Parameterized Complexity of Problems in Coalitional Resource Games*

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Abstract

Coalition formation is a key topic in multi-agent systems. Coalitions enable agents to achieve goals that they may not have been able to achieve on their own. Previous work has shown problems in coalition games to be computationally hard. Wooldridge and Dunne (Artifi. Intell. 2006) studied the classical computational complexity of several natural decision problems in Coalitional Resource Games (CRG) - games in which each agent is endowed with a set of resources and coalitions can bring about a set of goals if they are collectively endowed with the necessary amount of resources. The input of coalitional resource games bundles together several elements, e.g., the agent set Ag, the goal set G, the resource set R, etc. Shrot et al. (AAMAS 2009) examine coalition formation problems in the CRG model using the theory of Parameterized Complexity. Their refined analysis shows that not all parts of input act equal - some instances of the problem are indeed tractable while others still remain intractable.

We answer an important question left open by Shrot, Aumann, and Kraus by showing that the SC Problem (checking whether a Coalition is Successful) is W[1]-hard when parameterized by the size of the coalition. Then via a single theme of reduction from SC, we are able to show that various problems related to resources, resource bounds, and resource conflicts introduced by Wooldridge et al. are (i) W[1]-hard or co-W[1]-hard w.r.t the size of the coalition; and (ii) Para-NP-hard or co-Para-NP-hard w.r.t $|R|$. When parameterized by $|G|$ or $|R| + |Ag|$, we give a general algorithm which proves that these problems are indeed tractable.

I - Introduction

Coalitions

In multi-agent systems (MAS), where each agent has limited resources, the formation of coalitions of agents is a very powerful tool (Wooldridge 2009). Coalitions enable agents to accomplish goals they may not have been able to accomplish individually. As such, understanding and predicting the dynamics of coalitions formation, e.g., which coalitions are more beneficial and/or more likely to emerge, is a question of considerable interest in multi-agent systems. Unfortunately, a range of previous studies have shown that many of these problems are computationally complex (Wooldridge and Dunne 2004; 2006). However, as noted by Garey and Johnson (Garey and Johnson 1979), hardness results, such as NP-completeness, should merely constitute the beginning of research. NP-hardness just indicates that a general solution for all instances of the problem most probably does not exist. Still, efficient solutions for important sub-classes may well exist.

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Formal Model of Coalition Resource Games

The framework we use to model coalitions is the CRG model introduced in (Wooldridge and Dunne 2006), defined as follows. The model contains a non-empty, finite set $Ag = \{a_1, \ldots, a_n\}$ of *agents*. A *coalition*, typically denoted by C, is simply a set of agents, i.e., a subset of Ag. The *grand coalition* is the set of all agents, Ag. There is also a finite set of *goals* G. Each agent $i \in Ag$ is associated with a subset G_i of the goals. Agent i is *satisfied* if at least one member of G_i is achieved, and unsatisfied otherwise. Achieving the goals requires the expenditure of *resources*, drawn from the total set of resource types R. Achieving different goals may require different quantities of each resource type. The quantity $\mathbf{req}(g, r)$ denotes the amount of resource r required to achieve goal g. It is assumed that $\mathbf{req}(g, r)$ is a non-negative integer. Each agent is *endowed* certain amounts of some or all of the resource types. The quantity $\mathbf{en}(i, r)$ denotes the amount of resource r endowed to agent i. Again, it is assumed that $\mathbf{en}(i, r)$ is a non-negative integer. Formally, a *Coalition Resource Game* Γ is a $(n + 5)$-tuple given by

$$\Gamma = \langle Ag, G, R, G_1, G_2, \ldots, G_n, \mathbf{en}, \mathbf{req} \rangle$$

where:

- $Ag = \{a_1, a_2, \ldots, a_n\}$ is the set of agents
- $G = \{g_1, g_2, \ldots, g_m\}$ is the set of possible goals
- $R = \{r_1, r_2, \ldots, r_t\}$ is the set of resources
- For each $i \in Ag$, G_i is a subset of G such that any of the goals in G_i would satisfy i but i is indifferent between the members of G_i
- $\mathbf{en}$: $Ag \times R \to \mathbb{N} \cup \{0\}$ is the endowment function

- **req** : $G \times R \rightarrow \mathbb{N} \cup \{0\}$ is the requirement function

The endowment function **en** extends to coalitions by summing up endowments of its members as

$$\mathbf{en}(C, r) = \sum_{i \in C} \mathbf{en}(i, r) \qquad \forall r \in R$$

The requirement function **req** extends to sets of goals by summing up requirements of its members as

$$\mathbf{req}(G', r) = \sum_{g \in G'} \mathbf{req}(g, r) \qquad \forall r \in R$$

A set of goals G' *satisfies* agent i if $G_i \cap G' \neq \emptyset$ and satisfies a coalition C if it satisfies every member of C. A set of goals G' is *feasible* for coalition C if that coalition is endowed with sufficient resources to achieve all goals in G', i.e., for all $r \in R$ we have $\mathbf{req}(G', r) \leq \mathbf{en}(C, r)$. Finally we say that a coalition C is *successful* if there exists a non-empty set of goals G' that satisfies C and is feasible for it. In general, we use the notation $succ(C) = \{G' \mid G' \subseteq G,\ G' \neq \emptyset$ and G' is successful for $C\}$. The CRG models many real-world situations like the virtual organizations problem (Conitzer et al. 2006) and voting domains.

II - Problem Definitions and Previous Work

Problems Related to Coalition Formation

Shrot et al. 2009 considered the following four problems related to coalitions.

1. SUCCESSFUL COALITION (SC)
   Instance: A CRG Γ and a coalition C
   Question: Is C successful?
2. EXISTS A SUCCESSFUL COALITION OF SIZE k (ESCK)
   Instance: A CRG Γ and an integer k
   Question: Does there exist a successful coalition of size exactly k?
3. MAXIMAL COALITION (MAXC)
   Instance: A CRG Γ and a coalition C
   Question: Is every proper superset of C not successful?
4. MAXIMAL SUCCESSFUL COALITION (MAXS)
   Instance: A CRG Γ and a coalition C
   Question: Is C successful and every proper superset of C not successful?

The results from Shrot et al.2009 are summarized in the table below.

	SC	ESCK	MAXC, MAXSC
$\lvert G\rvert$	FPT	FPT	FPT
$\lvert C\rvert$	?	W[1]-H	W[1]-H
$\lvert R\rvert$	para-NP-H	?	para-NP-H
$\lvert Ag\rvert + \lvert R\rvert$	FPT	?	FPT

In this work we consider the problems which were defined by Wooldridge et al. 2006 but were not considered by Shrot et al. 2009. We define these problems in detail in the following sections.

Parameterized Complexity

The core idea of parameterized complexity is to single out a specific part of the input as a parameter and ask whether the problem admits an algorithm that is efficient in all but that parameter. The definitions in this section are from Flum and Grohe 2006 and Downey 2003.

Definition 1 *Let Σ be a finite alphabet.*

1. *A **parametrization** of Σ^* is a* mapping $\kappa : \Sigma^* \rightarrow \mathbb{N}$ *that is* polynomial time computable.
2. *A **parameterized problem** (over Σ) is a pair (Q, κ) consisting of a set $Q \subseteq \Sigma^*$ of strings over Σ and a parameterization κ of Σ^*.*

As stated, given a parameterized problem we seek an algorithm that is efficient in all but the parameter. This is captured by the notion of *fixed parameter tractability*, as follows:

Definition 2 *A parameterized problem (Q, κ) is fixed parameter tractable (FPT) if there exist an algorithm $\mathbb{A}$, a constant α, and a computable function f, such that $\mathbb{A}$ decides Q in time $f(\kappa(x))|x|^\alpha$.*

While the core aim of parameterized complexity is to identify problems that are fixed-parameter tractable, it has also developed an extensive complexity theory, allowing to prove hardness results, e.g., that certain problems are (most probably) not FPT. To this end, several parameterized complexity classes have been defined. Two of these classes are W[1] and Para-NP (PNP). There is strong evidence to believe that both these classes are not contained in FPT (much like NP is probably not contained in P). Thus, W[1]-hard and PNP-Hard problems are most probably not fixed-parameter tractable. The class W[1] can be defined by its core complete problem, defined as follows:

SHORT NONDETERMINISTIC TURING MACHINE COMPUTATION
Instance: A single-tape, single-head non-deterministic Turing machine M, a word x, and an integer k
Question: Is there a computation of M on input x that reaches the accepting state in at most k steps?
Parameter: k

Note that this definition is analogous to that of NP, with the addition of the parameter k.

Definition 3 *The class* W[1] *contains all parameterized problems FPT-reducible (defined hereunder) to Short-Nondeterministic-Turing-Machine-Computation.*

Definition 4 *A parameterized problem (Q, κ) is in Para-NP (PNP) if there exists a non-deterministic Turing machine M, constant α, and an arbitrary computable function f, such that for any input x, M decides if $x \in Q$ in time $\leq |x|^\alpha f(\kappa(x))$.*

Establishing hardness results most frequently requires reductions. In parameterized complexity, we use FPT-reduction, defined as follows:

Definition 5 *Let (Q, κ) and (Q', κ') be parameterized problems over the alphabets Σ and Σ' respectively. An FPT-reduction (FPT many-to-one reduction) from (Q, κ) to (Q', κ') is a mapping $R : \Sigma^* \rightarrow (\Sigma')^*$ such that:*

1. *For all $x \in \Sigma^*$ we have $x \in Q \Leftrightarrow R(x) \in Q'$.*
2. *R is computable in time $f(\kappa(x))|x|^\alpha$ for some constant α and an arbitrary function f.*

3. *There is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $\kappa'(R(x)) \leq g(\kappa(x))$ for all $x \in \Sigma^*$.*

III - Our Results & Techniques

We consider problems regarding resources bounds and resource conflicts which were shown to be computationally hard in Wooldridge et al. (2006) but were not considered in Shrot et al. We also solve three open questions posed in Shrot et al. by showing that

1. SC parameterized by $|C|$ is *W[1]-hard*
2. ESCK parameterized by $|Ag| + |R|$ is *FPT*
3. ESCK parameterized by $|R|$ is *PNP-hard*

We study the complexity of NR, SNR, CGRO, RPEGS, SCRB, and CC problems when parameterized by natural parameters $|G|$, $|C|$, $|R|$, and $|Ag|+|R|$. We also give a general integer program which with slight modifications for each problem shows that these problems are FPT when parameterized by $|G|$ or $|Ag| + |R|$ (except CC parameterized by $|Ag| + |R|$ which is open). We note that Shrot et al. showed that SC parameterized by $|R|$ is PNP-hard. We complete this hardness result by showing that SC parameterized by $|C|$ is W[1]-hard and thus answer their open question. Using these hardness results and via a single theme of parameter preserving reductions we show that hardness results for all of the above problems when parameterized by $|R|$ and $|C|$. We also show that Theorem 3.2 of (Shrot, Aumann, and Kraus 2009) is false - which claims that ESCK is FPT when parameterized by $|G|$. We give a counterexample to their proposed algorithm and show that the problem is indeed PNP-hard.
These results help us to understand the role of various components of the input and identify which ones actually make the input hard. Since all the problems we considered remain intractable when parameterized by $|C|$ or $|R|$, there is no point in trying to restrict these parameters. On the other hand, most of the problems are FPT when parameterized by $|G|$ or $|Ag| + |R|$ and thus we might enforce this restriction in real-life situations to ensure the tractability of these problems.

Due to the lack of space some of the proofs are omitted. The full version of the paper is available on arXiv.

IV - Problems Left Open in Shrot et al. (2009)

First we show that SC parameterized by $|C|$ is W[1]-hard.

Theorem 1 *SC is W[1]-hard when parameterized by $|C|$.*

Proof. We prove this by reduction from Independent Set (parameterized by size of independent set) which is a well-known W[1]-complete problem. Let $H = (V, E)$ be a graph with $V = \{x_1, \ldots, x_n\}$ and $E = \{e_1, \ldots, e_m\}$. Let k be a given integer. We also assume that H has no isolated points as we can just add those points to the independent set and decrease the parameter appropriately. We build a CRG Γ as follows:

$$\Gamma = \langle Ag, G, R, G_1, G_2, \ldots, G_k, \mathbf{en}, \mathbf{req} \rangle$$

where

- $Ag = \{c_1, \ldots, c_k\}$
- $G_i = \{g_i^1, \ldots, g_i^n\}$ for all $i \in [k]$
- $G = \bigcup_{i=1}^{k} G_i$
- $R = \{r_1, \ldots, r_m\}$
- For all $i \in [k], j \in [m]$, $\mathbf{en}(c_i, r_j) = 1$
- For all $i \in [k], j \in [m]$ and $\ell \in [n]$, we have $\mathbf{req}(g_i^\ell, r_j) = k$ if e_j and x_ℓ are incident in H and $\mathbf{req}(g_i^\ell, r_j) = 0$ otherwise

We claim that H has an independent set of size k if and only if the grand coalition Ag is successful in Γ. Note that $|Ag| = k, |G| = nk, |R| = m$ thus this reduction shows that the SC problem is W[1]-hard. □

We note that the SC problem can be solved in $O(|G|^{|C|} \times |R|)$ time (since we only need to check the subsets of size at most $|C|$ of G) and thus SC parameterized by $|C|$ is not PNP-hard. Now we answer the only remaining open problem by Shrot et al. by showing that ESCK parameterized by $|R|$ is PNP-hard.

Theorem 2 *Checking whether there exists a successful coalition of size k (ESCK) is PNP-hard when parameterized by $|R|$.*

Proof. We prove this by reduction from SC which was shown to be PNP-hard with respect to the parameter $|R|$ in Theorem 3.8 of (Shrot, Aumann, and Kraus 2009).
Let (Γ, C) be a given instance of SC. We consider an instance (Γ', k) of ESCK where $Ag' = C, R' = R, k = |C|$, and $G_i' = G_i \;\; \forall\, i \in C$. We claim that SC answers YES if and only if ESCK answers YES. Note that $|Ag'| = k, |G'| = |G|, |R'| = |R|$ thus this reduction shows that the ESCK problem is PNP-hard. □

V - Problems Related to Resources

For a coalition C, we recollect the notation we use: $succ(C) = \{G' \mid G' \subseteq G \; ; \; G' \neq \emptyset$ and G' both satisfies C and is feasible for it$\}$. In this section we show hardness results for three different problems related to resources.

Necessary Resource (NR)

The idea of a *necessary resource* is similar to that of a veto player in the context of conventional coalition games. A resource is said to be *necessary* if the accomplishment of any set of goals which is successful for the coalition would need a non-zero consumption of this resource. Thus if a necessary resource is scarce then the agents possessing the resource become important. We consider the NECESSARY RESOURCE problem: Given a coalition C and a resource r answer YES if and only if $\mathbf{req}(G', r) > 0$ for all $G' \in succ(C)$. NR was shown to be co-NP-complete in Wooldridge et al. 2006. We note that if C is not successful, then NR vacuously answers YES. We give a reduction from SC to $\overline{NR}$.

Lemma 1 *Given an instance (Γ, C) of SC we can construct an instance (Γ', C', r') of NR such that SC answers YES iff NR answers NO.*

	SC	ESCK	NR, CGRO, RPEGS	SNR	SCRB	CC
	(*NPC*)	(*NPC*)	(*co-NPC*)	(D^p*-complete*)	(*NPC*)	(*co-NPC*)
$\|G\|$	FPT	**PNP-hard**	**FPT**	**FPT**	**FPT**	**FPT**
$\|C\|$	**W[1]-hard**	W[1]-hard	**co-W[1]-hard**	**W[1]-hard**	**co-W[1]-hard**	**co-W[1]-hard**
$\|R\|$	PNP-hard	**PNP-hard**	**co-PNP-hard**	**PNP-hard**	**co-PNP-hard**	**co-PNP-hard**
$\|Ag\| + \|R\|$	FPT	**FPT**	**FPT**	**FPT**	**FPT**	?

Theorem 3 *The parameterized complexity status of Necessary Resource is as follows:*
FPT when parameterized by $|G|$
co-W[1]-hard when parameterized by $|C|$
co-PNP-hard when parameterized by $|R|$

Proof. When parameterized by $|G|$, we consider all $2^{|G|}$ subsets of G. For each subset, we can check in polynomial time if it is a member of $succ(C)$ and if it requires non-zero quantity of the resource given in the input.
The other two claims follow from Lemma 1, Theorem 3.8 in Shrot et al., and Theorem 1. □

Strictly Necessary Resource (SNR)

The fact that a resource is necessary does not mean that it will be used. Because the coalition in question can be unsuccessful and hence the resource is trivially necessary. So we have the STRICTLY NECESSARY RESOURCE problem: Given a coalition C and a resource r answer YES if and only if $succ(C) \neq \emptyset$ and $\forall\ G' \in succ(C)$ we have $\mathbf{req}(G', r) > 0$. SNR was shown to be strongly D^p-complete in Wooldridge et al. 2006. To prove the parameterized hardness results, we give a reduction from SC to SNR.

Lemma 2 *Given an instance* (Γ, C) *of SC we can construct an instance* (Γ', C', r') *of SNR such that SC answers YES iff SNR answers YES.*

Theorem 4 *The parameterized complexity status of Strictly Necessary Resource is as follows:*
FPT when parameterized by $|G|$
W[1]-hard when parameterized by $|C|$
PNP-hard when parameterized by $|R|$

Proof. When parameterized by $|G|$, we consider all $2^{|G|}$ subsets of G. For each subset, we can check in polynomial time if it is a member of $succ(C)$ and if it requires non-zero quantity of the resource given in the input.
The other two claims follow from Lemma 2, Theorem 3.8 in Shrot et al., and Theorem 1. □

(C, G_0, r)-Optimality (CGRO)

We may want to consider the issue of *minimizing* usage of a particular resource. If satisfaction is the only issue, then a coalition C will be equally happy between any of the goal sets in $succ(C)$. However in practical situations we may want to choose a goal set among $succ(C)$ which minimizes the usage of some particular *costly* resource. Thus we have the (C, G_0, r)-OPTIMALITY problem: Given a coalition C, resource r, and a goal set $G_0 \in succ(C)$ answer YES if and only if $\mathbf{req}(G', r) \geq \mathbf{req}(G_0, r)$ for all $G' \in succ(C)$. CGRO was shown to be strongly co-NP-complete in Wooldridge et al. 2006. To prove the parameterized hardness results, we give a reduction from SC to $\overline{CGRO}$.

Lemma 3 *Given an instance* (Γ, C) *of SC we can construct an instance* (Γ', C', G_0, r') *of CGRO such that SC answers YES iff CGRO answers NO.*

Theorem 5 *The parameterized complexity status of* (C, G_0, r)*-Optimality is as follows:*
FPT when parameterized by $|G|$
co-W[1]-hard when parameterized by $|C|$
co-PNP-hard when parameterized by $|R|$

Proof. When parameterized by $|G|$, we consider all $2^{|G|}$ subsets of G. For each subset, we can check in polynomial time if it is a member of $succ(C)$ and if it requires atleast $\mathbf{req}(G_0, r')$ quantity of resource r' where G_0 and r' are given in the input.
The other two claims follow from Lemma 3, Theorem 3.8 in Shrot et al., and Theorem 1. □

VI - Problems Related to Resource Bounds

R-Pareto Efficient Goal Set (RPEGS)

We use the idea of *Pareto Efficiency* to measure the optimality of a goal set w.r.t the set of all resources. In our model we say that a goal set G' is *R-Pareto Efficient* w.r.t a coalition C if no goal set in $succ_\Gamma(C)$ requires at most as much of every resource and strictly less of some resource. More formally we say that a goal set G' is *R-Pareto Efficient* w.r.t a coalition C if and only if $\forall\ G'' \in succ_\Gamma(C)$,

$$\exists\, r_1 \in R : \mathbf{req}(G'', r_1) < \mathbf{req}(G', r_1) \Rightarrow \exists\, r_2 \in R : \mathbf{req}(G'', r_2) > \mathbf{req}(G', r_2)$$

We note that G' is not necessarily in $succ(C)$. Thus we have the R-PARETO EFFICIENT GOAL SET problem: Given a coalition C and a goal set G_0 answer YES if and only if G_0 is *R-Pareto Efficient* w.r.t C. Wooldridge et al. 2006 show that RPEGS is strongly co-NP-complete. To prove the parameterized hardness results, we give a reduction from SC to $\overline{RPEGS}$.

Lemma 4 *Given an instance* (Γ, C) *of SC we can construct an instance* (Γ', C', G_0) *of RPEGS such that SC answers YES iff RPEGS answers NO.*

Theorem 6 *The parameterized complexity status of R-Pareto Efficient Goal Set is as follows:*
FPT when parameterized by $|G|$
co-W[1]-hard when parameterized by $|C|$
co-PNP-hard when parameterized by $|R|$

Proof. When parameterized by $|G|$, we consider all $2^{|G|}$ subsets of G. For each subset, we can check in polynomial time if it is a member of $succ(C)$ and if it shows that G_0 is not *R-Pareto Efficient*.
The other two claims follow from Lemma 4, Theorem 3.8 in Shrot et al., and Theorem 1. □

Successful Coalition With Resource Bound (SCRB)

In real-life situations we typically have a bound on the amount of each resource. A *resource bound* is a function $\mathbf{b} : R \rightarrow \mathbb{N}$ with the interpretation that each coalition has at most $\mathbf{b}(r)$ quantity of resource r for every $r \in R$. We say that a goal set G_0 *respects* a resource bound **b** w.r.t. a given CRG Γ iff $\forall\ r \in R$ we have $\mathbf{b}(r) \geq \mathbf{req}(G_0, r)$. Thus we have the Successful Coalition With Resource Bound problem: Given a coalition C and a resource bound **b** answer YES if and only if $\exists\, G_0 \in succ(C)$ such that G_0 respects **b**. Wooldridge et al. 2006 show that SCRB is strongly NP-complete. To prove the parameterized hardness results, we give a reduction from SC to $\overline{SCRB}$.

Lemma 5 *Given an instance (Γ, C) of SC we can construct an instance $(\Gamma', C', \boldsymbol{b}')$ of SCRB such that SC answers YES if and only if SCRB answers NO.*

Theorem 7 *The parameterized complexity status of Successful Coalition With Resource Bound (SCRB) is as follows:*
FPT when parameterized by $|G|$
co-W[1]-hard when parameterized by $|C|$
co-PNP-hard when parameterized by $|R|$

Proof. When parameterized by $|G|$, we consider all $2^{|G|}$ subsets of G. For each subset,we can check in polynomial time if it is a member of $succ(C)$ and if it requires non-zero quantity of the resource given in the input.
The other two claims follow from Lemma 5, Theorem 3.8 in Shrot et al., and Theorem 1. □

VII - Problems Related to Resource Conflicts

Conflicting Coalitions (CC)

When two or more coalitions desire to use some scarce resource, it leads to a *conflict* in the system. This issue is a classic problem in distributed and concurrent systems. In our framework we say that two goal sets are in *conflict* w.r.t a resource bound if they are individually achievable within the resource bound but their union is not. Formally a *resource bound* is a function $\mathbf{b} : R \rightarrow \mathbb{N}$ with the interpretation that each coalition has at most $\mathbf{b}(r)$ quantity of resource r for every $r \in R$. We say that a goal set G_0 *respects* a resource bound **b** w.r.t. a given CRG Γ if and only if $\forall\ r \in R$ we have $\mathbf{b}(r) \geq \mathbf{req}(G_0, r)$. We denote by $cgs(G_1, G_2, \mathbf{b})$ the fact that G_1 and G_2 are in conflict w.r.t **b**. Formally, $cgs(G_1, G_2, \mathbf{b})$ is defined as respects$(G_1, \mathbf{b}) \wedge$ respects$(G_2, \mathbf{b}) \wedge \neg$respects$(G_1 \cup G_2, \mathbf{b})$. Thus we have the CONFLICTING COALITIONS problem: Given coalitions C_1, C_2 and a resource bound **b** answer YES if and only if $\forall\ G_1 \in succ(C_1)$ and $\forall\ G_2 \in succ(C_2)$ we have $cgs(G_1, G_2, \mathbf{b})$. Wooldridge et al. 2006 show that CC is strongly co-NP-complete. To prove the parameterized hardness results, we give a reduction from SC to $\overline{CC}$.

Lemma 6 *Given an instance (Γ, C) of SC we can construct an instance $(\Gamma', C_1', C_2', \boldsymbol{b})$ of CC such that SC answers YES if and only if CC answers NO.*

Theorem 8 *The parameterized complexity status of Conflicting Coalitions (CC) is as follows:*
FPT when parameterized by $|G|$
co-W[1]-hard when parameterized by $|C|$
co-PNP-hard when parameterized by $|R|$

Proof. When parameterized by $|G|$, we consider all $2^{|G|}$ choices for G_1 and G_2. Given a choice (G_1, G_2) we can check in polynomial time if G_1 and G_2 are members of $succ(C_1)$ and $succ(C_2)$ respectively. Also we can check the condition $cgs(G_1, G_2, \mathbf{b})$ in polynomial time.
The other two claims follow from Lemma 6, Theorem 3.8 in Shrot et al., and Theorem 1. □

VIII - The Parameter $|Ag| + |R|$: Case of Bounded Agents plus Resources

Considering the results in previous sections, we see that even in the case that the size of the coalition or the number of resources is bounded the problem remains computationally hard. A natural question is what happens if the total number of agents plus resources is bounded? Shrot et al. 2009 show that by this parameterization the problems SC, MAXC, and MAXSC have FPT algorithms and they left the corresponding question for the ESCK open. We generalize the integer program given by Shrot et al. to give a FPT algorithm for ESCK. By using a similar approach we design FPT algorithms for the five other problems (NR, SNR, CGRO, SCRB, RPEGS) considered in this paper.

The integer program we define is a satisfiability problem. It consists of a set of constraints, and the question is whether there exists an integral solution to this set. Consider the following general integer program (which we call *GIP*):

$$\forall i \in Ag : \qquad \sum_{g \in G_i} \mathbf{x_g} \geq \mathbf{y_i} \qquad (1)$$

$$\forall r \in R : \sum_{g \in G} \mathbf{x_g} \times req(g, r) \leq \sum_{i \in Ag} \mathbf{y_i} \times en(i, r) \qquad (2)$$

$$\mathbf{y_i}, \mathbf{x_g} \in \{0, 1\}$$

In this setting for each $i \in Ag$, $\mathbf{y_i}$ indicates the event that agent i is participating in the coalition and for each $g \in G$, $\mathbf{x_g}$ indicates the event that goal g is achieved. One can show that any solution for GIP is a coalition of agents and a successful set of goals for that coalition.
The above integer program has $|Ag| + |R|$ constraints and in Flum and Grohe 2006 it is shown that INTEGER LINEAR PROGRAMMING is FPT wrt number of constraints or number of variables. We give tractable algorithms for each problem by adding a few constraints to our general integer program.

Theorem 9 *ESCK parameterized by $|Ag| + |R|$ is FPT.*

Proof. In ESCK we have to insure that exactly k number of agents will be selected. Therefore adding the constraint $\sum_{i \in Ag} \mathbf{y_i} = k$ to GIP gives the integer program for the ESCK problem. Since the number of constraints, (i.e., $|Ag| + |R| + 1$) is fixed the proof is complete. □

In the other five problems NR, SNR, CGRO, SCRB, and RPEGS the coalition C is given. Thus we change the variables $\mathbf{y_i}$'s to constants where $\mathbf{y_i}$ is set to 1 iff $i \in C$. We call this new integer program a **Fixed Coalition Integer Program** (FCIP). It is clear that a coalition is successful if and only if FCIP is satisfiable.

Theorem 10 *When parameterized by $|Ag| + |R|$, the NR, SNR, CGRO, SCRB, and RPEGS problems are FPT.*

Proof. Due to the lack of space we omit the details and we only mention the required changes for each problem.

- ***NR:*** For all goals $g \in G$ where $\mathbf{req}(g, r) > 0$ we change the variable x_g to zero.
- ***SNR:*** First we should check if the coalition is successful. If FCIP was satisfiable, for all goals $g \in G$ where $\mathbf{req}(g, r) > 0$ we have to set the variable x_g to zero.
- ***CGRO:*** We add the constraint $\sum_{g \in G} \mathbf{x_g} \times req(g, r) < \beta$ where $\beta = \mathbf{req}(G_0, r)$.
- ***SCRB:*** For every resource $r \in R$ we bound its usage by adding the constraint $\sum_{g \in G} \mathbf{x_g} \times \mathbf{req}(g, r) \leq \mathbf{b}(r)$.
- ***RPEGS:*** Since G_0 is given, $\mathbf{req}(G_0, r)$ is known to us. We write $|R|$ instances of FCIP such that in the FCIP for resource r, we have the constraint $\mathbf{req}(G', r) < \mathbf{req}(G_0, r)$, and $|R| - 1$ constraints $\mathbf{req}(G', r') \leq \mathbf{req}(G_0, r')$, one for each resource $r' \neq r$.

□

IX - Revisiting ESCK Parameterized by $|G|$

Shrot et al. (Shrot, Aumann, and Kraus 2009) show in Theorem 3.2 of their paper that ESCK parameterized by $|G|$ is FPT. We first show their proposed FPT algorithm is wrong by giving an instance when their algorithm gives incorrect answer. Then we show that in fact the problem is PNP-hard via a reduction from the independent set problem.

Counterexample to the Algorithm Given in Theorem 3.2 of Shrot et al. 2009

The algorithm is as follows:

1. For each $G' \subseteq G$
   - Let C' be set of all agents satisfied by G'
   - If $|C'| \neq k$, go to 1.
   - If G' is feasible for C', return TRUE
2. return FALSE

We give an instance Γ where the above algorithm gives an incorrect answer. Suppose $|Ag| > k$, each agent has 1 unit of endowment of each resource, each goal requires 0 of each resource, and $G_i = G$ for all agents $i \in Ag$. Thus each coalition is successful and $\forall\ G' \subseteq G$ we have $C' = Ag$ which means that $|C'| = |Ag| > k$ and so the algorithm answers NO while the correct answer is YES. Indeed by reducing Independent Set to a CRG instance with $|G| = 1$, we prove the following theorem.

Theorem 11 *ESCK parameterized by $|G|$ is PNP-hard.*

X - Directions for Future Work

The study of problems arising in coalitions of agents in multi-agents systems using the parameterized complexity paradigm was initiated by Shrot et al. In this paper we have tried to take a further step in this direction which we believe is still unexplored. There are various (classically) computationally hard problems which need to be better analyzed through the rich theory of parameterized complexity.
Both in Shrot et al. and this paper only the CRG model has been considered. In CRG model the status of CC parameterized by $|Ag| + |R|$ is left open. Alternatively one might consider other natural parameters like $|Ag|$ or try to examine other models like the QCG model (Wooldridge et al. 2004) through parameterized complexity analysis.

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