Hybrid Tractable Classes of Binary Quantified Constraint Satisfaction Problems

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Introduction

As a central problem in AI, the Constraint Satisfaction Problem (CSP), has been a popular paradigm to cope with complex combinational problems. A CSP consists of a set of variables and a set of constraints restricting the combinations of values taken by variables. The Quantified Constraint Satisfaction Problem (QCSP) is a generalization of the CSP, where variables are existentially or universally quantified. It is well-known that deciding the satisfiability of CSPs is NP-complete, while QCSPs have been proved to be PSPACE-complete. Hence, a great amount of attention has been drawn on identifying polynomial-time tractable classes of them. There are three main kinds of tractable classes for CSPs: structural classes; relational classes; hybrid classes (Dechter 2003). Many structural and relational tractable classes of QCSPs have been reported (Chen 2004; Gottlob, Greco, and Scarcello 2005), but there is little attention on hybrid tractable classes for QCSPs, which have achieved great success on CSPs (van Beek and Dechter 1995; Zhang and Freuder 2008; Cooper, Jeavons, and Salamon 2010). In this paper, we first identity a hybrid tractable class of QCSPs with the Broken-Triangle Property (BTP) (Cooper, Jeavons, and Salamon 2010), and then propose the broken-angle property for QCSPs to deal with the case that the variable ordering of the BTP differs from the ordering in the prefix, and thus identify a new tractable class. Next, we show that the property of min-of-max extensibility, a generalization of the BTP, can also be used to identify tractable classes of QCSPs in a similar way.

Preliminaries

A QCSP P can be described as a 5-tuple (V, Q, Π, D, C) where: V is a set of n variables. Q is a mapping from V to the set of quantifiers $\{\exists, \forall\}$. For each variable $x_i \in V$, $Q(x_i)$ is a quantifier ($\exists$ or $\forall$) associated with x_i. Π is a bijective mapping from V to the ordered set $\{1, ..., n\}$, which is used to specify the ordering of variables. For each variable $x_i \in V$, $j = \Pi(x_i)$ is a number in the ordered set, and the inverse mapping is denoted by $x_{\Pi^{-1}(j)}$. D is a mapping from V to a set of domains $D = \{D(x_1), ..., D(x_n)\}$. $C = \{C_1, ..., C_e\}$ is a set of e constraints. Each constraint $C_i \in C$ is defined as a pair $(vars(C_i), rel(C_i))$, where $vars(C_i) = (x_{j_1}, ..., x_{j_k})$ is a subset of V called the constraint scope; $rel(C_i)$ specifies the allowed combinations of values for the variables in $vars(C_i)$. For the simplicity, we always use the notion R_i to denote $rel(C_i)$ for short in the rest of the paper.

Let d be the maximum size of domains; P is binary if for each constraint $C_i \in C$, $|vars(C_i)| = 2$. Especially, we use C_{ij} to denote the binary constraint between variables x_i and x_j, and $R_{ij}(\alpha)$=$\{\beta | \beta$ supports α, and $\alpha \in D(x_i)$, $\beta \in D(x_j)\}$. As Π is introduced, our definition employs 5-tuple rather than 4-tuple in the literature (Gent et al. 2008). A QCSP $P = (V, Q, \Pi, D, C)$ represents the logical formula $\phi = Q(x_{\Pi^{-1}(1)})x_{\Pi^{-1}(1)}...Q(x_{\Pi^{-1}(n)})x_{\Pi^{-1}(n)}(C)$, where $Q(x_{\Pi^{-1}(1)})x_{\Pi^{-1}(1)}...Q(x_{\Pi^{-1}(n)})x_{\Pi^{-1}(n)}$ is the prefix of P.

We define $block(\Pi, x_i)$ as a maximal variable set such that for each x_j in it, $Q(x_i) = Q(x_j)$ and there does not exist a variable x_k such that x_k is between x_i and x_j with respect to Π and $Q(x_i) \neq Q(x_k)$ (x_j and x_i may be the same variable); $pre_\forall(\Pi, x_i)$ ($pre_\exists(\Pi, x_i)$) is a set consisting of all the universally (existentially) quantified variables before x_i with respect to Π but not in $block(\Pi, x_i)$; $suc_\forall(\Pi, x_i)$ ($suc_\exists(\Pi, x_i)$) is a set consisting of all the universally (existentially) quantified variables after x_i with respect to Π but not in $block(\Pi, x_i)$; $suc(\Pi, x_i)$ is a set consisting of all variables after x_i with respect to Π.

Gent et al. (2008) proposed the *quantified arc consistency*, and its consistency algorithm called QAC-2001, whose time complexity is $O(d^2 e)$.

Tractable classes based on the QBTP

The BTP was introduced by Cooper, Jeavons, and Salamon (2010). First, we provide the definition of the broken-triangle property for QCSPs and a basic class based on it.

Definition 1. *A binary QCSP $P = (V, Q, \Pi, D, C)$ satisfies the broken-triangle property (QBTP for short), if, for all triples of variables (x_i, x_j, x_k) such that $\Pi(x_i) < \Pi(x_j) < \Pi(x_k)$, if $(\alpha, \beta) \in R_{ij}$, $(\alpha, \gamma) \in R_{ik}$ and $(\beta, \theta) \in R_{jk}$, then either $(\alpha, \theta) \in R_{ik}$ or $(\beta, \gamma) \in R_{jk}$.*

Theorem 1. *Let $P = (V, Q, \Pi, D, C)$ be a binary QCSP. If P satisfies the QBTP, the satisfiability of P can be determined by a polynomial-time algorithm.*

Definition 1 restricts the variable ordering for the QBTP identical to the prefix ordering. We then show that a QCSP may also be tractable without such strong restriction. A tractable class allowing variables can shift in their own blocks is described as follows.

Definition 2. *Let $P^{\Pi} = (V, Q, \Pi, D, C)$ and $P^{\Delta} = (V, Q, \Delta, D, C)$ be binary QCSPs. Π is block-compatible with Δ if they satisfy the following conditions*:

1. *For each existentially quantified variable $x_i \in V$, $pre_{\forall}(\Pi, x_i) = pre_{\forall}(\Delta, x_i)$;*
2. *For each pair of universally quantified variables (x_i, x_j), if $\Pi(x_i) < \Pi(x_j)$, then $\Delta(x_i) < \Delta(x_j)$.*

In Definition 2, we only consider the ordering of existentially quantified variables, because constraints between universally quantified variables make no effect on the QBTP.

Theorem 2. *Given a binary QCSP $P^{\Pi} = (V, Q, \Pi, D, C)$, the satisfiability of P^{Π} can be determined by a polynomial-time algorithm if there exists a mapping Δ such that*:

1. Π is block-compatible with Δ;
2. $P^{\Delta} = (V, Q, \Delta, D, C)$ satisfies the QBTP.

Next, we discuss the case that existentially variables can shift between blocks. Shifting existentially quantified variables in front of universally quantified variables is trivial. We only need to consider the case that shifting existentially quantified variables after universally quantified variables.

Definition 3. *Let $P^{\Pi} = (V, Q, \Pi, D, C)$ and $P^{\Delta} = (V, Q, \Delta, D, C)$ be binary QCSPs, Π is semi-compatible with Δ if they satisfy the following conditions*:

1. *For each existentially quantified variable $x_i \in V$, $pre_{\forall}(\Pi, x_i) \subseteq pre_{\forall}(\Delta, x_i)$;*
2. *For each pair of universally quantified variables (x_i, x_j), if $\Pi(x_i) < \Pi(x_j)$, then $\Delta(x_i) < \Delta(x_j)$.*

To identify a new tractable class under semi-compatible orderings, we have to introduce the broken-angle property for QCSPs.

Definition 4. *A triple of variables (x_i, x_j, x_k) satisfies the broken-angle property under a binary QCSP $P^{\Pi} = (V, Q, \Pi, D, C)$ (QBAP for short) if*:

1. $\Pi(x_i) \leq \Pi(x_j) < \Pi(x_k)$;
2. *For each pair of (α, β) $(\alpha \in D(x_i), \beta \in D(x_j))$, if $(\alpha, \gamma) \in R_{ik}$ and $(\beta, \theta) \in R_{jk}$, then either $(\alpha, \theta) \in R_{ik}$ or $(\beta, \gamma) \in R_{jk}$.*

Theorem 3. *Given a binary QCSP $P^{\Pi} = (V, Q, \Pi, D, C)$ that is quantified arc consistent, P^{Π} is satisfiable if there exists a mapping Δ such that*:

1. *Π is semi-compatible with Δ;*
2. *$P^{\Delta} = (V, Q, \Delta, D, C)$ satisfies the QBTP;*
3. *For each existentially quantified variable x_k, for each pair of variables (x_i, x_j) such that $x_i, x_j \in dif(\Pi, \Delta, x_k)$ and $\Delta(x_i) \leq \Delta(x_j)$, the triple (x_i, x_j, x_k) satisfies the QBAP under P^{Δ}, where $dif(\Pi, \Delta, x_k) = suc_{\exists}(\Pi, x_k) - suc(\Delta, x_k)$.*

Such P^{Δ} is called a QBTP-adjoint problem of P^{Π}. Such Δ can be found in polynomial time.

Theorem 4. *Given a binary QCSP $P^{\Pi} = (V, Q, \Pi, D, C)$, there exists a polynomial-time algorithm to find a mapping Δ such that $P^{\Delta} = (V, Q, \Delta, D, C)$ is a QBTP-adjoint problem of P^{Π}. (or determine that no such mapping exists.)*

The Min-of-max Extendable Class

The min-of-max extendable (MME) class can be regarded as a generalization of the class with the BTP in classical CSPs (Cooper, Jeavons, and Salamon 2010). In this section, we introduce the definition of MME for QCSPs.

Definition 5. *Let $P^{\Pi} = (V, Q, \Pi, D, C)$ be a binary QCSP, where all variable domains are totally ordered. It is min-of-max extendable (QMME for short), if for all triples of variables (x_i, x_j, x_k) such that $\Pi(x_i) < \Pi(x_j) < \Pi(x_k)$, if $(\alpha, \beta) \in R_{ij}$, then assignments $(< x_i, \alpha >, < x_j, \beta >, < x_k, \gamma >)$ are consistent; a triple of variables (x_i, x_j, x_k) is extended QMME under the QCSP P^{Π} if $\Pi(x_i) \leq \Pi(x_j) < \Pi(x_k)$, and for each pair of (α, β) such that $\alpha \in D(x_i)$ and $\beta \in D(x_j)$, assignments $(< x_i, \alpha >, < x_j, \beta >, < x_k, \gamma >)$ are consistent , where $\gamma = min(max(R_{ik}(\alpha), max(R_{jk}(\beta))$.*

Theorem 5. *Let $P^{\Pi} = (V, Q, \Pi, D, C)$ be a binary quantified arc consistent QCSP whose variable domains are totally ordered. P^{Π} is satisfiable if there exists a mapping Δ such that*:

1. *Π is semi-compatible with Δ;*
2. *$P^{\Delta} = (V, Q, \Delta, D, C)$ is QMME;*
3. *For each existentially quantified x_k, for each pair of variables (x_i, x_j) such that $x_i, x_j \in dif(\Pi, \Delta, x_k)$ and $\Delta(x_i) \leq \Delta(x_j)$, (x_i, x_j, x_k) is extended QMME under P^{Δ}.*

In this paper, we have concentrated on identifying tractable classes of QCSPs by using hybrid techniques, such as the broken-triangle property and min-of-max extendability. We omit the proofs of them due to limitations of space. For more details, we refer to (Gao, Yin, and Zhou 2011).

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