Optimal Rectangle Packing on Non-Square Benchmarks

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Abstract

The rectangle packing problem consists of finding an enclosing rectangle of smallest area that can contain a given set of rectangles without overlap. We propose two new benchmarks, one where the orientation of the rectangles is fixed and one where it is free, that include rectangles of various aspect ratios. The new benchmarks avoid certain properties of easy instances, which we identify as instances where rectangles have dimensions in common or where a few rectangles occupy most of the area. Our benchmarks are much more difficult for the previous state-of-the-art solver, requiring orders of magnitude more time, compared to similar-sized instances from a popular benchmark consisting only of squares. On the new benchmarks, we improve upon the previous strategy used to handle dominance conditions, we define a variable order over non-square rectangles that generalizes previous strategies, and we present a way to adjust the sizes of intervals of values for each rectangle's x-coordinates. Using these techniques together, we can solve the new oriented benchmark about 500 times faster, and the new unoriented benchmark about 40 times faster than the previous state-of-the-art.

Introduction

Given a set of rectangles, our problem is to find all enclosing rectangles of minimum area that will contain them without overlap. We refer to an enclosing rectangle as a *bounding box*. The optimization problem is NP-hard, while the problem of deciding whether a set of rectangles can be packed in a given bounding box is NP-complete, via a reduction from bin-packing (Korf 2003). We introduce the *oriented constant-perimeter rectangle packing benchmark*, which is a simple set of increasingly difficult instances for this problem, where the task is to find all bounding boxes of minimum area that contain a set of rectangles of sizes 1xN, 2x(N-1), ..., (N-1)x2, Nx1. For example, Figure 1 is an optimal solution for N=23.

Rectangle packing has many practical applications. It appears when loading a set of rectangular objects on a pallet without stacking them. Various other cutting stock and layout problems also have rectangle packing at their core.

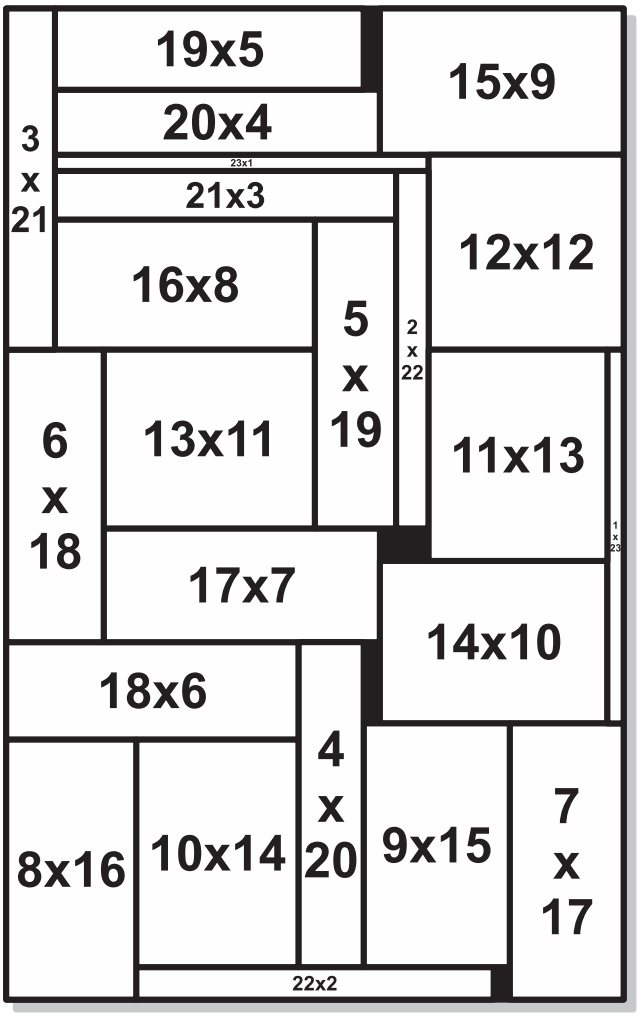
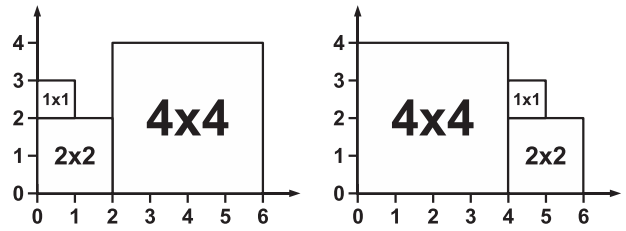


Figure 1: An optimal solution for N=23 with a bounding box of 38x61.



(a) A dominated position for the 4x4 square.

(b) An undominated position for the 4x4 square.

Figure 2: Example of dominance conditions.

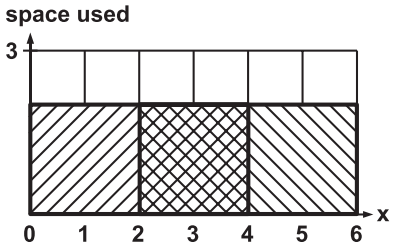


Figure 3: Assigning a 4x2 to [0,2].

Rectangle packing may also model some scheduling problems where tasks consume resources that must be allocated in contiguous chunks. For example, consider the task of scheduling when ships should berth at various locations along one long wharf. The time a ship remains docked may be represented as the width of a rectangle while the ship's length may be represented as its height. A rectangle packing solution then tells us where and when ships should be berthed. One can similarly model other problems where resources should be allocated contiguously to tasks that require them, such as memory or radio frequency spectrum.

Previous Work

Korf (2003) divided the rectangle packing problem into two subproblems: the *containment problem* and the *minimal bounding box problem*. The former tries to pack the given rectangles in a given bounding box, while the latter finds a bounding box of least area that can contain a given set of rectangles. The algorithm that solves the minimal bounding box problem calls the algorithm that solves the containment problem as a subroutine.

Containment Problem

Korf's (2003) absolute placement approach modeled rectangles as variables and positions in the bounding box as values. He introduced a set of dominance conditions to prune positions where large rectangles are too close to the sides of the bounding box. For example, imagine that we must pack the squares 4x4, 3x3, 2x2, and 1x1. In Figure 2a, the 4x4 placed at x=2 leaves a 2x4 gap against the left side of the bounding box in which the 3x3 cannot fit. Only the 2x2 and 1x1 can fit within the gap, and in fact they both can be placed entirely within the gap simultaneously.

Now notice that in any solution with the 4x4, 2x2, and 1x1 arranged as in Figure 2a, we can always rearrange them as in Figure 2b without disturbing any other squares. Thus, as long as we have tried placing the 4x4 at x=0, we would not have to try x=2, because if there were a solution at x=2 then we would have found it earlier when trying x=0. In general, a position for a rectangle is dominated if it leaves a gap in which all rectangles that can individually fit can also be packed together in the gap without protruding from it. This guarantees they can be swapped as in Figure 2 (Korf 2003).

In contrast to Korf's (2003) solver, Simonis and O'Sullivan's (2008) solver determined the x-coordinates of the rectangles prior to any of the y-coordinates, as suggested by Clautiaux et al. (2007). Their Prolog program used the *cumulative constraint* (Aggoun and Beldiceanu 1993), which requires that the sum of the heights of all rectangles overlapping any given x-coordinate never exceeds the height of the bounding box. Since we use some of these ideas, we review them here.

Simonis and O'Sullivan (2008) divided the domain of the x-coordinates for each rectangle into a set of intervals, the size of which was tuned experimentally. These x-intervals are explored in turn, and constrain the x-coordinates that may be assigned to the corresponding rectangle. Committing to an interval also induces a smaller rectangle representing the common intersecting area of placing the rectangle in any location in the interval, similar to the ideas of Beldiceanu et al. (2008). Larger intervals result in weaker constraint propagation (less pruning) but a smaller branching factor, while smaller intervals result in stronger constraint propagation but a larger branching factor.

For example, in Figure 3 a 4x2 rectangle assigned an x-interval of [0,2] consumes 2 units of area at each x-coordinate in [2,3], represented by the doubly-hatched area, regardless of where it is placed in the interval. This *compulsory part* (Lahrichi 1982) is a constraint common to all positions $x \in [0, 2]$ of the original 4x2 rectangle. If there were no feasible set of interval assignments, then finding individual x-values would be unnecessary. However, if we do find a set of interval assignments, then we will have to search for a set of single x-coordinate values. Simonis and O'Sullivan (2008) assigned (in order) x-intervals, single x-coordinates, y-intervals, and finally single y-coordinates.

Our previous solver (Huang and Korf 2009) outperformed that of Simonis and O'Sullivan by solving the y-coordinates differently. For every x-coordinate solution, we would solve a perfect packing instance to find the y-coordinates. A perfect packing problem is a rectangle packing problem with the property that the solution has no empty space, and such an instance is created by adding to the original set of rectangles a number of 1x1 rectangles equal to the area of empty space in the original instance.

Minimal Bounding Box Problem

A simple way to solve the minimal bounding box problem is as follows. The minimum height and width of any feasible bounding box is determined by the maximum height

and maximum width, respectively, of the given rectangles. The maximum width is found by computing a greedy solution with the minimum height, and the maximum height is found by a greedy solution with the minimum width. Such a greedy solution is computed by placing the rectangles in the first available position when scanning from left to right, and for each column scanning from bottom to top.

We then generate a bounding box for each pair of width and height in these ranges, rejecting any box whose area is less than the sum of the rectangle areas. For some box heights, some rectangles cannot be stacked on top of each other, so they must be placed side-by-side, giving us a minimum width. We reject any box whose width is less than the minimum width for its height, and reject any box whose height is less than the minimum height for its width. The remaining boxes are sorted in non-decreasing order of area, and tested one by one, until a feasible solution is found. If all optimal solutions are desired, we test all bounding boxes whose area equals that of the optimal solution.

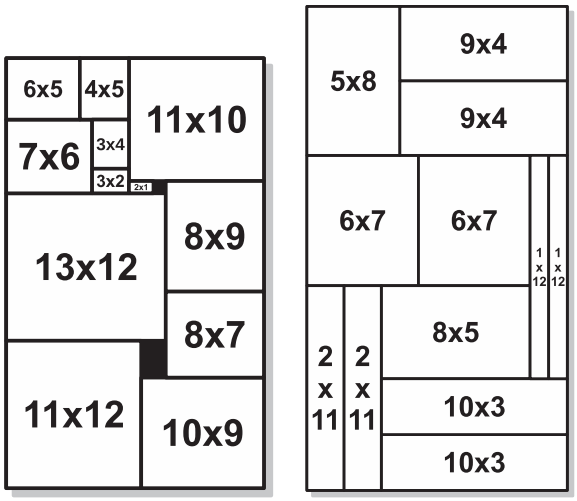
Rectangle Packing Benchmarks

The previous state-of-the-art used the *consecutive-square packing benchmark* (Korf 2003), a simple set of increasingly difficult instances for this problem, where the task is to find all bounding boxes of minimum area that contain a set of squares of sizes 1x1, 2x2, ..., up to NxN. Another benchmark used in the literature is the *unoriented rectangle packing benchmark* (Korf, Moffitt, and Pollack 2009), in which an instance is a set of rectangles of sizes 1x2, 2x3, ..., up to Nx(N+1), and rectangles may be rotated by 90-degrees.

Two Harder Benchmarks

We propose two new benchmarks that demonstrate limitations of using only the consecutive-square and unoriented rectangle packing benchmarks to compare previous solvers. The first is the *oriented constant-perimeter rectangle packing benchmark*, where instances are described as a set of rectangles of sizes 1xN, 2x(N-1), ..., (N-1)x2, Nx1, and rectangles may not be rotated. Given a parameter N, all rectangles are unique and have a perimeter of $2N$+2. The second is the *unoriented constant-perimeter rectangle packing benchmark*, where instances are described as a set of rectangles 1x(2N-1), 2x(2N-2), ..., (N-1)x(N+1), NxN, and rectangles may be rotated by 90-degrees. All rectangles here are unique and have a perimeter of $4N$.

Both of these new benchmarks are much more difficult than either the consecutive-square packing benchmark or the unoriented rectangle benchmark (Korf, Moffitt, and Pollack 2009). We tested our former state-of-the-art solver (Huang and Korf 2009) on both old and new benchmarks. Solving N=21 of our oriented constant-perimeter benchmark took almost four days, while instances with the same number of rectangles from the consecutive-square and unoriented rectangle packing benchmarks took only one second and eight seconds, respectively. Our unoriented constant-perimeter benchmark was even more difficult in that the largest size we could run using our previous solver was N=16, taking over 40 hours.



(a) Solution in a 21x35 bounding box for the unoriented instance 1x2, 2x3, ..., 11x12, 12x13.

(b) Solution in a 14x26 bounding box for the unoriented instance 1x12, 2x11, ..., 11x2, 12x1.

Figure 4: Examples of solutions for instances where rectangles may have the same dimensions.

Properties of Easy Benchmarks

There are several explanations for why the previous benchmarks are easier than our new ones. For the reasons we will subsequently explain, we have constructed our new benchmarks to describe instances consisting of rectangles with unique dimensions, without duplicates, and where most of the area is not captured by only a few rectangles.

Rectangles With Equal Dimensions In the old unoriented rectangle benchmark, all instances have rectangles that share some dimension with another rectangle. For example, Figure 4a is an optimal solution of N=12. In all optimal solutions up to the largest we can run, rectangles equal in some dimension tend to line up next to each other, forming larger rectangles and leaving little empty space. In Figure 4a, the 8x9 and 8x7 line up, as do the 6x5 with the 4x5, and the 3x4 with the 3x2. Note also that the 6x5, 4x5, 3x4, 3x2 and 7x6 together form a larger 10x11 rectangle with no empty space.

In fact, the optimal solutions in this benchmark all have much less empty space than similar-sized instances from the consecutive-square packing benchmark, where all rectangles have unique dimensions. Note that this is not an intrinsic property of a benchmark that is unoriented, but this happens to be a property that exists in the unoriented rectangle packing benchmark proposed by Korf et al. (2009).

We have also considered a benchmark whose instances contain duplicate rectangles. For example, take any instance from our oriented constant-perimeter benchmark, and simply allow the rectangles to be rotated 90-degrees. Figure 4b is a solution to one such instance. Here, there are always two identical rectangles in every instance, except for the occasional single square. Note that having duplicate rectangles is a special case of having rectangles with equal dimensions, which is why we see the rectangles lining up to form larger

rectangles. Furthermore, the presence of duplicates allows additional symmetry-breaking techniques such as forcing the x-coordinate value of one rectangle to be greater than or equal to that of its duplicate.

The optimal solution is found rapidly on instances with these properties. Since our solver tests bounding boxes in non-decreasing order of area, and since the optimal solutions of these easy instances have very little empty space, we test and find the optimal bounding boxes very early.

Rectangles With Little Area and Small Dimensions In both the consecutive-square and the unoriented rectangle packing benchmarks, a few large rectangles capture much of an instance's total area. Thus, the solver does not search too deeply before using up the allowable empty space. With little empty space, backtracking is very likely since we cannot find a place for the next rectangle. Therefore, small rectangles in these benchmarks may not matter for much of the search effort.

Previous benchmarks such as the consecutive-square packing benchmark described rectangles of small area with both small widths and small heights, and those of large area with large widths and heights, making it obvious to place large rectangles first because they are the most constrained variables as well as have the most constraining values.

By contrast, in our new benchmarks there is a trade-off between rectangles with large dimensions and those with large area. The widest rectangle in our oriented constant-perimeter benchmark has the smallest branching factor as we search for x-coordinates. However, it also has the least area, so during search it won't constrain the cumulative constraint much. This raises the non-trivial question of how to define a good variable order over non-square rectangles.

New Solution Strategies

We improve upon our previous state-of-the-art solver with the following techniques, which help in solving the new benchmarks.

Instances Without Dominance Conditions

On the consecutive-square packing benchmark, our previous solver used intervals for the x-coordinates only on undominated positions. For example, a 20x20 square in a bounding box of width 42 has an x-coordinate domain of x={0, [6,16], 22} because we specifically exclude the dominated positions x={[1,5], [17,21]} as we try different interval assignments. Here, the first x-interval assignment that we branch on will just be the single-valued interval x=[0,0], since it is useless to include any values from [1,5]. The same thing also occurs at the right side of the bounding box. Therefore, the solver only makes non-trivial interval assignments for the domain x={[6,16]}, representing the interior of the bounding box.

While this technique improved the performance for consecutive-square packing by a factor of five compared to leaving it out, it actually slowed the performance on our new benchmarks by the same factor. This occurs because the dominant positions, such as placing the rectangle flush against the left or right sides of the bounding box, are always explored by committing to single x-coordinate values.

In our constant-perimeter benchmarks, the 1xN rectangle can always partially fit in gaps left by other rectangles, but it must always protrude out of those gaps, thereby eliminating the dominance conditions we previously described. Without any dominated positions to account for, our old solver ends up committing to single x-coordinate values in a situation where it is more desirable to include those positions in a larger interval assignment. In addition to dynamically inferring dominance rules, our new solver detects when there are no dominated positions and adjusts the interval assignments to avoid this behavior.

Ordering Variables By Branching Factor

There is a natural variable order that arises from both the consecutive-square and unoriented rectangle packing benchmarks when using the strategy of picking the most constrained variable next. No matter which property we use to determine the most constrained variable – height, width, or area – all properties agree on placing rectangles in decreasing order of size. In our new benchmarks, however, it is not obvious what variable order to use.

We propose a variable order over rectangles of various aspect ratios by picking the most constrained variable first, to favor a smaller branching factor closer to the root of the search tree. For the oriented constant-perimeter benchmark, recall that we assign intervals to the x-coordinates before the y-coordinates, and like Simonis and Sullivan (2008) we use a constant factor times the rectangle width to define the interval size. This means the branching factor for the x-interval variables for a given rectangle is

$$b = \frac{B_w - r_w}{C r_w} = \frac{B_w}{C}\left[\frac{1}{r_w}\right] - \frac{1}{C}, \qquad (1)$$

where B_w is the bounding box width, r_w is the rectangle width, and C is a constant chosen experimentally. The numerator $B_w - r_w$ is the number of x-coordinate values that the rectangle may assume while still fitting in the bounding box, and the denominator Cr_w is the size of the interval we will be assigning to the given rectangle. For example, if C=0.75 then we would assign intervals of size three to a 4x2 rectangle.

We may drop the translational constant $-1/C$ as well as the positive scalar B_w/C, because doing so would not change the relative ordering of the rectangles prescribed by the function. This leaves us with $1/r_w$, which means that for the oriented benchmark we should place the rectangles in order of decreasing width.

For the unoriented constant-perimeter benchmark, our solver first tries all values for a particular x-interval, and then rotates the rectangle 90-degrees before trying another set of x-interval values. In this case the branching factor is

$$b = \frac{B_w - r_w}{C r_w} + \frac{B_w - r_h}{C r_h} = \frac{B_w}{C}\left[\frac{1}{r_w} + \frac{1}{r_h}\right] - \frac{2}{C}. \qquad (2)$$

As mentioned before, we can drop the positive scalar and translational constant, giving us

Size N	Optimal Solution	Empty Space	Boxes Tested	HK09 Time	NoDom Time	BrFactor Time	C=0.55 Time	HK10 Time
13	16×29	1.94%	7	:01	:00	:00	:00	:00
14	19×30, 15×38	1.75%	7	:02	:01	:00	:00	:00
15	24×29	2.30%	10	:16	:05	:01	:00	:00
16	23×36	1.45%	9	:57	:16	:02	:00	:00
17	24×41	1.52%	8	5:56	1:21	:27	:03	:02
18	24×48	1.04%	12	1:06:32	14:47	6:15	:32	:22
19	32×42, 24×56	1.04%	12	6:35:48	1:26:16	31:23	3:34	2:15
20	37×42	0.90%	11	1:18:51:34	7:36:09	1:51:10	13:06	7:51
21	35×51	0.78%	9	3:21:31:46	13:33:16	4:22:49	20:49	11:20
22	34×60	0.78%	15				14:22:03	9:12:37
23	38×61	0.78%	16					3:22:50:38

Table 1: Minimum-area bounding boxes containing all oriented rectangles 1xN, 2x(N-1), ..., (N-1)x2, Nx1.

$$\frac{1}{r_w} + \frac{1}{r_h} = \frac{r_w + r_h}{r_w r_h}. \quad (3)$$

Because all rectangles in a given instance have the same perimeter by definition, the numerator of the result in Equation 3 is constant. Therefore for our unoriented benchmark, we would place the rectangles in order of decreasing area.

Larger and Balanced Interval Sizes

Our old solver used an interval size that is 0.35 times the width of a given rectangle. We have found that larger interval sizes improve the performance of our solver on the new benchmarks, and use a value of C=0.55 instead.

As we assign larger intervals to the short and wide rectangles, the x-interval variables for these rectangles tend to have branching factors of three or less. Here, we should balance the sizes of these intervals so that the values assigned are equally constraining on their subtrees. For example, consider C=0.55, a rectangle of width 20, and its set of possible x-coordinate values [0,23]. Our previous state-of-the-art would try intervals containing eleven values each, first branching on x=[0,10] and x=[11,21], and then exploring the remaining interval x=[22,23]. This results in small compulsory parts in the first two branches, but a very large one in the third.

Since we must explore three branches anyway, we can balance the sizes of these interval assignments by exploring x=[0,7], x=[8,15], and x=[16,23]. Our solver first computes the branching factor induced by the global interval parameter C=0.55 for each rectangle, and then it balances the number of values in each interval assignment.

Experimental Results

Table 1 compares the performance of our previous state-of-the-art on the oriented constant-perimeter benchmark against different versions of our solver. Table 2 compares the same solvers using our unoriented constant-perimeter benchmark. The first column is the number of rectangles. The second gives the optimal bounding boxes (some instances have multiple optimal solutions). The third gives the fraction of empty space in the optimal solution. The fourth gives the number of bounding boxes that were tested to find all optimal solutions. The remaining columns represent the CPU times for each solver in the format of days, hours, minutes, and seconds. We wrote our solver in C++ and collected our data on a 2.93GHz Intel Core 2 Duo E7500 using only one core, one process, and one thread.

From left to right, each successive solver improves on the previous one by including an additional technique. The column called HK09 is data collected using our old code (Huang and Korf 2009). Since the previous literature did not define a variable order over non-squares, we used the order of decreasing area by default.

NoDom improves upon HK09 in that it detects when no dominated positions exist and therefore includes the dominant positions with the interval assignments. BrFactor improves upon NoDom in that it orders the oriented constant-perimeter benchmark by decreasing width and the unoriented constant-perimeter benchmark by decreasing area. C=0.55 improves upon BrFactor in that we use an interval size of 0.55 instead of C=0.35 as recommended in previous literature. Finally, HK10 improves upon C=0.55 in that it uses knowledge of the branching factor to rebalance the sizes of the interval assignments for the x-coordinates.

Notice that NoDom, BrFactor, and C=0.55 introduce techniques that reduce the branching factor, and so they have a greater effect on performance than HK10, whose new technique seeks to make the intervals assigned equally constraining. Our experiments reveal that these techniques interact with one another, and we note that without including dominated positions in the intervals, the performance gained from the other techniques appear muted. This interaction is also why we tune the global interval parameter C only after including the other techniques that affect the branching factor.

Ordering by branching factor made a significant difference for our oriented constant-perimeter benchmark but not for our unoriented benchmark. For the unoriented benchmark, we omit a column for BrFactor in Table 2 because ordering by branching factor prescribes ordering by decreasing area, which is what we gave the solver by default.

Note that the unoriented constant-perimeter benchmark

Size N	Optimal Solution	Empty Space	Boxes Tested	HK09 Time	NoDom Time	C=0.55 Time	HK10 Time
11	29×33	1.15%	12	:01	:00	:00	:00
12	21×59	1.37%	17	:20	:04	:01	:01
13	38×41	0.71%	13	1:45	:21	:06	:06
14	38×51, 17×114	0.67%	17	28:48	4:53	1:19	1:15
15	44×54	0.67%	21	1:43:01	11:36	3:33	2:34
16	45×64, 30×96[1]	0.83%	35	1:16:46:44	4:13:34	1:16:02	1:01:54
17	39×88, 52×66	0.44%	27		1:12:40:14	9:44:14	7:53:50
18	55×74	0.57%	35				2:02:10:38

Table 2: Minimum-area bounding boxes containing all unoriented rectangles 1x(2N-1), 2x(2N-2), ..., (N-1)x(N+1), NxN.

requires our solver to try approximately twice as many bounding boxes for a given parameter N than that required for our oriented benchmark. This is due to having $2N$-1 as the largest dimension in the unoriented benchmark while having N as the largest dimension in the oriented benchmark. Since we generate and test all bounding boxes with widths and heights within computed ranges, instances described with large integers result in more bounding boxes that must be tested. Thus, an instance with N rectangles in this benchmark is incomparable to an instance of N squares from the consecutive-square packing benchmark when evaluating benchmark difficulty. Still, the difference in the number of bounding boxes tested cannot fully account for the orders of magnitude of additional time that the previous state-of-the-art solver requires.

Finally, we also tested our new solver on the consecutive-square packing benchmark to see the effects of any extra overhead added by our improvements. Our new solver resulted in only a five percent speedup compared to our old solver, likely due to minor improvements in data structures, and balancing interval sizes.

In summary, using all of our techniques together, we can solve N=21 of the oriented constant-perimeter benchmark about 500 times faster and N=16 of the unoriented constant-perimeter benchmark about 40 times faster than the previous state-of-the-art.

Conclusion

We have introduced two new benchmarks, one oriented and one unoriented, that include rectangles of various aspect ratios. These new benchmarks avoid various properties of easy instances, which we have identified. We have also proposed several search strategies to improve performance on our new benchmarks. We improved upon the previous strategy used to handle dominance conditions, proposed a variable ordering heuristic based on increasing branching factor that generalizes previous strategies, tuned a global interval parameter, and introduced a method to balance the sizes of the intervals assigned to the x-coordinate variables.

Our experiments reveal that the previous state-of-the-art takes orders of magnitude more time to solve our new benchmarks compared to instances from the consecutive-square packing benchmark with the same number of rectangles. We therefore advocate the use of these benchmarks for research in optimal rectangle packing as they appear to be significantly more difficult than the benchmarks used in the previous literature. Finally, using all of our techniques together we are able to solve N=21 of the oriented constant-perimeter benchmark about 500 times faster and N=16 of the unoriented constant-perimeter benchmark about 40 times faster than the previous state-of-the-art.

[1] 40×72 and 48×60 are also optimal solutions.

Acknowledgments

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