Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence (AAAI-10)

Envy Quotes and the Iterated Core-Selecting Combinatorial Auction*

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Abstract

Using a model of agent behavior based around envy-reducing strategies, we describe an iterated combinatorial auction in which the allocation and prices converge to a solution in the core of the agents' true valuations. In each round of the iterative auction mechanism, agents act on *envy quotes* produced by the mechanism: hints that suggest the prices of the bundles they are interested in. We describe optimal methods of generating envy quotes for two different core-selecting mechanisms. Prior work on core-selecting combinatorial auctions has required agents to have perfect information about every agent's valuations to achieve a solution in the core. In contrast, here a core solution is reached even in the private information setting.

Introduction

The Vickrey-Clarke-Groves mechanism (VCG) is ubiquitous in theoretical mechanism design. In the standard setting, it is the revenue-maximizing mechanism among all incentive compatible efficient mechanisms (Krishna and Perry 1997). Unfortunately, in addition to being unwieldy to implement in practice, VCG suffers from a number of pathologies (Rothkopf, Teisberg, and Kahn 1990; Sandholm 2000; Ausubel and Milgrom 2006; Rothkopf 2007). These include revenue non-monotonicity, in which adding another bidder can lower revenues, and receiving an arbitrarily small fraction of the revenue achievable by posting prices.

To obtain higher revenues than VCG, one can explore inefficient mechanisms, which leads to combinatorial generalizations of the revenue-maximizing single-item auction (Myerson 1981). Revenue-maximizing mechanisms are unknown even for the (unrestricted) two-item setting, and in general a concise description of the revenue-maximizing combinatorial auction cannot exist (unless P=NP) because that design problem is NP-complete (Conitzer and Sandholm 2004). Some work has been done on automated mechanism design for finding high-revenue combinatorial auctions, but those approaches have not been used for large numbers of items (Likhodedov and Sandholm 2004; 2005). Even simple revenue-enhancement approaches like setting reserve prices require good knowledge of a prior distribution over agent valuations, which may or may not be available depending on the setting.

A different approach is to relax incentive compatibility. One recent stream of research in non-truthful mechanisms has involved *core-selecting combinatorial auctions*. Mechanisms of this class were used in the recent successful spectrum license auction in the United Kingdom (Cramton 2008a; 2008b; Day and Cramton 2008). These mechanisms mitigate the terrible worst-case revenue properties of the VCG mechanism without necessarily subscribing to a first-price mechanism (which motivates significant underbidding). Selecting an outcome in the core yields a host of desirable properties that VCG lacks (Parkes 2002), like revenue monotonicity and resistance to shill bidding (sybil attacks) (Yokoo 2006).

The word *iterative* has taken on a confusing double meaning in core-selecting combinatorial auction research. On the one hand, the auction process itself can be iterative, in which bids are solicited in a series of rounds until a termination condition is reached (for instance, no agent submits a new bid). This is the concept we explore in this paper. On the other hand, given a set of bids, a solution may be produced iteratively, e.g., by raising the price of bundles in ascending rounds in a specific way until reaching a point in the core. Examples of this latter process include (Parkes 1999), (Ausubel and Milgrom 2002), (Wurman, Zhong, and Cai 2004), and (Hoffman et al. 2006). Unfortunately, these techniques are too slow to be used in an explicitly multi-round auction, and so these mechanisms work by inputting valuations into proxy agents that bid on the behalf of auction participants. Thus, price increases are a function of iterating on the bids of these proxy agents rather than multiple iterative rounds of buyers changing their valuations. As a consequence these auctions are essentially one-shot.

One attempt to get around the computational constraints of a multi-round core-selecting auction is the clock-proxy auction (Ausubel, Cramton, and Milgrom 2006). This mechanism maintains (easy-to-compute) linear prices over items through a number of explicit iterative rounds before solving a final core-selecting round.

It would be ideal, however, to conduct a series of fully core-selecting rounds. Recent computational work by (Day and Raghavan 2007) has shown that *constraint generation*

*This work was supported by NSF grants IIS-0427858 and IIS-0905390.

can be used to clear large-scale core-selecting combinatorial auctions orders of magnitude faster than previously possible, making fully iterative approaches tractable in practice. In this paper we describe the implementation of a core-selecting mechanism over a number of explicitly iterative rounds.

Prior work by (Milgrom 2006) and (Day and Milgrom 2008) has studied some of the revenue and strategic properties of core-selecting combinatorial auctions. That literature has examined the case where bidders have perfect information about their own valuations and the valuations of others. They analyze a mechanism under which the bidder-optimal (revenue-pessimal) point in the core is selected. They emphasize that, with perfect information, this will be the outcome if side-payments between players are allowed in any core-selecting mechanism. They show that agents will adopt a set of strategies that involve *truncation*, that is, each agent shaves all her bids by a certain individual value.

Unfortunately, the perfect information assumption is seldom realistic in practice. One of the primary reasons for running an auction in the first place—as opposed to, for example, simply posting prices—is that there may be considerable uncertainty about agents' valuations. It is a poor modeling choice to assume the auctioneer has no prior information about the agents' valuations, while at the same time assuming that every agent exactly knows the private valuation of every other agent. The contribution of this paper is to provide a mechanism with all the benefits of a core-selecting combinatorial auction in a more plausible private-information setting.

We begin by formally establishing the properties of envy-freeness and envy-reduction, and show that a straightforward approach to applying these techniques to an iterated core-selecting combinatorial auction fails to produce an outcome in the core. We then introduce the driving concept behind this paper, *envy quotes*, which serve as estimates of the current prices of bundles the agent is losing. Implementing these envy quotes in an iterated setting, we prove that when agents act to reduce their envy of their envy quotes, the solution converges to the core of the agents' true valuations. Then, we discuss what happens when agents do not behave as we expect them to in our mechanism, showing that our mechanism has desirable safeguards to prevent returning a low revenue result, even when agents do not exactly follow the behavioral model we prescribe. Finally, we discuss how to generate these envy quotes in practice, showing a general method that works with any core-selecting combinatorial auction, an (inefficient) optimal technique for any core-selecting combinatorial auction, an optimal technique for the bidder-optimal core-selecting combinatorial auction, and an optimal technique for the bidder-pessimal core-selecting combinatorial auction, which is equivalent to a first-price mechanism.

The theory of envy-freeness

There exists a set of n agents and k items, and therefore $2^k - 1$ bundles. As is normal, we assume that agents have quasilinear utility, so that an agent's utility for receiving allocation a and paying π is $u(a) - \pi$.

Definition 1. *A (feasible)* outcome *is a set of allocations $a_1, \ldots, a_n$ and payments $\pi_1, \ldots, \pi_n$, where $a_i \cap a_j = \emptyset$ for $i \neq j$.*

Definition 2. *A* blocking coalition *is a group of agents G who can propose an alternate outcome (with allocations $a'_1, \ldots, a'_n$ and payments $\pi'_1, \ldots, \pi'_n$) in which only members of G win a bundle, such that for all $i \in G$, $u_i(a'_i) - \pi'_i \geq u_i(a_i) - \pi_i$ where the inequality is strict for at least one i, and $\sum_i \pi'_i > \sum_i \pi_i$.*

Definition 3. *An outcome is in* the core *if it induces no blocking coalitions.*

Definition 4. *An outcome is* efficient *if no other feasible outcome has a higher social welfare (sum of utilities) for the participants.*

It follows that every outcome in the core is efficient, because any inefficiencies would yield a blocking coalition (c.f. (Shapley and Shubik 1971; Day and Raghavan 2007)).

Definition 5. *Agent i (with allocation a_i and payment π_i)* envies *agent j (with allocation a_j and payment π_j) if $u_i(a_i) - \pi_i < u_i(a_j) - \pi_j$, where $(u_i(a_j) - \pi_j) - (u_i(a_i) - \pi_i)$ is the* amount of envy *(or just* envy*).*

Definition 6. *An agent plays a (myopic)* envy-reducing strategy *if, given a set of reports of the other agents, she modifies her type report to reduce her envy of some agent without lowering her utility. A group of agents play a (myopic)* group envy-reducing strategy *when no agent in the group lowers their utility and at least one agent changes her bid to reduce her envy of another agent.*

Definition 7. *An outcome is* envy-free *if no subset of agents prefers the allocation-payment pair of any other subset of agents to its own allocation-payment pair. If both subsets are restricted to consist of only individual agents, then we call that set* individually envy-free*.*

The set of individually envy-free points is at least as large as the set of envy-free points because the concept is less restrictive.

Definition 8. *An* envy-free fixed point *is a fixed point of a system where groups of agents follow envy-reducing strategies.*

Corollary 1. *Every envy-free fixed point is in the core (with respect to* true *valuations).*

Corollary 2. *Every envy-free fixed point is efficient (with respect to* true *valuations).*

Lemma 1 ((Leonard 1983)). *The revenue from VCG does not exceed that of the bidder-optimal (i.e., revenue-pessimal) outcome in the core.*

Corollary 3. *Every envy-free fixed point delivers at least as much revenue as the VCG mechanism.*

The path forward seems straightforward: Simply have agents iterate in an envy-reducing manner towards an envy-free solution in the core. This would yield desirable revenue properties while only requiring agents to have private information. However, as we show in the next section, the combinatorial nature of the problem complicates a simple model of individual envy-reduction.

Individual envy-reduction is insufficient

The following example shows how individual envy reduction can be insufficient for reaching a fixed point in the core.

Example 1. *Consider a two-item three-bidder problem, where bids (b_i) and true valuations (v_i) are given by the following table:*

Bundle	*Bid*	*Valuation*
A	$b_1 = 5, b_2 = 0, b_3 = 0$	$v_1 = 9, v_2 = 0, v_3 = 0$
B	$b_1 = 0, b_2 = 5, b_3 = 0$	$v_1 = 0, v_2 = 9, v_3 = 0$
AB	$b_1 = 5, b_2 = 5, b_3 = 15$	$v_1 = 9, v_2 = 9, v_3 = 15$

Every core solution awards item A to bidder one and item B to bidder two at a total price of between 15 and 18. But with these valuations and bids, every core-selecting combinatorial auction awards both items to the third agent at a price in [10,15]—a solution that is not in the true core. However, no agent individually *envies the allocation of any other: neither losing bidder would prefer to pay 10 for AB.*

Iterative combinatorial auctions with envy quotes

The problems illustrated by the above example arise because we are not properly expressing to an agent how her bid impacts the combinatorial nature of the allocation. As we have discussed, envy-free dynamics in the combinatorial setting only work when groups of agents work together. But this is undesirable, because it encourages collusion among bidders (which could lead to bad outcomes) or could be illegal, which is the case in many public goods auctions (Day and Raghavan 2007). In this section, we explore how to present *envy quotes* to agents regarding the clearing prices of bids they have lost, such that the envy quotes meaningfully reflect what actual prices are. In effect, we are changing the target of agents' envy from distinct bidders to the prices received by the winners of bundles in the agent's interest. Furthermore, we show that the fixed point of individuals reducing their envy on these quotes is in the core with respect to true valuations.

Our iterative scheme

We propose the following process:

1. Solicit bids from agents.
2. Compute current winners and payments according to some core-selecting combinatorial auction.
3. For each of her losing bids, an agent receives an envy quote, $p(S)$, in the form of "The bundle S is currently going for price $p(S)$".
4. Repeat steps 1 through 3 until no new bids are received.

When we provide an envy quote to an agent on the price of a bundle, we have competing objectives. On the one hand, the envy quote has to be low enough so that it does not cut into the core (which could lead to agents not envying outcomes they should legitimately envy). On the other hand, an envy quote should not be lower than the agent's bid on a bundle, in order to reflect the core-selecting nature of the mechanism. This leads us to the following definitions:

Definition 9. *An agent's* core support $c(S)$ *for a bundle S they are losing is the largest possible bid they could make and not change the current allocation or prices.*

It follows that the core support $c(S)$ is always less than the agent's quote on S, or the amount they would need to win the bundle.

Definition 10. *Let an agent bid $b(S)$ for bundle S and not win that bundle. The* envy quote *$p(S)$ satisfies $b(S) \leq p(S) \leq c(S)$.*

Now we can define what envy means in the envy quote context.

Definition 11. *Let an agent currently be winning bundle w at price π_w. She* envies *the envy quote $p(S)$ on a bundle she is losing, S, if $u(S) - p(S) > u(w) - \pi_w$.*

Proposition 1. *If the current solution in the iterated core-selecting combinatorial auction is not in the core (with respect to true valuations), then some agent has a bid that reduces her envy of the envy quote she receives on at least one bundle.*

Proof. Assume we are in a non-core state such that allocations are given by $a_1, \ldots, a_n$ and prices by $\pi_1, \ldots, \pi_n$. We will show that some agent has envy in this state, and that she has a bid to reduce that envy.

Since the solution is not in the core, there exists some blocking coalition in which at least one member of the coalition is strictly better off. Call that new solution $a'_1, \ldots, a'_n$, with prices $\pi'_1, \ldots, \pi'_n$. Without loss of generality, let agent 1 be strictly better off. We have $u_1(a_1) - \pi_1 < u_1(a'_1) - \pi'_1$, where the left hand side is non-negative, and $a'_1 \neq \emptyset$ by the restriction that no agent bids above her valuation. We will show that in the initial state, agent 1 is presented with an envy quote that induces envy. Let the envy quote of a'_1 in the original state be p. To show agent 1 has envy for a'_1, we must have $u_1(a_1) - \pi_1 < u_1(a'_1) - p$, for which it is sufficient to show that $p \leq \pi'_1$, which holds because envy quotes are always less than quotes. As an example of an envy-reducing strategy, the agent can increase her bid on a'_1 to $p + \epsilon$, where $0 < \epsilon < (u_1(a'_1) - p) - (u_1(a_1) - \pi_1)$, because either the agent's next envy quote on a'_1 must be higher so her envy of it is reduced, or she wins the item at a price of at most $p + \epsilon$, which gives her more utility. ■

Envy-reducing dynamics converge

In this section, we show that if agents respond to their envy quotes on items they are not winning, then prices to converge to a fixed point. Furthermore, it suffices that agents select such envy-reducing actions with positive probability. This kind of convergence result is standard in the matching market literature (c.f. (Roth and Vande Vate 1990)), and holds regardless of the path taken to the current set of prices.

Proposition 2. *Assume that bids must be from a finite set of discrete levels, where the difference between consecutive levels is at most ϵ (e.g., bids are in integer dollars). If at every state of the auction at least one agent has positive probability of selecting an action among those that reduce her envy of an envy quote on a bundle she is not winning, then* *any iterated core-selecting combinatorial auction converges to a fixed point with probability 1. In this fixed point, each agent's envy of the envy quote on any bundle is at most ϵ, each coalition's payoff is no less than its pessimal core payoff minus $|coalition| \cdot \epsilon$, and revenue is no less than that of the revenue-pessimal point of the core minus $n \cdot \epsilon$.*

Proof. The proof of Proposition 1 can be extended trivially to show that in the fixed point, no agent can have more than ϵ envy on any bundle, and that an ascending bid exists otherwise. Furthermore, the largest envy quotes on bundles of some core outcome are no smaller than the agents' bids in that core outcome minus ϵ. If this were not the case, some agent could reduce her envy by bidding ϵ higher for some bundle. Since some agent has an ascending bid at every non-fixed point, by assumption the agent will select such an action with positive probability. Since there are only a finite number of (agent, bundle, bid level)-triples, we can construct a finite number of steps to reach a state in which no agent has envy greater than ϵ of her envy quote on any bundle. Thus, these envy-reducing dynamics converge to such a fixed point with probability 1.

Suppose some coalition's payoff is less than its lowest core payoff minus $|coalition| \cdot \epsilon$. Then some agent in the coalition has at least ϵ envy of the core state. Because envy quotes are smaller than quotes, it follows that the agent has at least ϵ envy of the envy quote of the bundle she would receive in that state. Thus we are not in a fixed point, which contradicts our premise. Therefore, each coalition's payoff is at least its lowest core payoff minus $|coalition| \cdot \epsilon$.

Let r denote the revenue in the fixed point we reach. Let r_c denote the revenue from the core solution that we are near (not necessarily the revenue-pessimal core solution), and let $\overline{p_i}$ represent the largest envy quote received by any agent for agent i's bundle in the core solution we are near. Because the mechanism is core-selecting with respect to reported bids we have $r_c \geq \sum \overline{p_i}$, Since we are in a fixed point, we must have $r \leq \sum(\overline{p_i} + \epsilon)$, because were this not the case, an agent would have an envy-reducing play by bidding ϵ higher and thereby forcing the core solution. Letting $r_{pessimal} \leq r_c$ represent the revenue-pessimal core solution, it follows that $r_{pessimal} - n \cdot \epsilon \leq \sum \overline{p_i} \leq r$. Therefore, our revenue is no smaller than that of the revenue-pessimal core outcome minus $n \cdot \epsilon$. ∎

Robustness of the approach

Core-selecting combinatorial auctions are not incentive compatible: only when the VCG solution lies (at the bidder-optimal extrema) in the core will core-selecting mechanisms be incentive-compatible (Goeree and Lien 2009). Abandoning incentive compatibility comes a host of strategic concerns. How can we say how agents will play if they do not play truthfully? In this section, we explore the robustness of our mechanisms: what happens when agents either bid too little or too much, or attempt to otherwise manipulate the mechanism in ways that could be to their benefit. That is, what happens when agents fail—either due to willful manipulation or incompetence—to decrease their envy of envy quotes?

The optimal manipulation, if agents had perfect information and side payments were allowed, would be for each to shave her bid in a specific manner in order to achieve the core solution that minimizes the sum of the payments by the agents. This solution will coincide with the VCG solution if the VCG solution is in the core; this is the solution concept featured by (Ausubel and Milgrom 2002) and (Day and Raghavan 2007). But in our setting, agents do not have perfect information, and the setting is iterated.

An agent can, of course, shave her bid too much. This is the great fear of running a non-truthful mechanism—that agents, recognizing that they should shave their bids, will bid very little and the end result will be low revenue. As we show, however, we do not necessarily need to rely on agents being motivated only by envy to achieve solutions in the core.

Proposition 3. *If total revenues are less than the revenue-minimal point in the (true) core, then it is a weakly dominant strategy for some agent to increase her bid on a bundle she is losing.*

Proof. If total revenues are less than the revenue-minimal point in the core, then some agent has an envy-reducing bid on an envy quote they are receiving on a losing bundle. Moreover, since envy quotes may well coincide with a quote of the actual value she would need to pay to win the bundle, bidding in response to such a report is a weakly dominant strategy from a utility sense—if she captures the bundle at the higher price she will be better off utility-wise, and if she fails to capture the bundle she will be no worse off. ∎

Agent rationality provides a strong argument for our mechanism not returning a low revenue solution. On the other hand, if an agent shaves her bid too little, there might not be any straightforward way to achieve a better outcome for her. There may be a multitude of core outcomes with higher revenue, in which agents bid more than in the bidder-optimal solution. If an agent insufficiently shaves her bid, the mechanism might arrive at such a state. Since that outcome is in the core, only a global effort by a grand coalition of agents can force an outcome where the agents pay less.

In summary, our mechanism handles agents' mistakes in a revenue-optimizing way. If agents bid too low, self-interest will compel them to correct their bids. If agents bid too high, the structure of the core can lock agents into a high-revenue core solution that no agent can escape.

Another concern if agents do not play optimally is that, in an iterated setting, agents will move around in the state space of possible allocations, attempting to find advantageous outcomes in which other agents make errors that are beneficial to the agent that is causing the moving. One way of dealing with this possibility is to ignore it; as we have shown, only efficient outcomes can emerge as the fixed points of our iterated mechanisms. Therefore, the only way an agent will be able to take advantage of an inefficient outcome that yields low revenue for the auctioneer is for some other agent to not make an envy-reducing (and utility non-reducing) move that is made apparent to her by her envy quotes. Another possibility is to add an ascending clock to the auction, such that bids can only increase. This does not impact our convergence result, which relies on agents increasing their bids on bundles they are not winning.

Comparison with the ascending proxy auction

In this section, we discuss the differences between our approach and the iterative core-selecting combinatorial auction of (Ausubel and Milgrom 2002), the *ascending proxy auction (APA)*.

The most important difference involves bidder strategies. The APA mandates that bidders behave in a specific way to achieve a solution in the core, namely, that agents raise their losing bids by small ϵ in each iterative round. Since that elicitation process is slow and there is no guarantee agents will behave this way in practice, (Ausubel and Milgrom 2002) suggest that agents surrender their valuations to a proxy agent that bids in this manner on their behalf. Such a scheme loses out on perhaps the most important part of having an ascending auction in the first place; iterative auctions can produce higher revenues by making bidders feel more secure in their valuations (c.f. (Cramton 1998)). Furthermore, even with proxy agents, the APA is a very slow way to calculate a core solution from a set of bids (Day and Raghavan 2007).

Essentially, what envy quotes provide is a more efficient way of conveying price information to losing agents. As we discuss in the next section, we can formalize this argument. If an *optimal* scheme for generating envy quotes with a particular core-selecting mechanism is used, price information is conveyed to losing bidders as efficiently as possible. In contrast, the APA uses the *least* efficient scheme for conveying prices to losing bidders.

Additionally, our approach works with any core-selecting combinatorial auction, while the APA was designed only for the bidder-optimal core-selecting combinatorial auction, and our approach does not require an ascending clock.

Computing envy quotes

In this section we introduce the concept of optimal envy quotes and show how to compute optimal envy quotes for two different core-selecting mechanisms.

Trivial envy quotes for any core-selecting combinatorial auction

One simple but valid scheme for producing envy quotes is to give an agent losing bundle S with a bid of b, the envy quote of b. This trivially satisfies the envy quote definition. It is also fundamentally equivalent to the APA, because using this scheme the only feedback a bidder receives on a bundle they are losing is that they are losing the bundle at their current bid, and that a bid of ϵ higher may (or may not) win the bundle.

Recalling that envy quotes are bounded from above by the core support $c(S)$ (Definition 9), we have the following definition:

Definition 12. *A method for generating envy quotes is* optimal *for a core-selecting combinatorial auction if it always generates the largest possible envy quotes.*

It is never possible for an agent following their envy quotes to make a bid that does not change the final outcome; optimal envy quote methods are as efficient as possible and, for a given set of bids, will take the fewest interaction rounds before termination. Making envy quotes as large as possible ensures that a bidder will never place an uncompetitive bid. Conveying the best possible information about current prices to bidders allows them to make the best decisions about what bundles to bid on and how much to bid on them.

Optimal envy quotes for any core-selecting combinatorial auction

Intrinsically, the use of a core-selecting auction implies that, for each losing bid, there exists some threshold value (the optimal envy quote) above which the allocation and/or payments change and below which they do not. Because of this property, we can solve for the threshold value by treating the core-selecting process as a black box and using binary search. Letting v^* represent the sum total of the accepted bids of the current solution, we begin on the search interval $[0, v^* + \epsilon]$, and query as to whether the midpoint of the interval changes the current solution. If so, the midpoint becomes the new upper bound, and if not, the midpoint becomes the new lower bound. Each time this process is run, we produce an additional bit of accuracy with respect to finding the optimal envy quote.

Because it treats the core-selecting mechanism as a black box, this process can work with any core-selecting combinatorial auction, but is likely to be very slow, because it must be run multiple times for each losing bid. In the following sections, we develop techniques to solve for optimal envy quotes in a single step for two different core-selecting combinatorial auctions.

Optimal envy quotes for bidder-optimal core-selecting combinatorial auctions

We now describe a method for generating optimal envy quotes in the context of the recent core-selecting combinatorial auction of (Day and Raghavan 2007). The concept of selecting a bidder-optimal point in the core (a core solution that minimizes total payments) has also been featured in more recent papers (Day and Cramton 2008; Erdil and Klemperer 2009), which discuss different ways of selecting *which* bidder-optimal point in the core to return. To construct an optimal envy quote, we use a MIP to find the smallest price for that bundle that could be used to construct a blocking coalition. To compute an envy quote for agent i for bundle S, we do the following.

1. *Add each winners' accepted bid constrained by the price paid by the agent into the MIP*. For example, if an agent wins the bundle ABC at a price of 20, we add the constraint $\pi_A + \pi_B + \pi_C = 20$.
2. *For each bidder $j \neq i$, calculate j's surplus, subtract it from j's losing bids, and add those revised losing bids as constraints into the MIP using the sum of item prices*. To illustrate this, imagine a winning agent with surplus 3 that has a losing bid for the bundle AB for a price of 10. This would be added as $\pi_A + \pi_B \geq 10 - 3$.

3. *Add all of i's losing bids as constraints, without subtracting i's surplus*. For example, if i has a losing bid on AB for a price of 5, then we add the constraint $\pi_A + \pi_B \geq 5$.
4. *For each agent, link binary variables corresponding to each of their losing bids with an SOS1 constraint, so that any blocking coalition of bids involves at most one bid from each agent*. This ensures that none of the constraints (from Step 2 or Step 3) cut into the core.
5. *The objective of the MIP is to calculate the lowest bundle price (as sum of item prices) for S*. For instance, consider bundle AB. Our objective function is $\min \pi_A + \pi_B$. This is equivalent to designating the envy quote for a bundle along a hyperplane normal to the items in the bundle, and is therefore more flexible than imposing only item prices.

The solution to the objective, the minimum bundle price, is given to the agent as an envy quote for their losing bid.

Proposition 4. *The method described by this section is optimal for the bidder-optimal core-selecting combinatorial auction.*

Proof. Suppose our method produces an envy quote for a bundle S of p. We will show that a bid of $p + \epsilon$ will change the mechanism's allocation-payment pair.

The set of constraints in the MIP that we use to generate the envy quote define the set of all possible blocking coalitions involving an agent's bids exactly. Since the combinatorial auction selects the bidder-optimal point in the core, this implies the addition of a hyperplane corresponding to the items of S, $\sum_{x_i \in S} \pi_i = p + \epsilon$ cuts in to the core, which implies the existence of a blocking coalition at the current set of prices based on that bid. Because the mechanism selects the bidder-optimal point in the core, either the bid of $p + \epsilon$ would be part of a new winning coalition or it would change the prices charged by the mechanism. ■

One can make this MIP into an anytime algorithm by first adding the agent's losing bids as constraints, and then adding the constraints from the other agents into the MIP incrementally. After the optimization of every such addition, we have a valid envy quote, and those quotes increase as additional constraints are added. However, terminating the optimization prematurely will obviously not yield optimal envy quotes in every case.

Figure 1 provides graphical intuition for our method for generating envy quotes, and why it is superior to the simple universal method. In the figure, the bidder is receiving an envy quote on the bundle $\{x, y\}$, for which he is bidding 3. The actual core, defined by the losing bids of two other bidders, is supported by $x = 2, y = 3$. The trivial envy quoting mechanism would return the bidder's losing bid of 3, whereas the optimal quoting mechanism described in this section returns 5, the smallest value at which the hyperplane normal to $(1, 1)$ intersects the darker polytope.

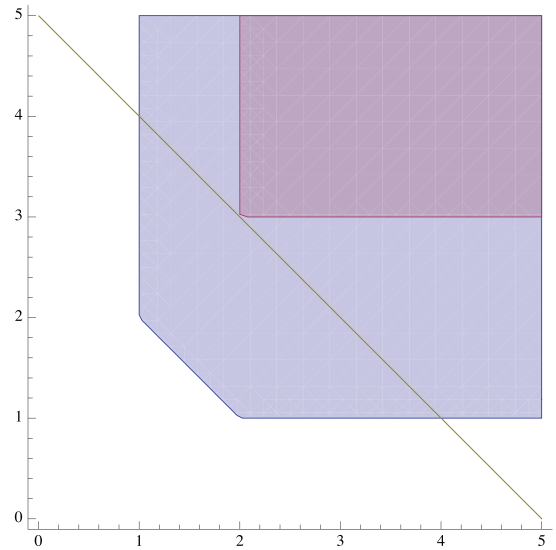

Figure 1: A simple graphical example of how smarter envy quotes can be more informative to bidders. Here, a bidder has losing bids of $\{x = 1, y = 1, x + y = 3\}$, while the actual core is supported by bids of $\{x = 2, y = 3\}$. The naïve envy quote formulation of returning only an agent's failed bids would return 3 for the bundle x, y, while our more sophisticated core generating MIP would return 5, as indicated by the line $x + y = 5$ intersecting the darker polytope.

Optimal envy quotes for the first-price combinatorial auction

We can also develop optimal envy-quoting methods for other core-selecting mechanisms. As an extreme (bidder-pessimal) example, consider a first-price mechanism, which charges winning bidders their bids, and selects the allocation that will maximize revenue. It is trivially core-selecting as long as no bidder bids above their true valuation for an item.

For this setting, the optimal envy quote scheme is to produce the quote for each losing bid, that is, the threshold value at which the agent would go from losing to winning the bundle in question. Any lower envy quote is not optimal, because an agent could bid higher without altering the mechanism's allocation-payment pair, and no envy quote can ever be higher than a quote. Solving for these quotes is NP-hard (Sandholm 2002).

Conclusion

The recently developed manipulable core-selecting combinatorial auction seems to be a promising future path for research from both a theoretical and practical standpoint. Theoretically, because it enforces a core outcome, it avoids the many pathologies of VCG, including terrible worst-case revenue properties and shill-bidding manipulations. Practically, core-selecting auctions are manipulable, closely matching the mechanisms seen in practice, as well as being efficient, which squares well with our intuitive sense of what an auction's outcome should be. However, previous models of the core-selecting combinatorial auction have either been too slow to clear practical-sized problems, or have made unrealistic assumptions about the amount of information available to bidders as a prerequisite to achieving these desirable results.

In this work, we expanded upon previous advances by designing an iterated version of the core-selecting combinatorial auction. We began by showing that when agents as a group follow envy-reducing strategies, the resulting fixed point is in the core of true values. However, we demonstrated that agents acting individually to reduce their envy would be insufficient to arrive at a fixed point in the core, because of the combinatorial nature of the auction.

To remedy this problem, we developed a system of *envy quotes*, where agents are given estimates on their losing bids of the prices at which those bundles are being won. We proved a convergence result, showing that when agents act to reduce their envy of their envy quotes, we achieve an outcome in the core of true values. Moreover, we discussed some of the safeguards the mechanism has for when agents do not play optimally within the mechanism. Our format makes it possible for agents to pay too much for their allocations, but that the mechanism cannot reach an allocation-payment pair where agents pay too little without some agent having a clear potentially utility-increasing (and not merely envy-reducing) bid.

Finally, we developed three different techniques for generating envy quotes. The trivial method of returning a losing agent's bid is valid, but not optimal, for *any* core-selecting combinatorial auction. We discussed the links between this inefficient method and the ascending proxy auction of (Ausubel and Milgrom 2002). We generated an optimal method for the bidder-optimal core-selecting combinatorial auction, and another optimal method for the bidder-pessimal core-selecting (first-price) combinatorial auction. Equipped with an optimal method for generating envy quotes, our iterative core-selecting combinatorial auction ensures, under very loose behavioral restrictions, a core solution in the fewest possible iterative rounds.

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