Human Spatial Relational Reasoning: Processing Demands, Representations, and Cognitive Model

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Abstract

Empirical findings indicate that humans draw inferences about spatial arrangements by constructing and manipulating mental models which are internal representations of objects and relations in spatial working memory. Central to the Mental Model Theory (MMT), is the assumption that the human reasoning process can be divided into three phases: (i) Mental model construction, (ii) model inspection, and (iii) model validation. The MMT can be formalized with respect to a computational model, connecting the reasoning process to operations on mental model representations. In this respect a computational model has been implemented in the cognitive architecture ACT-R capable of explaining human reasoning difficulty by the number of model operations. The presented ACT-R model allows simulation of psychological findings about spatial reasoning problems from a previous study that investigated conventional behavioral data such as response times and error rates in the context of certain mental model construction principles.

Introduction

Processing spatial information is among the most fundamental challenges that any intelligent system faces and is important in our everyday lives. But how is spatial information processed? Where is the focus of attention in processing qualitative information? In the following, we concentrate on the domain of relational reasoning problems as illustrated in Example 1:

(1a) The pliers are to the left of the hammer.
The pliers are to the left of the saw.
The screwdriver is to the right of the saw.
(1b) Is the screwdriver (always) to the right of the hammer?

The statements in Example 1a are referred to as *premises* and the tools are referred to as *terms*. With respect to the premises Example 1b poses a question concerning a possible constellation of terms and hence offers one possible *conclusion* that has to be verified by the reasoner.

Basically, there are two main cognitive, as well as mathematical, approaches to how humans solve such problems: syntactic-based theories and semantic-based theories. Rips for example, suggested that humans reason by applying formal transitivity rules to the premises (Rips 1994). According to this approach, human deduction can be compared to searching and finding mental proofs. Difficulties arise if a large number of rules must be applied to verify a conclusion.

An alternative approach is suggested by Huttenlocher. He argues that humans construct and inspect a spatial array that represents the state of affairs described in the premises (Huttenlocher 1968). This approach has been further developed to the Mental Model Theory (MMT) of relational reasoning (Johnson-Laird and Byrne 1991) and is generally the more accepted theoretical account of human reasoning in terms of empirical support. A mental model is a representation derived from the information given by the premises and hence is a structure in which the premises are true. Psychologically, a mental model is interpreted as an internal representation of terms and relations in spatial working memory that matches the state of affairs given by the premises. Central to the MMT is the assumption that the human reasoning process can be divided into three phases: (i) Mental model construction, (ii) model inspection, and finally (iii) model validation. In (i), when reading the premises, reasoners encounter new information and immediately use it for the incremental construction of the mental model. In (ii) they inspect the completed mental model to check if it is consistent with a putative conclusion. Finally, in (iii), if possible, they construct alternative models from the premises that refute the conclusion.

The reasoning process itself is supposed to remain unaffected by linguistic processes; once information has been encoded and transferred into some mental model the reasoning process relies on model manipulation only (Johnson-Laird and Byrne 1991; Goodwin and Johnson-Laird 2005).

There are cases in which it is possible to derive more than one mental model from a set of premises. In Example 1b the spatial description is not fully specified, because in the first two premises "the pliers are to the left of the hammer" and "the pliers are to the left of the saw" the exact relation between "saw" and "hammer" is not given. Such problems lead to the multiple model case illustrated in Example 2:

(2a)	pliers	hammer	saw
(2b)	pliers	saw	hammer

Both models in Example 2 fulfill the premises. Cognitive difficulty arises with the number of models which have to be maintained simultaneously (Johnson-Laird 2001). In this paper we work out the main assumptions found in the literature on human spatial (relational) reasoning and undertake a formalization of this in a computational model.

State-of-the-Art

Implementations of the MMT. From an AI perspective a formalization or implementation is necessary. A first algorithm implementing the MMT was presented by (Johnson-Laird and Byrne 1991). This algorithm is able to parse relational premises and identify objects within a coordinate system. Another model, the UNICORE model (Bara, Bucciarelli, and Lombardo 2001), has a similar implementation: The model is based on three considerations: A model must (i) include a grid of positions that are assigned to tokens (terms). Tokens must (ii) have a name, and some terms may (iii) stand in a certain relation to each other.

The cognitive architecture SOAR (Laird, Newell, and Rosenbloom 1987; Newell 1990) "uses neither mental logic nor mental models [...] but instead does a search through a space of possible inferences" (Thagard 2005). A different approach using mental models considers linguistic processes as central to the spatial reasoning process (Polk and Newell 1995). There are, to our knowledge, no implementations covering reasoning with preferred mental models.

Preferred Mental Model Theory. The approach we follow here is referred to as the Preferred Mental Model Theory (PMMT). The PMMT argues that in multiple model cases, human reasoners initially construct only one *Preferred Mental Model* (PMM). The PMM is easier to construct and easier to maintain in working memory compared to all other possible mental models (Rauh et al. 2005). Only if necessary in the model variation phase, is the PMM is varied to find alternative interpretations of the premises and thus accounts for the principle of economicity (Manktelow 1999). Consider the following sequence of abstract premises in Example 3:

(3) A is to the left of B
B is to the left of C
B is to the left of D
D is to the left of E

The resulting mental models that may later be inspected to check some putative conclusion will differ from each other depending on the applied construction principle.

Table 1: Integration principles.

Principle	Mental model	Processing demands
Free first fit (fff)	A B C **D E**	low
Mix	A B **D** C **E**	medium
First fit (ff)	A B **D E** C	high

Terms of the current premises are successively integrated only at free positions. This means that a position is skipped if it is already occupied with some other term that has been integrated in the course of processing a previous premise. The current term is then processed applying the *first-free-fit principle* (fff) and not the *first-fit principle* (ff) which would require preceding term shifting within the mental model. The fff-principle is hypothesized to be less expensive because it just skips occupied positions and requires no additional term shifting operations. Depending on the number of premises and terms a mixture of applicable principles is possible. Some terms may satisfactorily be integrated into the current mental model by the fff-principle. For other terms it may be necessary to apply the ff-principle. We refer to mental model construction that involves both principles as mix-principle. The processing costs for the mix-principle are accordingly hypothesized to be lower than a those of a model constructed merely by the ff-principle, but higher than in the mere fff-principle case.

Spatial Reasoning by Models (SRM). SRM is an approach that specifies a computational model implementing the PMMT (Ragni, Knauff, and Nebel 2005). An SRM-model consists of an (i) input device for the premises, (ii) a two-dimensional spatial array in which the mental model is constructed, inspected, and varied, and (iii) a focus that performs all operations and that can be compared to the read-/write head of the Turing machine. The focus operator performs all operations and is controlled by functions in which strategies for insertion principles can be defined. The SRM-model is able to explain experimental results by applying a standard cost measure for each necessary model operation (Ragni et al. 2007). Model predictions have been tested empirically (Ragni et al. 2007).

In the following, we illustrate how predictions by the SRM-model can be further evaluated. As this model shares features with the Turing machine we aimed for an implementation in a cognitive architecture that is able to reproduce behavioral findings. With the data at hand the SRM -model can be tested and at the same time provides a measure that accounts for cognitive complexity.

The Cognitive Architecture ACT-R. ACT-R is a modular theory and architecture of cognition with an underlying production system as procedural backbone. Production rules operate on symbolic representations of declarative memory items – so-called chunks (Anderson 2007). Modules are characterized by their functionality and each ACT-R module has an associated interface called buffer that allows the exchange information across modules. Both buffers and productions underlie a strict capacity limitation: Each module can keep only one chunk in a buffer at a time and only one production may be selected to modify the chunks. Any buffer manipulation is therefore a strictly serial process. Modules, however, may be active in parallel and consequently more than one chunk may be processed at a time, hence parallelism across modules is allowed.

On the sub-symbolic side, probabilistic processes direct behavior such as production selection from procedural and chunk selection from declarative memory. In example, the availability of chunks to the corresponding retrieval buffer depends on their level of activation, which is calculated on the basis of several aspects. Most notably, once a chunk has been created its initial activation starts decreasing following a fixed decay rate. Chunks may, however, receive additional activation determined by (i) the number of positive retrievals in the past and (ii) spreading activation from the chunk in the goal buffer per default.

Visual information provided by the environment is processed in the vision-module. The goal formulation is constructed in the goal module and for keeping and generating problem-specific sub goals the imaginal module is used.

Formalization

A *relational structure* is a tuple $(D, R_{i\in I})$ for an index set I consisting of a domain D – sometimes called discourse universe – and a set of usually binary relations R_i. For example, an expression *The cinema is left of the church* can be expressed by arrangement relations L, R, etc. over the domain of buildings. Complex expressions can be formed by using connectives like conjunctions and disjunctions (The cinema is to the left of the church *and* the church is in New York). A model $\mathcal{A}$ is *consistent* with a set of premises Φ over a relational language $\mathcal{L}$ (mathematically $\mathcal{A} \models \Phi$ with $\models$ denoting the *consequence relation*) if all expressions of Φ are true in $\mathcal{A}$. Then a conclusion Ψ can be derived from the premise set Φ (mathematically $\Phi \models \Psi$) if the following equivalence relations hold:

$$\begin{array}{lcl} \Phi \models \Psi & \Leftrightarrow & \text{All models of } \Phi \text{ are models of } \Psi. \\ & \Leftrightarrow & \text{There is no model } \mathcal{A} \text{ with } \mathcal{A} \models \Phi \text{ and} \\ & & \mathcal{A} \models \neg\Psi. \end{array}$$

A model $\mathcal{A}$ with the properties (i) $\mathcal{A} \models \Phi$ and (ii) $\mathcal{A} \models \neg\Psi$ is called a *counter-example*. It follows from (ii) that if there is a counter-example to Φ and Ψ then $\Phi \models \Psi$ cannot hold.

The classical (mathematical) consequence relation, however, does not explain how the initial PMM is constructed and varied. Therefore, supported by empirical evidence (Rauh et al. 2005; Ragni et al. 2007) for different calculi, the PMMT has been grounded mathematically. A PMM for a premise set Φ will be denoted as $\mathsf{PMM}(\Phi)$. The PMMT states that alternative models are generated by locally transforming the PMM (Rauh et al. 2005). Transformations can generally be described by the application of mathematical operators (e.g. exchanging adjacent terms (Ragni et al. 2007)) or the moving of terms according to a continuous transformation concept. Any set of applicable operators O is domain specific because there may be differing dimensions for distinct topological terms such as intervals or regions. Common to all operators, however, is that they change the relation of one term to another w.r.t. local transformations. Such continuous transformations can be conceptualized by transitions in so-called generalized neighborhood graphs (Ragni, Knauff, and Nebel 2005).

Definition 1 *The* image *of a model $\mathcal{A}$ w.r.t. an operator application o is*

$$img_o(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{B} = o(\mathcal{A})\}.$$

We set $\text{Transf}^1(\mathsf{PMM}(\Phi), o, \Phi) = \{\mathcal{B} \mid \mathcal{B} \models \Phi \text{ and } \mathcal{B} = img_o(\mathcal{A})\}$, the set of all models of Φ which can be derived by applying o at $\mathcal{A}$.

Definition 2 *For premises Φ, set of operators O and a preferred model $\mathsf{PMM}(\Phi)$, the* preferred consequence relation *is*

$$\begin{array}{lcl} \Phi \models_p \Psi & \Leftrightarrow & \mathsf{PMM}(\Phi) \models \Psi \textit{ and for each } \mathcal{B} \models \Phi \\ & & \textit{with } \mathcal{B} \in \textit{Transf}^1(\mathsf{PMM}(\Phi), O) \\ & & \textit{holds } \mathcal{B} \models \Psi. \end{array}$$

The *restricted consequence relation* $\models_{kp}$ reflects that humans might apply only a fixed number of operations and generate only a finite number of alternative models. There is empirical evidence for this (Ragni et al. 2007). $\text{Transf}^k(\mathsf{PMM}(\Phi), o, \Phi)$ describes the sequence of k applications of o at Φ.

Definition 3 *For premises Φ, set of operators O and a preferred model $\mathsf{PMM}(\Phi)$, the* restricted consequence relation *is*

$$\begin{array}{lcl} \Phi \models_{kp} \Psi & \Leftrightarrow & \mathsf{PMM}(\Phi) \models \Psi \textit{ and for each } \mathcal{B} \models \Phi \\ & & \textit{with } \mathcal{B} \in \textit{Transf}^k(\mathsf{PMM}(\Phi), O) \\ & & \textit{holds } \mathcal{B} \models \Psi. \end{array}$$

Computational Model

The previous considerations allow us to outline a first computational model. Such a model can be regarded as a necessary condition to the introduction of a complexity measure and will be defined as a quadruple (I, O, F, C), with:

I**, the input mechanism.** This process reads the premises from an external device. We assume that there is an external "parser" that supplies the correct meaning (Goodwin and Johnson-Laird 2005).

O**, the set of object names.**

F**, the spatial focus.** The focus operates on a spatial array and is initially set to position (0, 0). It can move right, left, forward and backward (Vandierendonck, Dierckx, and De Vooght 2004).

C**, the control process.** This process is responsible for the control of the focus and other executive functions.

We ignore natural language processing components for spatial inference expressions because these are not central to the actual reasoning process in the mental model theory (Johnson-Laird and Byrne 1991). In addition, there is no principle reason that premises have to be presented as sentences. A theory that expects input from an external language processing device is more flexible; it can also account for data from experiments in which premises were presented as figures rather than verbally (Fangmeier et al. 2006). The third and fourth components of our definition reflect the main theoretical assumptions of our theory (cf. Figure 1). The third component states that a model is represented in spatial working memory, which is conceptualized as a spatial array. The spatial array is a two-dimensional grid structure, in which the terms can be inserted, moved, and removed by the fourth component, the focus. This focus (a similar concept was proposed by (Vandierendonck, Dierckx, and De Vooght 2004)) is a manipulation device able to perform all operations on the spatial array.

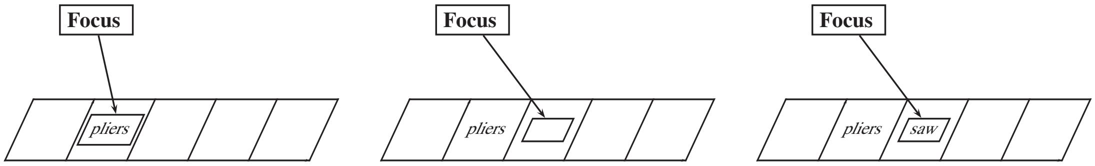


Figure 1: The process model. The three figures depict the processing of the premise: ”The pliers are to the left of the saw”. The focus inserts the terms successively by movements of the focus. The control process is illustrated in Figure 2.

Mental Model Construction. The construction process begins with the first premise and an empty array. The input mechanism provides us with the terms and the relation. The focus of the SRM-model places the RO of the premise first, then moves one cell according to the direction of the relation and places the LO in the next free cell. In Example 1b, $pliers$ is inserted in the array first, then the focus moves to the right and inserts the $hammer$. The model has to check the type of each new premise and insert the term(s) according to the specific case.

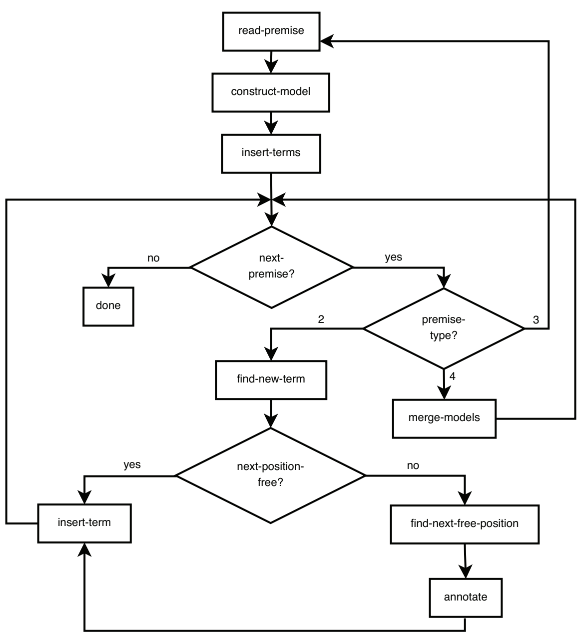


Figure 2: The mental model construction phase. The SRM-model inserts terms into mental models according to the type of the respective premise.

Different premise types can be distinguished: A premise with one new term, and one term already contained in the model, is called premise of Type 2. Type 3 premises are those, in which no term is already contained in the model. If both terms are already given but in different sub models, the premise is called Type 4.

For Type 2 premises the SRM-model proceeds as follows (cf. Figure 2): If the new term in the premise cannot be placed according to the given relation, the focus annotates the object by adding the relational information of the new term and inserts the new term at the first-free position (the already defined fff-principle). This annotation is necessary for the model variation case, where additional models are generated. In Example 1 the second premise is of Type 2, because the $pliers$ is already in the model, the saw is inserted to the right of $hammer$ according to the fff-principle, and gets the annotation ’Right of the pliers’. The next processed premise is also of Type 2 and the $screwdriver$, which is not in the model, is inserted directly to the right of the saw, but because the saw has an annotation, the $screwdriver$ must be annotated too.

For Type 3 premises the SRM proceeds as follows: As both terms are new, a new model is generated into which both terms are entered (the SRM-model proceeds as in the first-premise case). For example, for a premise ’A left of B’ and a premise ’C right of D’, two models have to be generated to contain each premise. So sub model AB would be the first and DC the second sub model.

For Type 4 premises the SRM proceeds as follows: Both terms are contained in different sub models, both sub models have to be merged according to the relation of the premise. Now the construction phase is complete. Annotations are only used for Type 2 premises, i.e. in the construction process an annotation is made only for indeterminate object positions.

Model Inspection. After mental model construction, the *inspection phase* checks a (given) putative conclusion. The focus moves to the first given object (RO), and from there it inspects the model according to the relation in order to find the second object (LO). The search process terminates since the model is bounded by its number of objects.

Model Variation. *Model variation* is necessary to check if a conclusion is valid. The variation process starts from the generated PMM in which the putative conclusion holds. Other mental models are generated by a focus process and the use of the annotations. The focus performs local transformations to generate a counter-example (to the putative conclusion). The focus checks whether one of the terms in the conclusion is annotated. Annotations of objects specify the positional relation to reference terms. These reference terms are called *anchors*. If the annotations of one of the terms include the information of the putative conclusion (the relation and the other object) then the putative conclusion holds. If none of the conclusions' terms appear in the annotations the conclusion holds.

If there is an annotation to one object (and not to the other), as in the example conclusion '$screwdriver$ is to the right of the $hammer$', the only object of the conclusion to be moved is $screwdriver$. If the object to be moved has an anchor, it may be necessary to move the anchor first. If both objects have annotations, then first the LO of the putative conclusion is exchanged. LO is exchanged into the direction of RO until its anchor is reached. If, thereby, an inconsistent model is generated, the algorithm stops and returns false, otherwise true. The SRM adheres to the principle of local transformations by successively exchanging neighbored objects. For other relations, other principles can be used.

Processing Demands. The above described computational model provides a process model for generating, inspecting and alternating PMMs. Having a computational model for mental models, we can assign unit costs to each operation (Ragni, Knauff, and Nebel 2005). The more operations a task requires the more difficult it is (according to this measure based on our model). To test this SRM prediction we investigate the problems in Table 1. The PMM requires less operations than any other model, as for premises of type 2 the new object is inserted at the next free position (fff-principle). Such operations are easier to perform than copying and moving each object to insert the new object according to the ff-principle. If we have to alternate the mental model in the variation phase, it should be easier to generate the mix model than the ff-model. There is, however, evidence that a human reasoner does not generate all possible models. Our model explains this in that specific annotations cannot be retrieved (e.g. if they are least recently used). To achieve this, we must implement our SRM-model in the cognitive architecture ACT-R which will extend our SRM with activation functions and make predictions about response times and error rates. Then we can test the predictions of our computational model SRM.

An ACT-R Model of the SRM

Figure 3 illustrates how the ACT-R model constructs a PMM according to the fff-principle. The parameters for activation noise, latency and retrieval threshold deviated from the default values. Once parameters had been set they were held constant across runs.

The imaginal module is responsible for building up premises and mental models from information provided by the visual module. Due to the capacity limitation described above, the imaginal buffer has to be cleared when a new premise is to be built up. The incomplete PMM holding the related terms that have been thus far seen and integrated has

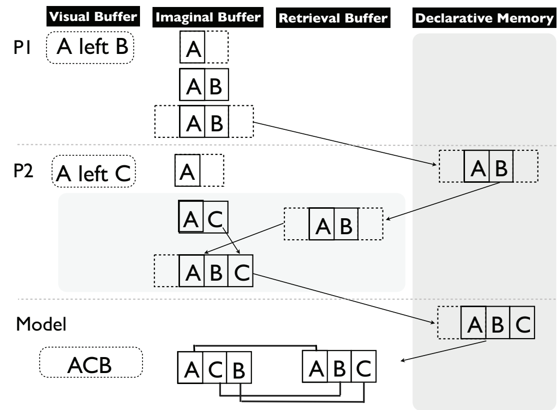


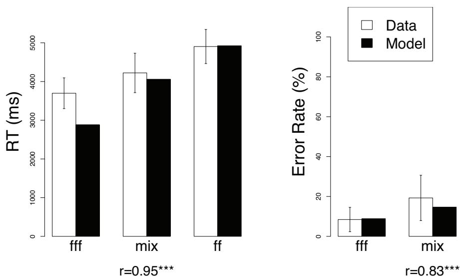
Figure 3: The ACT-R model processing the premises (P1: A is left of B; P2: A is left of C): Buffer columns show the different buffers each holding the respective chunks. The right column shows relevant declarative memory contents.

to be released to declarative memory. From there it can be retrieved again in a subsequent step and the spatial relational information concerning the terms is again accessible from the retrieval buffer.

The contents of the imaginal buffer that at this moment hold the information of the cur- rent premise can be integrated into the PMM because information can be transferred across buffers. There is, however, a certain chance of forgetting the chunk representing the PMM. Such a retrieval failure is one possible source of error. In general, the probability of a retrieval failure increases the more time has passed since the last retrieval of the corresponding chunk.

In the processing example illustrated in Figure 3 the conclusion consists of a mental model that at first has to be built up analogous to the premises. The PMM accordingly has to be released to declarative memory from where it is retrieved again immediately. Now residing in the retrieval buffer, comparison processes with the conclusion model in the imaginal buffer can take place. The conclusion model, however, contradicts the PMM. Consequently, to verify the conclusion model the term constellation in the PMM has to be varied. The ACT-R model encodes previous integration options, i.e. those determined by the ff-principle, in so-called annotated premises. Any premise that contains information that allows an ff-integration of a term is said to get annotated before it is released to declarative memory. In case it is necessary to vary some PMM a specific retrieval request for an appropriate annotated premise can be made to return the necessary information for reconstructing the PMM.

Empirical Evaluation. Figure 4 compares data from 21 participants with predictions based on 10.000 model runs. Response times and error rates refer to 12 experimental trials presenting mental models that had to be validated according to the fff-, mix- or ff-principle. Correlations between model predictions and human data show a significant effect for both response times, $r = .95$, $p < .001$, and error rate, $r = .83$, $p < .001$. Correlations were computed on a by task aggregate level resulting in 12 data points for each human and model means.



(a) Response times (b) Error rates

Figure 4: Response times and error rates for possible valid mental models according to the fff-, mix- and ff-principles. Error bars show 95% confidence intervals. Asterisks mark significant correlations at a $p < .001$ (***) level.

Conclusion and Outlook

The formal complexity of problems is typically determined by space and time requirements of the algorithm solving them. Different models are equivalent with respect to specific complexity classes (Papadimitriou 1994). Cognitive complexity in turn is more difficult to identify. In this article, we worked out and formalized central assumptions of a psychological theory for spatial relational reasoning, the Preferred Mental Model Theory. To be able to predict operation costs a computational model was necessary. On the basis of the SRM-approach we implemented a computational model in the cognitive architecture ACT-R to test the cognitive adequacy of the predictions. Memory allocation strategies in ACT-R followed the least recently used principle. This explains why in some cases certain mental models cannot be retrieved. A comparison of model predictions and behavioral data resulted in substantial correlations (cf. Figure 4). The presented ACT-R model is not an attempt to cover all aspects of deductive reasoning and MMT, but rather to add to a methodology. Namely to formalize psychological theories by using methods of knowledge representation. To identify the formal aspects of problems, and to transform such models into cognitive models in a cognitive architecture. Future work must investigate predictions for two-dimensional spatial problems (Johnson-Laird and Byrne 1991).

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