A Belief Revision Framework for Revising Epistemic States with Partial Epistemic States

Jianbing Ma[1]**, Weiru Liu**[1] **and Salem Benferhat**[2]

[1] School of Electronics, Electrical Engineering and Computer Science,
Queen's University Belfast, Belfast BT7 1NN, UK
{jma03, w.liu}@qub.ac.uk
[2] CRIL-CNRS, UMR 8188, Faculté Jean Perrin, Université d'Artois,
rue Jean Souvraz, 62307 Lens, France
benferhat@cril.fr

Abstract

Belief revision performs belief change on an agent's beliefs when new evidence (either of the form of a propositional formula or of the form of a total pre-order on a set of interpretations) is received. Jeffrey's rule is commonly used for revising probabilistic epistemic states when new information is probabilistically uncertain. In this paper, we propose a general epistemic revision framework where new evidence is of the form of a partial epistemic state. Our framework extends Jeffrey's rule with uncertain inputs and covers well-known existing frameworks such as ordinal conditional function (OCF) or possibility theory. We then define a set of postulates that such revision operators shall satisfy and establish representation theorems to characterize those postulates. We show that these postulates reveal common characteristics of various existing revision strategies and are satisfied by OCF conditionalization, Jeffrey's rule of conditioning and possibility conditionalization. Furthermore, when reducing to the belief revision situation, our postulates can induce most of Darwiche and Pearl's postulates.

Introduction

Belief revision (Alchourrón, Gärdenfors, and Makinson 1985; Katsuno and Mendelzon 1991) performs belief change on an agent's beliefs when new evidence is received. It has been observed that a pure logic-based revision framework, e.g., AGM postulates based framework, may lead to some counterintuitive results in iterated revision. As a result, revision on epistemic states should be introduced accordingly (Darwiche and Pearl 1997; Benferhat et al. 2000; Nayak, Pagnucco, and Peppas 2003; Booth and Meyer 2006; Jin and Thielscher 2007), etc.

However, in most of these research efforts, new evidence is still represented as a *propositional* formula, not an epistemic state. Therefore, these methods do not fully implement a revision that reflects the effect of epistemic states, e.g., new information could be uncertain (Darwiche and Pearl 1997; Delgrande, Dubois, and Lang 2006). Although effort has been made to address this problem in a couple of papers (e.g., (Benferhat et al. 2000)), in which new evidence is represented as a full epistemic state. The revision methods proposed still cannot manage the strengths over partitions on a set of interpretations, which, in probability or

possibility settings, is already accomplished by Jeffrey's rule (Benferhat et al. 2010). That is, we need to develop a revision framework which can deal with new information with strengths that could be modeled by partial epistemic states similar to the probability counterparts of Jeffrey's rule.

Jeffrey's rule is widely applied when an agent's current belief and new evidence are both modeled in probability measures. More precisely, in Jeffrey's rule, the prior state is a probability distribution representing an agent's current beliefs or generic knowledge whilst new evidence is a partial probability measure solely on a partitioned subsets of the world. Similar strategies were also proposed for ordinal conditional functions (OCFs) (Spohn 1988; Williams 1994), for possibility measures (Dubois and Prade 1988; Benferhat et al. 2010), etc. However, despite of the need to handle new, input information with strengths that may be present in different forms, to the best of our knowledge, there does not exist a common revision strategy (and its corresponding postulates) to address this issue. In another words, can we develop a general revision framework that subsumes these individual revision strategies (in different frameworks) with a set of common postulates? A significant advantage of this, if achievable, is to facilitate further understanding of the nature of revision, regardless of which formalism may be deployed to represent an agent's beliefs and new uncertain evidence.

To answer this question, we first propose a framework to represent an agent's epistemic beliefs, which generalizes various definitions of epistemic states in the literature (e.g., a weighted formula (Jin and Thielscher 2007), a total pre-order (Benferhat et al. 2000), an OCF-based epistemic state (Meyer 2000; Spohn 1988; Williams 1994), a probability measure (Halpern 2003), etc). This framework takes inspirations from Jeffrey's rule of conditioning under uncertain inputs. We then investigate how a set of rational postulates should be derived to regulate revision operators defined from this framework and provide representation theorems for these postulates. We prove that these postulates are satisfied by OCF conditionalization, possibility conditionalization, and most significantly Jeffrey'rule of conditioning.

Our main objective of defining a general iterated revision framework is to implement the revision of an agent's current beliefs (represented as a full epistemic state) with new, uncertain evidence (represented as a partial epistemic state).

The key difference, compared with logic-based iterated belief/epistemic revision, is to allow the strengths of prior beliefs and evidence to determine the result of revision.

Furthermore, we investigate the relationships between this general framework with logic base belief/epistemic revision, especially with Darwiche and Pearl's (DP's) belief revision framework (Darwiche and Pearl 1997). We prove that when reducing to the belief revision situation, our postulates can induce most of DP's postulates.

The rest of the paper is organized as follows. We provide the preliminaries and Jeffrey's rule in Sec. 2 and 3 respectively. In Sec. 4, formal definitions of epistemic space and epistemic state are introduced. In Sec. 5, we propose a set of postulates for epistemic revision and their corresponding representation theorems. In Sec. 6, we discuss how our framework subsumes existing revision strategies. Finally, we conclude the paper in Sec. 7.

Preliminaries

We consider a propositional language $\mathcal{L}$ defined on a finite set $\mathcal{A}$ of propositional atoms, denoted by p, q, r etc. An interpretation ω (or possible world) is a function $\mathcal{A}$ to $\{0, 1\}$. The set of all possible worlds defined on $\mathcal{A}$ is denoted as W. ω is a model of (or satisfies) ϕ iff $\omega(\phi) = 1$, denoted as $\omega \models \phi$. The set of models for ϕ is denoted as $Mod(\phi)$. $\phi \vdash \psi$ iff $Mod(\phi) \subseteq Mod(\psi)$ and $\phi \equiv \psi$ iff $Mod(\phi) = Mod(\psi)$.

$\{A_1, \ldots, A_n\}$ is a *partition* of set W iff $\bigcup_{i=1}^{n} A_i = W$ and for $i \neq j$, $A_i \cap A_j = \emptyset$. For convenience, we also call $\{\mu_1, \ldots, \mu_n\}$ a *partition* of set W when $\{A_1, \ldots, A_n\}$ is a partition and for any A_i, $Mod(\mu_i) = A_i$. A partition $\{B_1, \ldots, B_k\}$ (resp., $\{\phi_1, \ldots, \phi_k\}$) is called a **refinement** of partition $\{A_1, \ldots, A_n\}$ (resp., $\{\mu_1, \ldots, \mu_n\}$) if $\forall i, 1 \leq i \leq k, \exists j, 1 \leq j \leq n$, s.t. $B_i \subseteq A_j$ (resp., $\phi_i \models \mu_j$).

Jeffrey's Rule

In probability theory framework, revision is achieved by Jeffrey's rule (Jeffrey 1965).

Definition 1 *(Jeffrey's rule) Let P be the prior probability distribution on W and $\mathscr{F} = \{\mu_1, \ldots, \mu_n\}$ be a partition of W with $P(\mu_i) \neq 0$ for all μ_i. Assume that a new piece of evidence gives a probability measure $(W, \mathscr{F}, P^{\mathscr{F}})$ such that $P^{\mathscr{F}}(\mu_i) = \alpha_i, 1 \leq i \leq n$. Then Jeffrey's Rule revises P with $P^{\mathscr{F}}$ with operator $\circ_p$ and obtains*

$$(P \circ_p P^{\mathscr{F}})(w) = \alpha_i P(w)/P(\mu_i) \quad for \ w \models \mu_i \quad (1)$$

Jeffrey's rule revises the prior probability distribution P to P' given an uncertain input with probabilities bearing on a partition of W. It produces a unique distribution that satisfies the following two equations (Chan and Darwiche 2005):

$$P'(\mu_i) = P^{\mathscr{F}}(\mu_i) = \alpha_i \quad (2)$$

which shows that the new information is preserved and

$$\forall \mu_i, \forall \phi \vdash \mu_i, P(\phi|\mu_i) = P'(\phi|\mu_i) \quad (3)$$

which states that the revised (new) probability distribution P' must retain the degree of conditional probability of any event ϕ that implies μ_i.

Epistemic Space and Epistemic State

In order to define a general revision framework with uncertain input, we first provide formal definitions of epistemic space and epistemic state. Let D denote an infinite set of values with two special elements $\bot, \top$ in D, and there is a total pre-order $\leq_D$ on D such that $\forall x \in D, \bot \leq_D x \leq_D \top$.

Extension functions

To assist the definition of epistemic state, we first define the notion of **extension function**. A function f associating a value in D to every finite tuple of values in D is called an extension function if it satisfies

Identity $f(x) = x$
Minimality $f(x_1, \ldots, x_k) = \bot$ iff $x_1 = \ldots = x_k = \bot$
Monotonicity $f(x, y) \geq_D f(x)$

An extension function is similar to an aggregation function in (Konieczny, Lang, and Marquis 2004) (where an aggregation function aggregates a set of non-negative integers to a single non-negative integer) in the sense that both of them attempt to associate a set of values to a single value within a given domain. The differences between them are (i) an extension function is defined on D instead of a set of integers; and (ii) it satisfies the Monotonicity property above instead of the *Non-decreasingness* property below.

Non-decreasingness If $x \leq_D y$, then
$$f(x_1, \ldots, x, \ldots, x_k) \leq_D f(x_1, \ldots, y, \ldots, x_k)$$

Note that *Monotonicity* property and *Non-decreasingness* property define two different classes of extension functions for which we will investigate their relationships future in a longer version of the paper.

Example 1 *Let $D = \{2^a 3^b | a \in N, b \in N \cup \{\infty\}\}$ such that $\leq_D$ is defined as arithmetic $\geq$, $\top = 1$, and $\bot = \infty$. Let f be defined as $f(2^{a_1} 3^{b_1}, \ldots, 2^{a_n} 3^{b_n}) = 2^{min(a_1, \ldots, a_n)} 3^{min(b_1, \ldots, b_n)}$. Obviously, f satisfies Identity, Minimality and Monotonicity, but f does not satisfy the non-decreasingness property. For example, $4 \leq_D 3$, but $f(4, 9) = 1 >_D 3 = f(3, 9)$.*

Partial and full epistemic states

Now we define epistemic spaces and epistemic states which are similar to the definition of probability spaces and probability measures.

Definition 2 *A **partial epistemic space** is a tuple $(W, \mathscr{F}, \Phi, D, f)$, where $\mathscr{F}$ is a partition of W, Φ is a mapping from $\mathscr{F}$ to D, called a **partial epistemic state**, and f is an extension function. Φ can be extended from $\mathscr{F}$ to $2^{\mathscr{F}}$ by f such that for $A_1, \ldots, A_k \in \mathscr{F}$, $\Phi(\bigcup_{i=1}^{k} A_i) = f(\Phi(A_1), \ldots, \Phi(A_k))$[1].*

An epistemic space $(W, \{\{w_1\}, \ldots \{w_n\}\}, \Phi, D, f)$ is a special case of partial epistemic space $(W, \mathscr{F}, \Phi, D, f)$ where the partition of W is the set of all singleton sets. To differentiate the former from the latter, we call $(W, \{\{w_1\}, \ldots \{w_n\}\}, \Phi, D, f)$ a **full epistemic space** and

---

[1]It would be more accurate to use $\{A_1, \ldots, A_k\}$ instead of $\bigcup_{i=1}^{k} A_i$, but as $A_i \cap A_j = \emptyset$ when $i \neq j$, we simply use $\bigcup_{i=1}^{k} A_i$.

its corresponding Φ a **full epistemic state.** Note that a partial epistemic state Φ such that $\Phi(A) = \alpha$, $\Phi(\overline{A}) = \beta$ is not equivalent to a full epistemic state Ψ such that $\forall w \in A, \Psi(w) = a$, $\forall w \notin A, \Psi(w) = b$ and $\Psi(A) = \alpha$, $\Psi(\overline{A}) = \beta$ by f_Ψ. For example, in probability theory, a probability measure P with $P(\{man, woman\}) = 0.8$ does not mean $P(\{man\}) = P(\{woman\}) = 0.4$. So partial epistemic states can not be encoded by full epistemic states.

Obviously, if Φ is a probability measure ($D = [0,1]$, f is $+$), then the above definition degenerates to the definition of probability space.

In the rest of the paper we will use Φ, Ψ, Θ etc (possibly with a subscript) to denote an epistemic state.

Literally, although there have been many papers focusing on epistemic revision and merging, there does not exist a commonly accepted definition of epistemic state. In some papers (e.g. (Darwiche and Pearl 1997)), no formal definitions of epistemic state are given, though the concept is used. In some other papers, definitions for epistemic states are mainly based on plausibility orderings on possible worlds (Meyer 2000; Spohn 1988; Williams 1994; Benferhat et al. 2000; Jin and Thielscher 2007), etc. It is easy to see that an epistemic state as a plausibility ordering can be induced from a *full epistemic state*. That is, for any full epistemic state Φ, it encodes the $\leq_\Phi$ ordering between interpretations as $\forall w, w', w \leq_\Phi w'$ iff $\Phi(w) \geq_D \Phi(w')$.

Furthermore, Def. 2 not only generalizes the notion of probability space, it also takes definitions of OCFs (when D is set a set of ordinals and $f = min$) and possibility measures (when D is $[0,1]$ and $f = max$) as special cases. Therefore, Def. 2 indeed provides a general framework to model epistemic states defined in different formalisms.

Value $\Phi(A)$ can be interpreted as an agent's *epistemic firmness* on A. Note that $\Phi(A)$ encodes *all* the information an agent provides on A, in particular, if the agent changes the value $\Phi(\overline{A})$ while maintaining $\Phi(A)$ unchanged, we should consider that the agent maintains its belief on A, despite that the agent has changed its belief on $\overline{A}$[2].

Intuitively, the *Minimality* property of f, when considered in Def. 2, ensures that if an agent thinks the true world is definitely not in a particular set, then the true world should not be in any of its subsets, and vice versa, i.e., $\Phi(\bigcup_{i=1}^{k} A_i) = \bot$ iff $\Phi(A_1) = \ldots = \Phi(A_k) = \bot$. The *Monotonicity* property indicates that if $A \subseteq B$, then $\Phi(A) \leq_D \Phi(B)$, especially when $\Phi(A)$ is interpreted as a kind of plausibility value (epistemic firmness) of A. This property is very similar to Axiom A1: if $A \subseteq B$ then $Pl(A) \leq Pl(B)$ for a plausibility measure Pl (Friedman and Halpern 1995) which was also mentioned in (Halpern 2003).

Parallel to probability theory, probability distributions are applied (and discussed) more frequently than their corresponding probability spaces. In the following, most of the time we will only mention epistemic states without explicitly discussing their corresponding epistemic spaces too.

---

[2]Usually, constraints are placed on $\Phi(A)$ and $\Phi(\overline{A})$, e.g., if Φ is a probability measure, then $\Phi(A) + \Phi(\overline{A}) = 1$, or if Φ is an OCF , then $min(\Phi(A), \Phi(\overline{A})) = 0$. Here for generality, we do not assume any constraints on $\Phi(A)$ and $\Phi(\overline{A})$.

Example 2 *Let $W = \{w_1, w_2, w_3\}$ and a partition $\mathscr{F}$ on W be $\{\{w_1, w_2\}, \{w_3\}\}$. Also, let $D = \{Good, Neutral, Bad\}$ be the set of values with $\bot = Bad <_D Neutral <_D Good = \top$ and $f = max$ (f satisfies Identity, Minimality and Monotonicity). Let Φ define the following mapping: $\Phi(\{w_1, w_2\}) = Good$ and $\Phi(\{w_3\}) = Bad$, then $(W, \mathscr{F}, \Phi, D, f)$ is a partial epistemic space.*

Entailment of epistemic states

As W, D and f are assumed to be clear and unchanged throughout, an epistemic state Φ defined on $\mathscr{F}$ is denoted as $\Phi^{\mathscr{F}}$.

By abuse of notations, we also write $\Phi^{\mathscr{F}}(\mu) = \alpha$ when $\Phi^{\mathscr{F}}(A) = \alpha$ and $Mod(\mu) = A$, i.e., a proposition is assigned a plausibility value which is the value assigned to the set of its models. In the following, we will always use propositions rather than their corresponding sets of models.

We define the entailment of epistemic states as follows.

Definition 3 *Let $\Phi^{\mathscr{F}_1}$ and $\Psi^{\mathscr{F}_2}$ be two epistemic states, then $\Phi^{\mathscr{F}_1}$ entails $\Psi^{\mathscr{F}_2}$, denoted as $\Phi^{\mathscr{F}_1} \models \Psi^{\mathscr{F}_2}$, iff $\mathscr{F}_1$ is a refinement of $\mathscr{F}_2$ and $\forall \mu \in \mathscr{F}_2$, $\Phi^{\mathscr{F}_1}(\mu) = \Psi^{\mathscr{F}_2}(\mu)$.*

Note that each element μ of $\mathscr{F}_2$ is necessarily the union of several elements of $\mathscr{F}_1$. Hence $\Psi_1^{\mathscr{F}}(\mu)$ can always be evaluated and defined using Definition 2.

Example 3 *Let $W = \{w_1, w_2\}$ and $f = +$, $\Phi^{\mathscr{F}_1}$ be such that $\mathscr{F}_1 = \{\{w_1\}, \{w_2\}\}$, $\Phi^{\mathscr{F}_1}(\{w_1\}) = 1$, $\Phi^{\mathscr{F}_1}(\{w_2\}) = 2$, $\Psi^{\mathscr{F}_2}$ be such that $\mathscr{F}_2 = \{\{w_1, w_2\}\}$, $\Psi^{\mathscr{F}_2}(\{w_1, w_2\}) = 3$, then we have $\Phi^{\mathscr{F}_1} \models \Psi^{\mathscr{F}_2}$.*

In the rest of the paper, to differentiate, a full epistemic state will be represented without a superscript describing a partition (e.g., Φ) and a partial epistemic state always with a superscript describing a partition (e.g., $\Psi^{\mathscr{F}}$).

Postulates for Iterated Epistemic Revision
Intuitions on Revision & Postulates

Motivated by the principle of Jeffrey's rule on conditioning on probability spaces and the ideal requirement that only the strengths of prior beliefs and evidence should determine the outcome of belief revision (Darwiche and Pearl 1997), we propose the following constraints on revision in our epistemic space framework.

- Revision should be focused on a full epistemic state (representing prior beliefs or generic knowledge) revised by a partial epistemic state (representing a new, uncertain input). This is the spirit of Jeffrey's rule (revising a probability distribution with an uncertain input) and existing revision frameworks (e.g., prior beliefs are total pre-orders whilst an input is a propositional formula). Hence we use full epistemic states to encode current beliefs and partial epistemic states to encode new, uncertain inputs.

- Only the strengths of beliefs and new evidence determine the outcome of revision. This is the main argument in (Darwiche and Pearl 1997). This postulate is intuitively in agreement with the *Neutrality with respect to the intensity scale* condition proposed in (Dubois and Koning

1991) which says in a social choice scenario, an aggregation function should not depend on the semantic meanings of a set of social choice functions, but only focus on their intensities (numerical values in $[0, 1]$) of choices.

- New, most recent evidence has the priority. This is explained as that new evidence is preserved. This is also explained under the context that when two pieces of new information happen to have the same partition $\mathscr{F}$ (which implies that both pieces of evidence refer to the same sets of hypotheses) but with different strengths of belief on them, then the most recent evidence overrules the previous one (as the latter (evidence) is assumed to represent the most recently received (and acceptable) information about a static situation).

Based on these constraints, we propose the following four postulates. Let $\circ$ be a revision operator.

ER1 If Φ is a full epistemic state and $\Psi^{\mathscr{F}}$ is a partial epistemic state, then $\Phi \circ \Psi^{\mathscr{F}}$ is a full epistemic state.
Explanation: This postulate suggests that the revision operator $\circ$ is a mapping $\Omega \times \Upsilon \to \Omega$.

ER2 $\Phi \circ \Psi^{\mathscr{F}} \models \Psi^{\mathscr{F}}$.
Explanation: New evidence is preserved.

ER3 For any $\mu \in \mathscr{F}$, and $\mu' \in \mathscr{F}'$, if $\Phi(\mu) = \Phi'(\mu')$ and $\Psi^{\mathscr{F}}(\mu) = \Psi'^{\mathscr{F}'}(\mu')$, then for $\psi \models \mu$ and $\psi' \models \mu'$, $(\Phi \circ \Psi^{\mathscr{F}})(\psi) = (\Phi' \circ \Psi'^{\mathscr{F}'})(\psi')$ iff $\Phi(\psi) = \Phi'(\psi')$.
Explanation: This postulate implements the constraint that the strengths of beliefs and evidence determine the outcome of revision. More specifically, as evidence $\Psi^{\mathscr{F}}$ (resp. $\Psi'^{\mathscr{F}'}$) provides no information on ψ (resp. ψ') directly, the only information related to ψ (resp. ψ') is μ (resp. μ') as $\psi \models \mu$ (resp. $\psi' \models \mu'$), so the strength of ψ (resp. ψ') after revision should only rely on its own strength before revision and the strengths of μ (resp. μ') before and after revision. **ER3** can be viewed as the counterpart of Equation (3).

ER4 $\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}} = \Phi \circ \Theta^{\mathscr{F}}$.
Explanation: W.r.t. the same hypotheses with different strengths, the latest evidence overrules previous ones.

Probabilistic revision by Jeffrey's rule is an example that follows the above four postulates.

Relationship with Darwiche and Pearl's postulates
For comparison, here we review the six modified AGM postulates and four additional postulates on iterated revision in (Darwiche and Pearl 1997).

The six modified AGM postulates are

R1 $\Phi \circ \mu$ implies μ.
R2 If $\Phi \wedge \mu$ is satisfiable, then $\Phi \circ \mu \equiv \Phi \wedge \mu$.
R3 If μ is satisfiable, then $\Phi \circ \mu$ is also satisfiable.
R4 If $\Phi_1 = \Phi_2$ and $\mu_1 \equiv \mu_2$, then $\Phi_1 \circ \mu_1 \equiv \Phi_2 \circ \mu_2$.
R5 $(\Phi \circ \mu) \wedge \phi \vdash \Phi \circ (\mu \wedge \phi)$.
R6 If $(\Phi \circ \mu) \wedge \phi$ is satisfiable, then $\Phi \circ (\mu \wedge \phi) \vdash (\Phi \circ \mu) \wedge \phi$.

Here, Φ (possibly with a subscript) stands for an epistemic state[3] and μ, ϕ are propositional formulae. $\circ$ is a revision operator. For simplicity, when Φ is embedded in a

---

[3](Darwiche and Pearl 1997) did not provide an explicitly def-

propositional formula, it stands for its belief set $Bel(\Phi)$, for example, $\Phi \wedge \phi$ means $Bel(\Phi) \wedge \phi$.

The four additional postulates for iterated revision are

C1 If $\alpha \models \mu$, then $(\Phi \circ \mu) \circ \alpha \equiv \Phi \circ \alpha$.
C2 If $\alpha \models \neg\mu$, then $(\Phi \circ \mu) \circ \alpha \equiv \Phi \circ \alpha$.
C3 If $\Phi \circ \alpha \models \mu$, then $(\Phi \circ \mu) \circ \alpha \models \mu$.
C4 If $\Phi \circ \alpha \not\models \neg\mu$, then $(\Phi \circ \mu) \circ \alpha \not\models \neg\mu$.

Obviously, ER2 is a straightforward generalization of R1, whilst ER1 extends R3 in the epistemic revision situation where new evidence is also an epistemic state. ER3, however, not only generalizes R4, but also is a key characteristic postulate of revision considering with strengths of beliefs and evidence. ER4 is closely related to C1 and C2, but in general it does not imply C1 and C2. To obtain a generalization of C1 and C2, postulate ER4 should be strengthened as follows.

ER4* $\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'} = \Phi \circ \Theta^{\mathscr{F}'}$ where partition $\mathscr{F}'$ is a refinement of partition $\mathscr{F}$.

There are no obvious generalizations for R5 and R6 in our postulates, because the conjunction of two formulae (for two belief sets) used in DP postulates is hardly generalizable on epistemic revision in our framework. In another words, the conjunction of two epistemic states are undefinable.

As for postulate R2, the following proposition shows why we do not need to provide a separate postulate as its generalization.

Proposition 1 *Let Φ be a full epistemic state, $\Psi^{\mathscr{F}}$ be a partial epistemic state and $\circ$ be an epistemic revision operator satisfying **ER2** and **ER3**. For any $\mu \in \mathscr{F}$, if $\Phi(\mu) = \Psi^{\mathscr{F}}(\mu)$, then $\forall\phi \models \mu$, $\Phi(\phi) = (\Phi \circ \Psi^{\mathscr{F}})(\phi)$.*

This proposition shows that when both **ER2** and **ER3** hold, then if new evidence is partially consistent with the prior state, then the consistent part is not changed, which can be seen as an extension of R2.

Representation Theorems

In order to establish the representation theorems, we need to define the **retentive** and **conductive** operators on D.

Definition 4 *An operator $\ominus$ defined on D is called **retentive** if for any $a_1, a_2, b_1, b_2 \in D$ s.t. $a_1 \leq_D a_2$, $b_1 \leq_D b_2$, then the following statement holds: If $a_1 \ominus a_2 = b_1 \ominus b_2$ and $a_2 = b_2$, then $a_1 = b_1$.*

The word *retentive* here intuitively means that when eliminating the equivalent second operands, the equivalence is still retained for the first operands.

Definition 5 *An operator $\ominus$ defined on D is called **conductive** if for any $a_1, a_2, a_3, b_1, b_2, b_3 \in D$ s.t. $a_1 \leq_D a_2 \leq_D a_3$, $b_1 \leq_D b_2 \leq_D b_3$, then the following statement holds: If $a_1 \ominus a_3 = b_1 \ominus b_3$, and $a_2 \ominus a_3 = b_2 \ominus b_3$, then $a_1 \ominus a_2 = b_1 \ominus b_2$.*

---

inition on epistemic states, except that an epistemic state can be understood as an agent's current beliefs together with the relative plausibility orderings of possible worlds.

The word *conductive* here intuitively means that when eliminating same items from two equations (the second operands in both the first and second equations), the remaining operands can be combined naturally to form a new equation.

The retentive requirement specifies how to remove equivalent item in one equation and the conductive requirement tells how to remove same items from multiple equations to form a new equation.

There are many concrete retentive and conductive operators, for example, if $\ominus$ is the subtraction ('-') or division ('/') operator in mathematics, then it is retentive and conductive.

Theorem 1 *A revision operator $\circ$ satisfies postulates **ER1-ER4** iff there exists a **retentive** operator $\ominus$ defined on D such that for any full epistemic state Φ and any epistemic state $\Psi^{\mathscr{F}}$, $\forall \mu \in \mathscr{F}$ and $\forall \phi \models \mu$, $(\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Psi^{\mathscr{F}}(\mu)$ and $(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$.*

To some extent, this theorem shows *how minimal change* happens during epistemic revision in the sense that it preserves a kind of *distance* based on $\ominus$ (although $\ominus$ is not constructively given). Furthermore, evidently $(\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Psi^{\mathscr{F}}(\mu)$ and $(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$ are counterparts of Equation (2) and (3), respectively. As mentioned before, Jeffrey's rule yields a unique distribution that satisfies Eq. (2) and Eq. (3). Here this theorem shows that our postulates yield a unique full epistemic state that satisfies the two counterparts too. Therefore, this theorem presents a generalization of Jeffrey's rule.

With postulates **ER1-ER3** and **ER4***, we get the following representation theorem.

Theorem 2 *A revision operator $\circ$ satisfies postulates **ER1-ER3** and **ER4*** iff there exists a **retentive and conductive** operator $\ominus$ defined on D such that for any full epistemic state Φ and any epistemic state $\Psi^{\mathscr{F}}$, $\forall \mu \in \mathscr{F}$ and $\forall \phi \models \mu$, $(\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Psi^{\mathscr{F}}(\mu)$ and $(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$.*

An instance of revision operator

Let $W = \{(a,b) | a \in N, b \in N \cup \{\infty\}\}$, $D = \{2^a 3^b | a \in N, b \in N \cup \{\infty\}\}$ such that $\leq_D$ is defined as the arithmetic $\geq$, $\top = 1$, and $\bot = \infty$. Let f be defined as $f(2^{a_1}3^{b_1}, \ldots, 2^{a_n}3^{b_n}) = 2^{min(a_1,\ldots,a_n)}3^{min(b_1,\ldots,b_n)}$ and Φ be such that $\Phi((a,b)) = 2^a 3^b$. Let a new piece of evidence taken on partition $\mathscr{F} = \{\mu_1, \ldots, \mu_n\}$ be such that $\Psi^{\mathscr{F}}(\mu_i) = \alpha_i$, $1 \leq i \leq n$, then we can define a revision operator $\circ_n$ as

$$(\Phi \circ_n \Psi^{\mathscr{F}})(w) = \alpha_i \Phi(w) / \Phi(\mu_i) \text{ for } w \models \mu_i \quad (4)$$

$\circ_n$ also satisfies postulates **ER1-ER4** and **ER4***, and f satisfies the *Monotonicity* Property but not the *Non-decreasingness* property.

Comparison with related revision strategies

Jeffrey's rule: In probability theory, if we view $\circ_p$ (Equation (1)) as a revision operator, then we have

Proposition 2 *The revision operator $\circ_p$ defined in Equation (1) satisfies postulates **ER1-ER4** and **ER4***.*

OCF conditionalization: An OCF (Spohn 1988) κ is a function from a set of possible worlds to the set of ordinals with $\kappa^{-1}(0) \neq \emptyset$. It can be extended to a set of propositions as $\kappa(\mu) = min_{w \models \mu} \kappa(w)$. Given κ as the prior OCF, $\mathscr{F} = \{\mu_1, \ldots, \mu_n\}$ as a partition and a new piece of evidence as $\lambda^{\mathscr{F}}(\mu_i) = \alpha_i$, $1 \leq i \leq n$ s.t. $min_{1 \leq i \leq n}(\alpha_i) = 0$, then the conditionalization of κ w.r.t $\lambda^{\mathscr{F}}$ is

$$(\kappa \circ_c \lambda^{\mathscr{F}})(w) = \alpha_i + \kappa(w) - \kappa(\mu_i) \text{ for } w \models \mu_i \quad (5)$$

If we consider the conditionalization operator $\circ_c$ as a revision operator, then Equation (5) can be seen as a revision strategy, and we have

Proposition 3 *The revision operator $\circ_c$ defined in Equation (5) satisfies postulates **ER1-ER4** and **ER4***.*

Possibility conditionalization: Similar result holds for a so-called *qualitative* possibility conditionalization defined in (Dubois and Prade 1993) which is similar to OCF conditionalization.

Logic-based iterated belief/epistemic revision: To compare with other logic-based (iterated) belief revision frameworks, we need to ensure that each epistemic state has a non-empty belief set, hence to exclude epistemic states with empty belief sets.

Definition 6 *Let $\Phi^{\mathscr{F}}$ be a partial epistemic state and μ be any propositional formula. $\Phi^{\mathscr{F}}$ is said to satisfy the **Maximality** property iff $\Phi^{\mathscr{F}}$ satisfies $\Phi^{\mathscr{F}}(W) = \top$, and $\Phi^{\mathscr{F}}(\mu) = \top$ iff $\exists \phi \in \mathscr{F}, \phi \models \mu, \Phi^{\mathscr{F}}(\phi) = \top$.*

Particularly, if Ψ is a full epistemic state and μ is any propositional formula, Ψ satisfies the Maximality property iff it satisfies $\Psi(W) = \top$, and $\Psi(\mu) = \top$ iff $\exists w \models \mu$, $\Psi(w) = \top$. Now we can define the belief set of an epistemic state as follows.

Definition 7 *Let $\Phi^{\mathscr{F}}$ be a partial epistemic state which satisfies property Maximality, then its belief set $Bel(\Phi^{\mathscr{F}})$ is defined as*

$$Bel(\Phi^{\mathscr{F}}) = \{\mu : \Phi^{\mathscr{F}}(\mu) = \top, \mu \in \mathscr{F}\} \quad (6)$$

In other words, the belief set of an epistemic state (with property Maximality) is the set of propositions with a plausibility value $\top$. An alternative but not equivalent, weaker definition of belief set is $Bel(\Phi^{\mathscr{F}}) = \{\Phi^{\mathscr{F}}(\mu) > \Phi^{\mathscr{F}}(\neg\mu)\}$. In the following, we only concentrate on epistemic states with non-empty belief sets. We can prove that the definition of entailment on epistemic states generalizes the classical definition of entailment on beliefs of epistemic states.

Proposition 4 *Let $\Phi^{\mathscr{F}_1}$ and $\Psi^{\mathscr{F}_2}$ be two epistemic states, if $\Phi^{\mathscr{F}_1} \models \Psi^{\mathscr{F}_2}$, then $\bigvee Bel(\Phi^{\mathscr{F}_1}) \vdash \bigvee Bel(\Psi^{\mathscr{F}_2})$.*

For convenience, we use $\Delta^{\mathscr{F}_\mu}$ to denote a partial epistemic state such that its corresponding partition $\mathscr{F}_\mu$ is $\mathscr{F}_\mu = \{\mu, \neg\mu\}$, and the values are $\Delta^{\mathscr{F}_\mu}(\mu) = \top, \Delta^{\mathscr{F}_\mu}(\neg\mu) <_D \top$ ($\Delta^{\mathscr{F}_\mu}(\neg\mu)$ can be any value in D other than $\top$). Hence we have $Bel(\Delta^{\mathscr{F}_\mu}) = \{\mu\}$. In the following, we use $\Delta^{\mathscr{F}_\mu}$ to encode new evidence where in logic-based revision frameworks, e.g., (Darwiche and Pearl 1997), etc, new evidence is simply represented as a single formula μ.

Theorem 3 *Let Φ be a full epistemic state, μ be a propositional formula and $\circ$ be an epistemic revision operator, if $\circ$ satisfies postulates* **ER1-ER3** *and* **ER4***, *then we have*

$$Mod(Bel(\Phi \circ \Delta^{\mathscr{F}_\mu})) = min(Mod(\mu), \leq_\Phi), and$$

C1* *If $\alpha \models \mu$, then $Bel(\Phi \circ \Delta^{\mathscr{F}_\mu} \circ \Delta^{\mathscr{F}_\alpha}) = Bel(\Phi \circ \Delta^{\mathscr{F}_\alpha})$.*

C2* *If $\alpha \models \neg\mu$, then $Bel(\Phi \circ \Delta^{\mathscr{F}_\mu} \circ \Delta^{\mathscr{F}_\alpha}) = Bel(\Phi \circ \Delta^{\mathscr{F}_\alpha})$.*

This theorem shows that the belief set from epistemic revision on an epistemic state Φ with $\Delta^{\mathscr{F}_\mu}$ is equivalent to the belief set from belief revision on Φ with formula μ. It also reveals that our revision postulates imply DP's iterated belief revision postulates C1 and C2. Furthermore, this theorem[4] also conclude that our postulates (**ER1-ER3, ER4***) indeed imply **R1-R6** when epistemic states have belief sets.

However, it can be shown that postulates C3, C4, Recalcitrance (Nayak, Pagnucco, and Peppas 2003) and Independence (Jin and Thielscher 2007) do not hold in our framework as our framework also enables revision on the same beliefs with different strengths which is also suggested in (Brent 1997) that *one might revise one's epistemic commitments without thereby revising one's beliefs.*

REE Axioms: A set of axioms (i.e., REE*1-REE*4, REE*It) for characterizing iterated revision of full epistemic states (total pre-orders) by full epistemic states was presented in (Benferhat et al. 2000) as follows.

REE*1 $\Phi \circ \Psi \models \Psi$
REE*2 If $\Phi \wedge \Psi$ is consistent, then $\Phi \circ \Psi \equiv \Phi \wedge \Psi$
REE*3 If Ψ is consistent, then $\Phi \circ \Psi$ is consistent
REE*4 If $\Psi_1 \equiv \Psi_2$, then $\Phi \circ \Psi_1 \equiv \Phi \circ \Psi_2$
REE*It $(\Phi \circ \Theta) \circ \Gamma \equiv \Phi \circ (\Theta \circ \Gamma)$

Similarly, here an epistemic state Φ embedded in a formula stands for $Bel(\Phi)$.

The following result presents the relationship between the REE Axioms and our postulates.

Proposition 5 *A revision operator $\circ$ satisfying* **REE*1-4** *and* **REE*It** *also satisfies* **ER1-3** *and* **ER4***.

However, the converse is false. This is not surprising since the framework of (Benferhat et al. 2000) leads to a unique solution.

Conclusion

In this paper, we have proposed a general definition of epistemic states and studied its revision strategy where new, uncertain evidence is represented as a partial epistemic state. A set of epistemic revision postulates and their corresponding representation theorems were then provided from which we can recover several well-known revision strategies including Jeffrey's probabilistic kinematics and the revision of full epistemic states by full epistemic states.

For our future work, we will investigate belief expansion and contraction in our epistemic framework. We also plan to apply our approach to computer security such as alert correlation.

References

Alchourrón, C. E.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet functions for contraction and revision. *J. Sym. Log.* 50:510–530.

Benferhat, S.; Konieczny, S.; Papini, O.; and Pérez, R. P. 2000. Iterated revision by epistemic states: Axioms, semantics and syntax. In *ECAI'00*, 13–17.

Benferhat, S.; Dubois, D.; Prade, H.; and Williams, M. A. 2010. A general framework for revising belief bases using quatitative jeffrey's rule. *Funda. Inform.* 99:1–22.

Booth, R., and Meyer, T. 2006. Admissible and restrained revision. *JAIR* 26:127–151.

Brent, H. 1997. *Belief revision and epistemic value*. Department of Philosophy, University of Calgary: https://dspace.ucalgary.ca/handle/1880/26655.

Chan, H., and Darwiche, A. 2005. On the revision of probabilistic beliefs using uncertain evidence. *Artif. Intel.* 163:67–90.

Darwiche, A., and Pearl, J. 1997. On the logic of iterated belief revision. *Artif. Intel.* 89:1–29.

Delgrande, J.; Dubois, D.; and Lang, J. 2006. Iterated revision as prioritized merging. In *KR'06*, 210–220.

Dubois, D., and Koning, J. L. 1991. Social choice axioms for fuzzy set aggregation. *Fuz. sets and Sys.* 43:257–274.

Dubois, D., and Prade, H. 1988. Representation and combination of uncertainty with belief functions and possibility measures. *Computational Intelligence* 4:244–264.

Dubois, D., and Prade, H. 1993. Belief revision and updates in numerical formalisms: An overview, with new results for the possibilistic framework. In *IJCAI*, 620–625.

Friedman, N., and Halpern, J. 1995. Plausibility measures: A user's guide. In *Proceedings of Eleventh Conference on Uncertainty in Artificial Intelligence (UAI-95)*, 175–18. San Francisco, CA: Morgan Kaufmann.

Halpern, J. 2003. *Reasoning about Uncertainty*. MIT Press.

Jeffrey, J. 1965. *The logic of decision*. McGraw-Hill Press.

Jin, Y., and Thielscher, M. 2007. Iterated belief revision, revised. *Artif. Intel.* 171:1–18.

Katsuno, H., and Mendelzon, A. O. 1991. Propositional knowledge base revision and minimal change. *Artif. Intel.* 52:263–294.

Konieczny, S.; Lang, J.; and Marquis, P. 2004. Da^2 merging operators. *Artif. Intel.* 157:49–79.

Meyer, T. 2000. Merging epistemic states. In *PRICAI'00*, 286–296.

Nayak, A. C.; Pagnucco, M.; and Peppas, P. 2003. Dynamic belief revision operators. *Artif. Intel.* 146:193–228.

Spohn, W. 1988. Ordinal conditional functions: A dynamic theory of epistemic states. *Causation in Decision, Belief Change, and Statistics* 2:105–134.

Williams, M. A. 1994. Transmutations of knowledge systems. In *KR'94*, 619–629.

---

[4]Compared to Darwiche and Pearl's results, we also need to give a total preorder $\leq_\Phi$ and a faithful assignment from Φ to $\leq_\Phi$, here we omit their descriptions due to the limitation of space.