavior* 65(2):289–317.
- Aumann, R. 1959. Acceptable points in general cooperative  $n$ -person games. In Tucker, A. W., and Luce, R. D., eds., *Contributions to the Theory of Games*, volume 4. Princeton University Press.
- Bhawalakar, K., and Roughgarden, T. 2011. Welfare Guarantees for Combinatorial Auctions with Item Bidding. In *Proceedings of the 22nd ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 700–709. SIAM.
- Buchfuhrer, D.; Schapira, M.; and Singer, Y. 2010. Computation and incentives in combinatorial public projects. In *Proceedings of the 11th ACM Conference on Electronic Commerce (ACM EC)*, 33–42. ACM.
- Cavallo, R. 2006. Optimal Decision-Making with Minimal Waste: Strategyproof Redistribution of VCG Payments. In *Proceedings of the 5th International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS 2006)*, 882–889. ACM.
- Christodoulou, G.; Kovács, A.; and Schapira, M. 2008. Bayesian Combinatorial Auctions. In *Proceedings of the 35th International Colloquium on Automata, Languages and Programming (1) (ICALP)*, volume 5125 of *LNCS*, 820–832. Springer.
- Clarke, E. H. 1971. Multipart Pricing of Public Goods. *Public Choice* 11:17–33.
- de Keijzer, B.; Markakis, E.; Schäfer, G.; and Telelis, O. 2013. Inefficiency of Standard Multi-unit Auctions. In *Proceedings of the 21st European Symposium on Algorithms (ESA)*, volume 8125 of *LNCS*, 385–396. Springer.
- Dughmi, S. 2011. A Truthful Randomized Mechanism for Combinatorial Public Projects via Convex Optimization. In *Proceedings of the 12th ACM Conference on Electronic Commerce (ACM EC)*, 263–272. ACM.
- Feige, U. 2009. On Maximizing Welfare when Utility Functions are Subadditive. *SIAM Journal on Computing* 39(1):122–142.
- Feldman, M.; Fu, H.; Gravin, N.; and Lucier, B. 2013. Simultaneous Auctions are (almost) Efficient. In *Proceedings of the 45th ACM Symposium on the Theory of Computing (ACM STOC)*, 263–272. ACM.
- Groves, T. 1973. Incentives in Teams. *Econometrica* 41(4):617–631.
- Guo, M.; Naroditskiy, V.; Conitzer, V.; Greenwald, A.; and Jennings, N. R. 2011. Budget-Balanced and Nearly Efficient Randomized Mechanisms: Public Goods and beyond. In *Proceedings of the 7th Workshop on Internet and Network Economics (WINE)*, volume 7090 of *LNCS*, 158–169. Springer.
- Guo, M.; Markakis, E.; Apt, K. R.; and Conitzer, V. 2013. Undominated Groves Mechanisms. *Journal of Artificial Intelligence Research (JAIR)* 46:129–163.
- Hassidim, A.; Kaplan, H.; Mansour, Y.; and Nisan, N. 2011. Non-price Equilibria in Markets of Discrete Goods. In *Proceedings of the 12th ACM Conference on Electronic Commerce (ACM EC)*, 295–296. ACM.
- Lehmann, B.; Lehmann, D. J.; and Nisan, N. 2006. Combinatorial auctions with decreasing marginal utilities. *Games and Economic Behavior* 55(2):270–296.
- Lucier, B.; Singer, Y.; Syrgkanis, V.; and Tardos, E. 2013. Equilibrium in Combinatorial Public Projects. In *Proceedings of the 9th International Conference on Web and Internet Economics (WINE)*, volume 8289 of *LNCS*, 347–360. Springer.
- Mas-Colell, A.; Whinston, M.; and Green, J. R. 1995. *Microeconomic Theory*. Oxford University Press.
- Maskin, E., and Riley, J. 2000. Equilibrium in Sealed High Bid Auctions. *The Review of Economic Studies* 67:439–454.
- Moore, J. 2006. *General Equilibrium and Welfare Economics: An Introduction*. Springer.
- Moulin, H. 1988. *Axioms of Cooperative Decision Making*. Cambridge University Press.
- Naroditskiy, V.; Guo, M.; Dufton, L.; Polukarov, M.; and Jennings, N. R. 2012. Redistribution of VCG Payments in Public Project Problems. In *Proceedings of the 8th International Workshop on Internet and Network Economics (WINE)*, volume 7695 of *LNCS*, 323–336. Springer.
- Nemhauser, G. L.; Woolsey, L. A.; and Fischer, M. L. 1978. An Analysis of Approximation for Maximizing Submodular Functions – II. *Mathematical Programming Study* 8:73–87.
- Papadimitriou, C. H.; Schapira, M.; and Singer, Y. 2008. On the Hardness of Being Truthful. In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (IEEE FOCS)*, 250–259. IEEE Computer Society.
- Schapira, M., and Singer, Y. 2008. Inapproximability of Combinatorial Public Projects. In *Proceedings of the 4th International Workshop on Internet and Network Economics (WINE)*, volume 5385 of *LNCS*, 351–361. Springer.
- Syrgkanis, V., and Tardos, E. 2013. Composable and Efficient Mechanisms. In *Proceedings of the 45th ACM Symposium on Theory of Computing (ACM STOC)*, 211–220. ACM.
- Vickrey, W. 1961. Counterspeculation, Auctions, and Competitive Sealed Tenders. *Journal of Finance* 16(1):8–37.