Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence (AAAI-10)

Nonmanipulable Randomized Tournament Selections

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Abstract

Tournament solution concepts, selecting winners based on a pairwise dominance relation are an important structure often used in sports, as well as elections, and argumentation theory. Manipulation of such choice rules by coalitions of agents are a significant problem in most common rules. We deal with the problem of the manipulation of randomized choice rules by coalitions varying from a single agent, to two or more agents. We define two notions of coalitional manipulations of such choice rules based on whether or not utility is transferable. We show useful choice rules satisfying both notions of non-manipulability, and for the transferable utility case provide bounds on the level of Condorcet consistency.

Introduction

Participants in sporting tournaments often act selfishly in order to increase their potential gains. Situations such as a team deliberately throwing a match, or several teams colluding to fix the outcome of a match are common and undesirable events in such tournaments. This introduces the need for tournament designs that are resistant to such manipulations.

A *tournament* is simply a complete pairwise dominance relation over a set of agents A. The concept of a rule selecting candidates from tournaments has been introduced by Moulin (1986). In that paper, and in many works since, a choice rule was defined as selecting a nonempty subset of the alternatives, members of which are tied for winning the tournament. Such set selection rules, such as the Top Cycle and Tournament Equilibrium Set (Schwartz 1990) make no attempt to break such ties. Our notion of tournament choice rules explicitly assigns winning probabilities to each alternative in A. These can be interpreted as an actually lottery to determine the winner, or a prescribed division of a monetary prize (Felsenthal and Machover 2007).

Choice rules from tournaments are directly tied to voting, where a tournament is used to represent the pairwise majority relation in an election. That is, an alternative x wins a match against alternative y if a majority of the voters prefer x to y. As observed by the Marquis de Condorcet (1785), such majority relationships frequently admit cycles. Any tournament may arise from an election with at least three voters (McGarvey 1953), therefore tournament choice rules are equivalent to election methods that only consider pairwise majority.

The topic of incentives in the context of selecting a winner from a tournament has been recently studied by Altman, Procaccia, and Tennenholtz (2009). That paper introduced choice rules that select an alternative given a tournament, that is, functions from the set of all tournaments over A to A. The players in this setting are the alternatives. In their model, as in this paper, an alternative cannot announce that it dominates an alternative that it loses to. However, an alternative can lose on purpose to another alternative. In other words, if $x \in A$ dominates $y \in A$, x can cheat by reversing the outcome of the match between itself and y, but y cannot reverse the outcome of this match unless x agrees.

Altman, Procaccia, and Tennenholtz have studied manipulations of such deterministic choice rules by a single agent throwing a match, or by a pair of agents fixing a match. They have shown a strong negative result for the existence of deterministic Condorcet-consistent non-manipulable choice rules.

In this paper we extend this framework to consider randomized choice rules which allow lotteries among the alternatives. Randomized choice rules can be used as a fair way of breaking ties in a tournament, and can also be viewed as a means of dividing a title or prize amongst the co-winners of a tournament.

We consider two distinct ways of extending the notion of non-manipulability to the randomized setting: In the weaker version, Pareto non-manipulability (PNM), we consider only manipulations where all manipulators stand to gain, or at least not lose from the manipulation. In contrast, Strong non-manipulability (SNM) requires immunity to any manipulation where the sum of the manipulators' utilities increases.

We show that PNM is easily satisfiable by choice rules that apply proper randomization. In fact, we show that a stronger variant of the Condorcet condition is also satisfiable. With SNM, however, we show that Condorcet consistency is impossible, and proceed to bound the probability a Condercet winner can be guaranteed to win.

Related Work. Much work has been done on axiomatic characterizations of tournament choice rules that select single winners, sets of winners, and probability distributions over winners (as we do in this paper). Our work can be seen as contributing to this body of work by considering incentive-based axioms, i.e. nonmanipulability. For a detailed review of this research see (Laslier 1997).

In the context of elections, nonmanipulability of randomized voting rules has been shown to practically impossible (Gibbard 1977). That is, if there are at least three alternatives, any nonmanipulable voting rule must be a mixture of ones that are unilateral, i.e. dictatorships, and those that are duple, i.e. restrictive to a pair of outcomes. Our work on manipulability is inherently different, as in our case the manipulators are the alternatives who can collude to increase the chances of any or all of them winning. Moreover, Gibbard (1977) considers general voting rules from preference profiles to distributions on candidates, while our tournament solutions are limited to working on the pairwise dominance relation.

An example of a paper that directly studies manipulation of tournament choice rules is the work of Dutta, Jackson, and Le Breton (2002). They examined a specific class of choice rules, in the context of voting. Specifically, Dutta, Jackson, and Le Breton investigated a setting where alternatives can decide whether or not to enter the election; they characterize the set of alternatives that can be outcomes of the election in equilibrium.

Structure of the paper. We begin by formally defining the notions of tournaments, choice rules, and manipulations. In the following sections we present our results of Pareto and strong manipulations. Finally, we conclude and provide directions for future research.

Tournaments and Manipulations

Let A be a set of alternatives. A *tournament* T over A is a complete asymmetric binary relation over A. In other words, for every two distinct alternatives $x, y \in A$, exactly one of the following holds: xTy (read: x *dominates* y), or yTx. We denote the set of tournaments over A by $\mathcal{T}$.

A common visual way to represent tournaments is via graphs. A tournament $T \in \mathcal{T}$ corresponds to a directed graph $G = (V, E)$, where $V = A$, and the directed edge from x to y is in E if and only if xTy. In other words, G is an orientation of the complete graph on A. In the sequel we will use the terms alternatives, participants, nodes, and agents interchangeably.

We can now define our main focus of this paper: tournament choice rules.

Definition 1. A *choice rule* is a mapping $r : \mathcal{T} \mapsto \Delta(A \cup \{\varepsilon\})$ that maps each tournament to a probability distribution over the alternatives, with a possibility of selecting no alternative (ε). A tournament choice rule is called *strict* if $r_T(\varepsilon) = 0$ for all $T \in \mathcal{T}$. A tournament choice rule is called *deterministic* if $r_T(a) \in \{0, 1\}$ for all $T \in \mathcal{T}$ and $a \in A$.

A simple requirement from a randomized choice rule is to be independent of the names of the alternatives. This requirement is reasonable since ties could always be broken using randomization.

Definition 2. A choice rule r is called *anonymous* if for every tournament $T \in \mathcal{T}$ and for every permutation π of A: $\pi(r_T) \equiv r_{\pi(T)}$.

We observe that any choice rule can be made into an anonymous choice rule by randomly permuting the names of the alternatives before applying a choice rule. Therefore, in the remainder of this paper we will implicitly limit our discussion to anonymous choice rules.

We would expect a desirable choice rule to use randomization only when there is some kind of tie for winner. In particular, we would like a clear-cut winner, which beats all other participants to always win the tournament.

Definition 3. Alternative x is a *Condorcet winner* in $T \in \mathcal{T}$ if xTy for all $y \in A \setminus \{x\}$.

A tournament choice rule r is called *Condorcet consistent* if for all $T \in \mathcal{T}$ such that c is a Condorcet winner in T: $r_T(c) = 1$.

Similarly, we can define the notion of a Condorcet loser, which should never win a tournament:

Definition 4. Alternative x is a *Condorcet loser* in $T \in \mathcal{T}$ if yTx for all $y \in A \setminus \{x\}$.

A tournament choice rule r is called *Condorcet loser consistent* (CLC) if for all $T \in \mathcal{T}$ such that c is a Condorcet loser in T: $r_T(c) = 0$.

Our main focus in this paper is resistance to manipulations by a pair of participants. Such manipulations involve the flipping of an edge in the tournament graph. This brings rise to the following definition of adjacency:

Definition 5. We say that T and T' are *S-adjacent* for some $S \subseteq A$ if and only if the two tournaments disagree only on edges in S, that is for all $\{z, w\} \not\subseteq S$: $zTw \Leftrightarrow zT'w$.

The most simple manipulation a single participant may consider is throwing a match, that is losing a match they could have won. Resistance to this kind of manipulation coincides with the well-known property of *monotonicity*, which means winning a match cannot reduce the winner's probability of winning the tournament.

Definition 6. A choice rule r is *monotonic* if and only if for all $T \neq T' \in \mathcal{T}$ that are $\{x, y\}$-adjacent and xTy: $r_T(x) \geq r_{T'}(x)$. Such a rule is *strictly monotonic* if these inequalities are strict.

When considering manipulations by a pair of participants, we consider two distinct utility models. When utility is not transferable, a manipulation is valid only if neither of the manipulators are worse off and at least one of the manipulators is better off. We call this Pareto Manipulation:

Definition 7. A choice rule r is *Pareto non-Manipulable for coalitions of size k (k-PNM)* if for every $S \subseteq A$ such that $|S| \leq k$ and for every $T, T' \in \mathcal{T}$ that are S-adjacent: $\exists x \in S : r_T(x) > r_{T'}(x)$ or $\forall x \in S : r_T(x) \geq r_{T'}(x)$. We say that r is PNM if it is k-PNM for any $k \in \mathbb{N}$.

If we assume transferable utility, a much stronger manipulation resistance is required, as the manipulators only care about their combined probability of winning the tournament:

Definition 8. A choice rule r is *Strongly non-Manipulable for coalitions of size k (k-SNM)* if for every $S \subseteq A$ such that $|S| \leq k$ and for every $T, T' \in \mathcal{T}$ that are S-adjacent: $\sum_{x \in S} r_T(x) = \sum_{x \in S} r_{T'}(x)$. We say that r is SNM if it is k-SNM for any k.

Note that in the case of deterministic choice rules the above definitions of non-manipulability are equivalent, and both variants of non-manipulability are extensions of the non-manipulability property for deterministic choice rules.

Pareto Manipulation

When restricting our attention to Pareto manipulations where all manipulators must gain from the manipulation, the use of strict randomized choice rules allows to easily obtain Condorcet consistency in addition to other desirable properties.

The intuition behind Condorcet consistency can be extended to any tournament using the concept of a top cycle:

Definition 9. A set of alternatives $C \subseteq A$ is the *top cycle* in $T \in \mathcal{T}$ [notation: $C = TC(T)$] if for all $x \in C, y \in A \setminus C$: xTy and C is the minimal such set.

The top cycle is well-defined since the set of all alternatives A satisfies the condition and the minimal set is unique. We can now define the following special case of Condorcet consistency.

Definition 10. A choice rule r satisfies the *top cycle condition* (TCC) if for every tournament $T \in \mathcal{T}$: $r_T(x) > 0$ implies $x \in TC(T)$. r further satisfies the *strong top cycle condition* if for every tournament T: $x \in TC(T)$ also implies $r_T(x) > 0$.

Another useful property of a choice rule is consistent behavior between different sizes of tournaments.

Definition 11. A choice rule r satisfies the *Condorcet loser removal condition* (CLR) if for every tournament $T \in \mathcal{T}$ such that there exists a Condorcet loser $c \in A$: $r_T \equiv r_{T \setminus \{c\}}$. That is, r assigns the same probability distribution for T and T where c is removed.

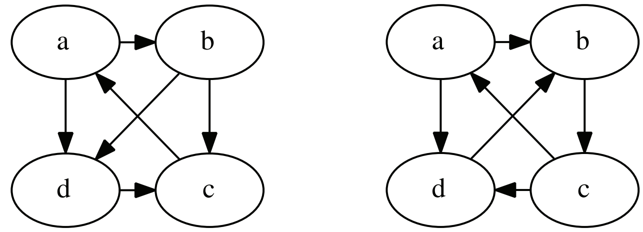
Note that for strict choice rules, the CLR implies that a Condorcet loser must be chosen with zero probability.

We shall now present several examples of strict choice rules that satisfy 2-PNM, monotonicity, strong TCC, and CLR:

- The *Iterative Condorcet* rule (ICR) is defined algorithmically as follows: Let T be the input tournament. If x is a Condorcet winner in T output x and stop. Otherwise, remove a random alternative $y \in A$ and repeat.
- The *Randomized Voting Caterpillar* (RVC) selects a random permutation π of A and then selects the winner of the voting caterpillar defined by π. That is, the winner of iterative pairwise matches between the previous winner and next participant according to π. See (Fischer, Procaccia, and Samorodnitsky 2009) for a formal definition.
- The *Top Cycle Rule* (TCR) simply assigns probability $\frac{1}{|TC(T)|}$ for all elements in $TC(T)$ and zero otherwise.

The three rules defined above are all different. For example, for four alternatives, the rules assign the following probabilities in the tournament T where no Condorcet winner nor loser exist (Figure 1a):

- ICR: $r_T(a) = \frac{5}{12}, r_T(b) = \frac{1}{3}, r_T(c) = \frac{1}{6}, r_T(d) = \frac{1}{12}$.
- RVC: $r_T(a) = \frac{5}{12}, r_T(b) = \frac{1}{4}, r_T(c) = \frac{1}{4}, r_T(d) = \frac{1}{12}$.
- TCR: $r_T(a) = \frac{1}{4}, r_T(b) = \frac{1}{4}, r_T(c) = \frac{1}{4}, r_T(d) = \frac{1}{4}$.



(a) T (b) T'

Figure 1: Size 4 tournament with no Condorcet winner or loser

We shall now prove the properties of these choice rules:

Proposition 12. *ICR, RVC, and TCR satisfy* 2*-PNM, monotonicity, strong TCC, and CLR. Moreover, TCR satisfies PNM while ICR and RVC do not satisfy* 3*-PNM.*

Proof. (Sketch) Monotonicity of the ICR and RVC is a direct result of each of the rules being randomized on are monotone. Monotonicity of TCR is due to the fact that reversing an edge may never decrease the size of the top cycle without removing the loser from the top cycle, and may never allow the loser into the top cycle unless it was already in the top cycle.

Strong TCC is satisfied by TCR by definition. ICR and RVC also satisfy strong TCC due to the fact that only members of the top cycle may be selected (since once all but one of the top cycle members are removed the remaining member is a Condorcet winner), and due to the minimality of the top cycle there exists a RVC and a removal strategy such that each member of the top cycle is selected.

All the choice rules satisfy CLR because they are defined solely based on the top cycle.

ICR and RVC satisfy 2-PNM since they are both strictly monotone within the top cycle. Therefore, any edge that is flipped within such that the loser was in the top cycle will strictly reduce the probability of the loser being chosen, and hence such a manipulation is not possible. An edge flipped outside the top cycle obviously has no effect.

To see TCR satisfies PNM, note that a successful manipulation must either reduce the size of the top cycle while all coalition members remain within, or add a new manipulator to the top cycle while keeping the top cycle of the same size. Both cases require the manipulators to remove a non-manipulator from the top cycle. For that to happen, all alternatives directly or indirectly dominated by that manipulator must also be removed from the top cycle, and this is only possible if at least one manipulator is removed from the top cycle, in contradiction to PNM.

RVC does not satisfy 3-PNM since in the tournament T in Figure 1a, a manipulation by $\{b, c, d\}$ could manipulate the tournament to tournament T' in Figure 1b, where (due to symmetry) $r_{T'}(b) = r_T(c) = \frac{1}{4} = r_T(b)$, $r_{T'}(c) = r_T(a) = \frac{5}{12} > r_T(c)$, $r_{T'}(d) = r_T(d) = \frac{1}{12}$.

To show ICR does not satisfy 3-PNM we must look at a tournament of size at least 6 (see Figure 2). Here, $\{a, b, c\}$ can manipulate by flipping their internal edges. Specifically,

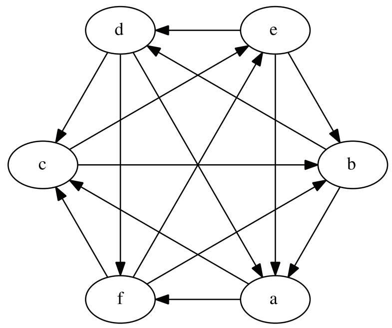


Figure 2: 3-Manipulable tournament for ICR

this manipulation strictly increases the probability of *all* of these alternatives. □

Strong Non-manipulability

In this section we discuss the stronger notion of manipulations where utility is transferable. We begin by noting that the impossibility theorem of Altman, Procaccia, and Tennenholtz (2009) for deterministic rules equally applies to strongly non-manipulable randomized rules.

Proposition 13. *There exist no* 2*-SNM Condorcet-consistent choice rules.*

This impossibility result requires us to weaken either the non-manipulability or Condorcet-consistency requirement. We have already discussed a weaker version of non-manipulability, therefore we will now focus on choice rules that are approximately Condorcet consistent. Specifically, we aim to maximize the approximation ratio while maintaining nonmanipulability.

Definition 14. A choice rule r has *Condorcet consistency value* $\alpha : \mathbb{N} \mapsto [0,1]$ if for all $T \in \mathcal{T}$ such that c is a Condorcet winner in T: $r_T(c) \geq \alpha(|A|)$.

The following simple choice rule could serve as a lower bound for the Condorcet consistency value of SNM choice rules.

Proposition 15. *There exists a SNM, strictly monotone, CLC, strict choice rule which has Condorcet consistency value of* $\frac{2}{n}$.

Proof. The required choice rule simply selects two alternatives uniformly at random, and then chooses the winner of the pairwise match between them. Equivalently, every alternative is chosen with probability

$$r_T(x) = \frac{2}{n(n-1)} \cdot |\{y : xTy\}|.$$

The required conditions are trivially satisfied. □

This lower bound is tight in the case where non-manipulability is required for coalitions of size 3:

Theorem 16. *There exists no choice rule that is* 3*-SNM and has a Condorcet consistency value* $\alpha > \frac{2}{n}$.

Proof. Assume for contradiction choice rule r satisfying the conditions of the theorem. Let T be a tournament of size n where for all $i < j \in A$: jTi. Consider any three agents $i-1, i, i+1 \in A$. Consider the tournament T' which is $\{i-1, i+1\}$-adjacent to T such that $(i-1)T'(i+1)$. This manipulation creates a 3-cycle in the graph: $iT'(i-1)T'(i+1)T'i$, where agents $i-1, i, i+1$ are completely symmetric. Since this is a manipulation by $i-1$ and $i+1$, we have $r_{T'}(i+1) + r_{T'}(i-1) = r_T(i+1) + r_T(i-1)$, and by symmetry $r_{T'}(i+1) = r_{T'}(i-1) = \frac{1}{2}[r_T(i+1) + r_T(i-1)]$. However, this could also be seen as a manipulation by the trio $(i-1, i, i+1)$, and hence $2r_T(i) = r_T(i+1) + r_T(i-1)$. Compounded with the fact that the sum of all $r_T(i)$ must be ≤ 1, we get that $r_T(n) \leq 2/n$, in contradiction to our assumption. □

For coalitions of size 2, we know the following upper bound on α:

Proposition 17. *There exist no* 2*-SNM choice rules that have Condorcet-consistency value* $\alpha > \frac{2}{3}$.

Proof. Assume for contradiction a choice rule r that satisfies the requirements, and consider a tournament T of arbitrary size n with a top cycle of three elements $\{a, b, c\}$. Any pair of these elements may manipulate to make one of them a Condorcet winner, gaining at least α. Hence,

$$\begin{aligned} r_T(a) + r_T(b) &> \frac{2}{3} \\ r_T(a) + r_T(c) &> \frac{2}{3} \\ r_T(b) + r_T(c) &> \frac{2}{3}, \end{aligned}$$

and therefore $r_T(a) + r_T(b) + r_T(c) > 1$, which is a contradiction. □

Given a tournament size n, the problem of finding a choice rule with a maximal Condorcet consistency value for 2-SNM tournaments of that size could be expressed as a linear program:

$$\begin{aligned} \max \quad & \alpha \text{ s.t.} \\ p_{iT} - \alpha \quad \geq \quad & 0 \text{ if } i \text{ is the Condorcet winner in } T \\ p_{iT} + p_{jT} \quad & \\ -p_{iT'} - p_{jT'} \quad = \quad & 0 \text{ where } T, T' \text{ are } \{i,j\}\text{-adjacent} \\ \sum_i p_{iT} \quad \leq \quad & 1 \; \forall T \\ p_{iT} \quad \geq \quad & 0 \; \forall i, T \end{aligned}$$

Additional constraints such as strictness, monotonicity, and CLC could also be expressed in this linear program:

$$\begin{aligned} \sum_i p_{iT} \quad = \quad & 1 \; \forall T \\ p_{iT} - p_{iT'} \quad \geq \quad & 0 \text{ where } T, T' \text{ are } \{i,j\}\text{-adjacent and } iTj \\ p_{iT} \quad = \quad & 0 \text{ if } i \text{ is the Condorcet loser in } T \end{aligned}$$

The size of the aforementioned LP could be significantly reduced by requiring the rule to be anonymous, and hence

Strict	CLC	Mon.	3	4	5	6
no	no	no	$2/3$	$2/3$	0.6548	0.6548
no	no	yes	$2/3$	$2/3$	0.6471	0.6471
no	yes	no	$2/3$	$2/3$	0.6471	0.6472
no	yes	yes	$2/3$	$2/3$	0.6471	0.6471
yes	no	no	$2/3$	$1/2$	0.5667	0.6196
yes	no	yes	$2/3$	$1/2$	0.5333	0.5513
yes	yes	no	$2/3$	$1/2$	0.4667	0.5375
yes	yes	yes	$2/3$	$1/2$	0.4667	$1/2$
Lower Bound			$2/3$	$1/2$	$2/5$	$1/3$

Table 1: Best obtainable Condorcet consistency values under various restrictions

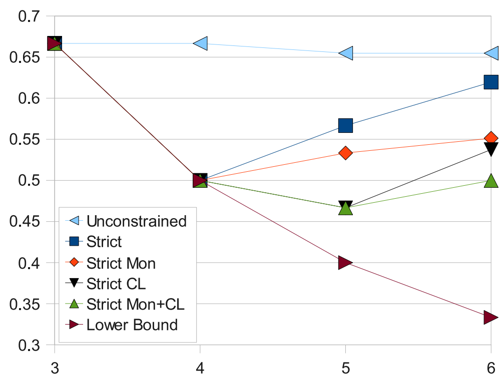


Figure 3: Best obtainable Condorcet consistency values under various restrictions

we could define variables and rules only once for each structurally unique tournament.

Using an LP solver to solve these LPs for small values of n, we see that the aforementioned lower bound of $\frac{2}{n}$ is tight for $n = 3$, and for $n = 4$ when strictness is required. Even more surprisingly, for larger values of n the empirically calculated best value for α is larger than $\frac{1}{2}$, even when strictness is enforced. Best obtainable α values for $n = 3, 4, 5, 6$ can be seen in Table 1 and Figure 3.

Our experimental results show that at least for small values of n, the $\frac{2}{3}$ upper bound is nearly attainable when not requiring strict choice rules, even though it could not be reached for $n > 4$. When strictness is required these values vary significantly with n but seem to be well over the $\frac{2}{n}$ lower bound.

A specific interesting randomized non-strict choice rule for $n = 4$ allows for an α value of $\frac{2}{3}$. This rule is defined as follows: If a Condorcet winner exists, is selected with probability of $\frac{2}{3}$, and no alternative is selected with probability $\frac{1}{3}$. In the case where there is a top-cycle of three alternatives, all get chosen with probability $\frac{1}{3}$. In the remaining case, the probabilities are assigned according to RVC. That is, for Figure 1a, the rankings are $r_T(a) = \frac{5}{12}$, $r_T(b) = r_T(c) = \frac{1}{4}$, $r_T(d) = \frac{1}{12}$. Any pair of agents that can become a Condorcet winner or two members of a 3-top cycle sum up to $\frac{2}{3}$, as required, and any pair that can become a member of a 3-top cycle and a Condorcet loser sum up to $\frac{1}{3}$.

Conclusions and Future Work

In this paper we have expanded the notion of nonmanipulable tournament choice rules to encompass two major extensions: First, we discuss randomized choice rules, which may assign varying weights for each alternative to win, or even make no choice at all. Second, we consider manipulations by coalitions of more than only two agents.

We have presented two different notions of manipulation that are consistent with the deterministic case. The weaker notion, Pareto Non-Manipulability, could be easily attained, even for coalitions of arbitrary size, by allocating positive probability to all alternatives in the top cycle of the tournament.

Our stronger notion of manipulation is still always possible when the choice rule can be randomized, and hence we are forced to weaken Condorcet consistency. However, unlike the deterministic case, we can attempt to maximize the probability of a Condorcet winner being selected. We have shown a tight bound for manipulations by coalitions of three or more agents, and have experimentally explored manipulations by pairs of agents.

Many open questions still remain. For Pareto manipulations, finding a classification theorem for nonmanipulable choice rules, both for pairs and for arbitrary size coalitions, is still an open problem. For strong manipulations, a gap still remains between the lower bound for Condorcet consistency of $\frac{2}{n}$ and the known upper bound of $\frac{2}{3}$.

Acknowledgments

The authors would like to thank Ariel Procaccia, Jon Kleinberg, and Moshe Tennenholtz for useful discussions.

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