Logics for Sizes with Union or Intersection

Caleb Kisby,[1] **Saúl A. Blanco,**[1] **Alex Kruckman,**[2] **Lawrence S. Moss**[3]
[1]Department of Computer Science, Indiana University, Bloomington, IN 47408, USA
{cckisby, sblancor}@indiana.edu
[2]Department of Mathematics and Computer Science, Wesleyan University, Middletown, CT 06459, USA
akruckman@wesleyan.edu
[3]Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
lmoss@indiana.edu

Abstract

This paper presents the most basic logics for reasoning about the sizes of sets that admit either the union of terms or the intersection of terms. That is, our logics handle assertions $\mathsf{All}\ x\ y$ and $\mathsf{AtLeast}\ x\ y$, where x and y are built up from basic terms by either unions or intersections. We present a sound, complete, and polynomial-time decidable proof system for these logics. An immediate consequence of our work is the completeness of the logic additionally permitting $\mathsf{More}\ x\ y$. The logics considered here may be viewed as efficient fragments of two logics which appear in the literature: Boolean Algebra with Presburger Arithmetic and the Logic of Comparative Cardinality.

1 Introduction

Reasoning about the sizes of sets is common in both human and artificial reasoning. It is also common, both in "real-world" human settings and in artificial systems, to reason in this way about unions and intersections of sets. In human settings, this use of union or intersection is often reflected by natural language phrases such as "animals *or* plants" (for union), or "mammals *with* paws" (for intersection). For example, one might reason that if *all cats are mammals that purr* and *there are at least as many cats as purring things*, then it follows that *all purring things are cats*.

In this paper, we examine logics which capture the most basic fragments of reasoning about the sizes of finite sets alongside union or intersection. Our two main logics handle assertions $\mathsf{All}\ x\ y$ (*all x are y*) and $\mathsf{AtLeast}\ x\ y$ (*there are at least as many x as y*). In the first logic, terms may be formed using union, whereas in the second logic, terms may be formed using intersection.

We emphasize that these fragments are the most basic because we wish to reflect one of the primary lessons of cognitive science: That computationally light systems are the most cognitively plausible ones (Moss and Raty 2018). Accordingly, the two main logics we present are decidable in polynomial time.

The logics considered here are part of a broader enterprise of *natural logic* (Moss 2015; Moss and Raty 2018; van Benthem 2008). One of the main goals of this program is to demonstrate that components of natural language inference that can be modeled at all can be modeled by decidable logical systems. Also, our work aims at obtaining *complete* logical systems for its fragments, with an eye toward efficient computer implementations. Union and intersection are common points of interest and inspiration in natural language semantics (Keenan and Faltz 1985), and reasoning about the sizes of sets has been a common subject of investigation in natural logic (Moss and Raty 2018; Pratt-Hartmann 2008). This paper contributes to this goal of natural logic: we provide complete axiomatizations of fragments of reasoning about sizes with either union or intersection, and we show that these logics are decidable in polynomial time.

Related Work. There has been recent work at the confluence of logic and artificial intelligence on systems for reasoning about sizes, often including reasoning about union and intersection. The closest system to ours in the literature is the polynomial-time decidable logic of All, Some, $\mathsf{AtLeast}$, More, with set complement as a term forming operation (but not including union or intersection), which was investigated in (Moss 2016).

If one were to add *strict* cardinality comparison, set complement, and propositional connectives $\wedge$, $\vee$, and $\neg$ on top of our work, the resulting logic would look like the Logic of Comparative Cardinality $\mathsf{CardCompLogic}$ in (Ding, Harrison-Trainor, and Holliday 2018). Our logics are restricted further by requiring that all variables x have a finite interpretation (in $\mathsf{CardCompLogic}$ one may express this using the predicate $Fin(x)$). As shown in (Ding, Harrison-Trainor, and Holliday 2018), $\mathsf{CardCompLogic}$ is NP-complete, so what our logics lose in expressive power they recoup in efficiency. Additionally, the method of proof in (Ding, Harrison-Trainor, and Holliday 2018) is rather different from ours.

The logics presented in this paper may also be viewed as fragments of Boolean Algebra with Presburger Arithmetic (BAPA) (Kuncak, Nguyen, and Rinard 2006). BAPA is a two-sorted logic, allowing both *set relations* in the language of Boolean Algebra and *numerical relations* in the language of Presburger Arithmetic. These two sorts are connected by a set cardinality function $|s|$ mapping sets to numbers. BAPA

builds formulas from set relations and numerical relations via propositional connectives $\wedge, \vee, \neg$, and quantifiers over sets and numbers. The two main logics we present capture the sublogic of BAPA involving subsets, equality, *nonstrict* cardinality comparison, and union or intersection terms, without propositional connectives or quantifiers. Decidability in BAPA is also NP-complete (Kuncak and Rinard 2007), so again our logics present a more efficient fragment of the more expressive logic.

Reasoning about subsets and sizes alongside union and intersection is also relevant to the description logic community. In (Baader and Ecke 2017), the authors incorporate the language of BAPA into the DL $\mathcal{ALC}$ in order to allow constraints on the cardinalities of concepts.

Our Contribution. The main contribution of this paper is our axiomatization of reasoning about sizes of sets alongside either union or intersection, and the resulting polynomial decidability of these logics. This efficiency result is in contrast to other logics involving reasoning of this kind (e.g. the NP-complete logics BAPA and CardCompLogic).

2 Our Logics

We focus our discussion on the two logics we call $\mathcal{A}^{\cup}(\text{card})$ and $\mathcal{A}^{\cap}(\text{card})$. We begin by defining the syntax and semantics of these systems.

Terms in our syntax may either be *basic terms* or *binary terms*. We use symbols $a, b, c, \ldots$ to denote basic terms and symbols $x, y, z, \ldots$ to denote terms that may be either basic or binary. If a and b are basic terms, then $a \cup b$ and $a \cap b$ are binary terms. Note that we do not allow nested terms like $(a \cup b) \cup c$ (we discuss this choice in the following subsection).

The *sentences* which we consider are $\mathsf{All}\ x\ y$ and $\mathsf{AtLeast}\ x\ y$, where x and y are terms. Note that in our logics we do not build up more complex sentences using propositional connectives or quantifiers; every sentence is one of these two operators applied to a pair of terms.

We now provide our terms and sentences with their semantics. A *model* $\mathcal{M}$ consists of a set M (the *universe* of $\mathcal{M}$), together with an interpretation function which assigns to each basic term a a subset $[\![a]\!] \subseteq M$. We extend the interpretation function to binary terms $a \cup b$ and $a \cap b$ by $[\![a \cup b]\!] = [\![a]\!] \cup [\![b]\!]$ and $[\![a \cap b]\!] = [\![a]\!] \cap [\![b]\!]$.

Our sentences are given the expected semantics:

$$\begin{array}{lll} \mathcal{M} \models \mathsf{All}\ x\ y & \text{iff} & [\![x]\!] \subseteq [\![y]\!] \\ \mathcal{M} \models \mathsf{AtLeast}\ x\ y & \text{iff} & |[\![x]\!]| \geq |[\![y]\!]| \end{array}$$

For a set Γ of sentences and another sentence φ, we have that $\Gamma \models \varphi$ if every *finite* model $\mathcal{M}$ that satisfies the sentences in Γ satisfies φ. Note the finiteness assumption about $\mathcal{M}$; this is weaker than the usual logical consequence notion in logic.

$\mathcal{A}^{\cup}(\text{card})$ and $\mathcal{A}^{\cap}(\text{card})$ differ in the terms and proof systems they use. Both logics employ natural-deduction style rules. The full table of rules is shown in Figure 1. In particular, $\mathcal{A}^{\cup}(\text{card})$ is the logic of $\mathsf{All}\ x\ y$ and $\mathsf{AtLeast}\ x\ y$, with binary union terms, but no binary intersection terms, using rules (AXIOM), (BARBARA), (MIX), (SIZE), (TRANS) in addition to (UNION-L), (UNION-R), and (UNION-ALL). $\mathcal{A}^{\cap}(\text{card})$ is the analogous logic, but with binary intersection terms instead of binary union terms, and using rules (INTER-L), (INTER-R), and (INTER-ALL) in place of (UNION-L), (UNION-R), and (UNION-ALL). Note that because we do not allow nested terms, the rules for union and intersection involve variables a, b, and c representing basic terms, while the other rules involve variables x, y, and z representing arbitrary terms.

For a set Γ of sentences in one of these logics and another such sentence φ, we say that φ is *provable* from Γ, written $\Gamma \vdash \varphi$, whenever φ may be obtained from the sentences in Γ from natural deduction via the rules for that logic. When we speak of the *decidability* of a logic, we refer to the problem of determining whether or not $\Gamma \vdash \varphi$ as a function of Γ and ϕ, when Γ is finite.

A logic is *sound* if whenever $\Gamma \vdash \varphi$ it follows that $\Gamma \models \varphi$. We say a logic is *complete* if the converse holds for finite Γ: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$. We only consider finite Γ in the definition of completeness because our logics $\mathcal{A}^{\cup}(\text{card})$ and $\mathcal{A}^{\cap}(\text{card})$ are not compact. One may verify that each of the rules in Figure 1 is individually sound for our semantics. Hence, the selected set of rules for each of our logics is sound.

Remark 2.1. The expected facts about set union and intersection are provable from the rules of $\mathcal{A}^{\cup}(\text{card})$ and $\mathcal{A}^{\cap}(\text{card})$. For example, symmetry of $\cup$ follows:

$$\dfrac{\dfrac{}{\mathsf{All}\ b\ (a \cup b)}\ (\text{UNION-R}) \qquad \dfrac{}{\mathsf{All}\ a\ (a \cup b)}\ (\text{UNION-L})}{\mathsf{All}\ (b \cup a)\ (a \cup b)}\ (\text{UNION-ALL})$$

The assumption that our models are finite is reflected in (MIX); this rule is not sound for infinite models.

Remark 2.2. It is worth mentioning the related logics $\mathcal{A}^{\cup}$ and $\mathcal{S}^{\cup}$. The logic $\mathcal{A}^{\cup}$ is simply the All-fragment of $\mathcal{A}^{\cup}(\text{card})$. $\mathcal{S}^{\cup}$ extends this All-fragment by admitting the sentence former $\mathsf{Some}\ x\ y$ (*some x is y*), with the semantics that $\mathcal{M} \models \mathsf{Some}\ x\ y$ whenever $[\![x]\!] \cap [\![y]\!] \neq \emptyset$. $\mathcal{S}^{\cup}$ additionally borrows the (SOME), (CONVERSION), and (DARII) rules from (Moss 2016). $\mathcal{A}^{\cup}$ and $\mathcal{S}^{\cup}$ are both complete.

Logics with Arbitrarily Large Terms

The reader might object that while in the Introduction we claim to capture basic reasoning about unions and intersections, the logics we address only allow *binary* (unnested) unions and intersections. This restriction has the advantage of simplifying our proof of polynomial decidability in Section 6. Although it may not initially be obvious, the completeness (and polynomial decidability, see Remark 6.3 below) of $\mathcal{A}^{\cup}(\text{card})$ and $\mathcal{A}^{\cap}(\text{card})$ with arbitrarily large finite terms follows from their completeness with only binary terms.

To see this, we reduce arbitrary terms to binary terms in the natural way, iteratively replacing binary subterms of a complex term by fresh basic terms. For example, $a \cup ((b \cup c) \cup d)$ becomes $a \cup (t_1 \cup d)$, which becomes $a \cup t_2$. Given Γ and φ with arbitrary terms, we define $\Gamma^\star$ and $\varphi^\star$ by reducing all terms appearing in Γ and φ in this way, and then adding additional sentences to $\Gamma^\star$: For every fresh term t replacing a binary term, say $a \cup b$, in either Γ *or* φ, we include in $\Gamma^\star$ the sentences $\mathsf{All}\ t\ (a \cup b)$ and $\mathsf{All}\ (a \cup b)\ t$.

$$\frac{}{\textsf{All}\ x\ x}\ (\textsc{axiom}) \qquad \frac{\textsf{All}\ x\ y \quad \textsf{All}\ y\ z}{\textsf{All}\ x\ z}\ (\textsc{barbara})$$

$$\frac{\textsf{All}\ x\ y \quad \textsf{AtLeast}\ x\ y}{\textsf{All}\ y\ x}\ (\textsc{mix}) \qquad \frac{\textsf{All}\ x\ y}{\textsf{AtLeast}\ y\ x}\ (\textsc{size}) \qquad \frac{\textsf{AtLeast}\ x\ y \quad \textsf{AtLeast}\ y\ z}{\textsf{AtLeast}\ x\ z}\ (\textsc{trans})$$

$$\frac{}{\textsf{All}\ a\ (a \cup b)}\ (\textsc{union-l}) \qquad \frac{}{\textsf{All}\ b\ (a \cup b)}\ (\textsc{union-r}) \qquad \frac{\textsf{All}\ a\ c \quad \textsf{All}\ b\ c}{\textsf{All}\ (a \cup b)\ c}\ (\textsc{union-all})$$

$$\frac{}{\textsf{All}\ (a \cap b)\ a}\ (\textsc{inter-l}) \qquad \frac{}{\textsf{All}\ (a \cap b)\ b}\ (\textsc{inter-r}) \qquad \frac{\textsf{All}\ a\ b \quad \textsf{All}\ a\ c}{\textsf{All}\ a\ (b \cap c)}\ (\textsc{inter-all})$$

Figure 1: The rules for the logics $\mathcal{A}^{\cup}(\text{card})$ and $\mathcal{A}^{\cap}(\text{card})$. In addition to the rules above the line, $\mathcal{A}^{\cup}(\text{card})$ uses ($\textsc{union-l}$), ($\textsc{union-r}$), and ($\textsc{union-all}$), whereas $\mathcal{A}^{\cap}(\text{card})$ uses ($\textsc{inter-l}$), ($\textsc{inter-r}$), and ($\textsc{inter-all}$).

One can check that this transformation does work, i.e.

$$\begin{array}{llll} \Gamma \models \varphi & \Longrightarrow & \Gamma^{\star} \models \varphi^{\star} & \\ & \Longleftrightarrow & \Gamma^{\star} \vdash \varphi^{\star} & \text{(by Section 3)} \\ & \Longrightarrow & \Gamma \vdash \varphi & \text{(allowing arbitrary terms)} \end{array}$$

3 Completeness of $\mathcal{A}^{\cup}(\text{card})$

In this section, we prove the completeness of the logic $\mathcal{A}^{\cup}(\text{card})$. We will return to $\mathcal{A}^{\cap}(\text{card})$ in Section 5. First, we present a representation lemma that will later be used to build a model of any finite set Γ of sentences in $\mathcal{A}^{\cup}(\text{card})$. In logics with sentential negation $\neg$ and a proof rule of *reductio ad absurdum*, such a representation lemma is tantamount to completeness. But $\mathcal{A}^{\cup}(\text{card})$ has neither of these, and so more work will be needed. This extra work will be presented subsequently.

Representation Lemma

Since we restrict our attention to binary terms, we may model the All- and AtLeast-relationships provable from Γ by corresponding relations on pairs of basic terms; this is the content of our representation lemma. We represent the problem in this way in order to argue that model-building in $\mathcal{A}^{\cup}(\text{card})$ can be done in polynomial time.

In order to state this lemma, we must first define the appropriate relations on pairs. Let BT be a finite set of basic terms. We fix a linear order $<$ on BT. We define the set of pairs under discussion as Pairs $= \{(a, b) : a, b \in BT \text{ and } a \leq b\}$.

Definition 3.1. A *suitable* pair of relations on Pairs is a pair of relations $(\preceq, \Subset)$ such that for all pairs p and q, and all basic terms a, b, c, and d,

1. $\preceq$ and $\Subset$ are preorders on Pairs. (That is, they are reflexive and transitive.)
2. $\preceq$ is linear: either $p \preceq q$ or $q \preceq p$ (and possibly both).
3. If $a < b$ in the fixed ordering on BT, then $(a, a) \Subset (a, b)$. If $b < a$, then $(a, a) \Subset (b, a)$.
4. If $(a, a) \Subset (c, d)$ and $(b, b) \Subset (c, d)$ and $a < b$, then $(a, b) \Subset (c, d)$.
5. If $p \Subset q$, then $p \preceq q$.
6. If $p \Subset q$ and $q \preceq p$, then $q \Subset p$.

For $p, q \in$ Pairs, we often write $\prec$ to denote the *strict part* of $\preceq$, i.e. $p \prec q$ whenever $p \preceq q$ but $q \not\preceq p$.

Here is an example: Let $\mathcal{N}$ be any model, and define $(a, b) \preceq (c, d)$ iff $|[\![a]\!] \cup [\![b]\!]| \leq |[\![c]\!] \cup [\![d]\!]|$, and $(a, b) \Subset (c, d)$ iff $[\![a]\!] \cup [\![b]\!] \subseteq [\![c]\!] \cup [\![d]\!]$. This gives a suitable pair of relations. Our representation lemma shows that every suitable pair of relations arises in this way. In order to build such models, we use families of sets corresponding to basic terms, defined as follows:

Definition 3.2. A *BT-family* is a family of finite sets $S = (S_a)_{a \in BT}$. For a BT-family S, we write $S_{a,b}$ for $S_a \cup S_b$. We also write $s_{a,b}$ for the number $|S_a \cup S_b|$. We also write s_a for $s_{a,a}$ (i.e., for $|S_a|$).

For $p \in$ Pairs, say with $p = (a, b)$, we often write S_p instead of $S_{a,b}$.

Definition 3.3. A BT-family S is *$\preceq$-preserving* if, for all $p, q \in$ Pairs, $p \preceq q$ implies that $s_p \leq s_q$. S is *$\preceq$-reflecting* if $s_p \leq s_q$ implies that $p \preceq q$.

Similarly, S is *$\Subset$-preserving* if $p \Subset q$ implies that $S_p \subseteq S_q$, and S is *$\Subset$-reflecting* if $S_p \subseteq S_q$ implies that $p \Subset q$.

Every BT-family S determines a model $\mathcal{M}$ as follows. Its universe M is $\bigcup_a S_a$. This set is finite. For a basic term a, let $[\![a]\!] = S_a$. So for union terms $a \cup b$, we automatically have $[\![a \cup b]\!] = S_a \cup S_b = S_{a,b}$.

We may now state our representation lemma. In the next section, we use it to prove the completeness of $\mathcal{A}^{\cup}(\text{card})$.

Lemma 3.4 (Representation Lemma)**.** *Let $(\preceq, \Subset)$ be a suitable pair of relations on Pairs. Then there is a BT-family of sets S such that for all $p, q \in$ Pairs,*

$$p \preceq q \quad \textit{iff} \quad s_p \leq s_q \tag{1}$$

$$p \Subset q \quad \textit{iff} \quad S_p \subseteq S_q \tag{2}$$

That is, S preserves and reflects $\preceq$ and $\Subset$.

Completeness

Theorem 3.5. *The logic $\mathcal{A}^{\cup}(\text{card})$ is complete.*

The rest of this section is devoted to the proof. We want to show that if Γ is a finite set of sentences and $\Gamma \nvdash \varphi$, then $\Gamma \not\models \varphi$. Our plan is to use Lemma 3.4 to build a model of Γ where φ is false. Note that we may assume φ has the form $\mathsf{AtLeast}\ (a \cup b)\ (c \cup d)$ or $\mathsf{All}\ (a \cup b)\ (c \cup d)$ for some basic terms a, b, c, and d; for example, $\mathsf{AtLeast}\ a\ c$ is provably equivalent to $\mathsf{AtLeast}\ (a \cup a)\ (c \cup c)$.

We first define relations $\leq_\Gamma$ and $\subseteq_\Gamma$ on Pairs by

- $(a, b) \leq_\Gamma (c, d)$ iff $\Gamma \vdash \mathsf{AtLeast}\ (c \cup d)(a \cup b)$
- $(a, b) \subseteq_\Gamma (c, d)$ iff $\Gamma \vdash \mathsf{All}\ (a \cup b)\ (c \cup d)$

for all (a, b) and (c, d) in Pairs.

Note that $(\leq_\Gamma, \subseteq_\Gamma)$ has all the properties of a suitable pair, except for linearity of $\leq_\Gamma$. We use the following fact to extend $\leq_\Gamma$ to a linear preorder. The statement is a bit stronger than a straightforward linearization, for purposes that will become clear at the end of this section.

Proposition 3.6. *Let $x^\star$ be a fixed pair. We can extend $\leq_\Gamma$ to a linear preorder $\preceq_\Gamma$ over Pairs such that for all $y \in$ Pairs, if $y \not\leq_\Gamma x^\star$, then $x^\star \prec_\Gamma y$.*

If φ is $\mathsf{AtLeast}\ (a \cup b)\ (c \cup d)$, let $x^\star$ be the pair (a, b). Otherwise, choose $x^\star$ arbitrarily. Let $\preceq_\Gamma$ be the linear preorder obtained from Proposition 3.6 using this $x^\star$. One can verify the following proposition.

Proposition 3.7. *$(\preceq_\Gamma, \subseteq_\Gamma)$ is a suitable pair of relations on Pairs.*

We may now apply Lemma 3.4. Take BT to be the set of basic terms appearing in Γ and φ. There is a BT-family of sets $S = (S_a)_{a \in BT}$ such that for all $(a, b), (c, d) \in$ Pairs,

$$\begin{array}{cc} \Gamma \vdash \mathsf{All}\ (a \cup b)\ (c \cup d) & \Gamma \vdash \mathsf{AtLeast}\ (c \cup d)\ (a \cup b) \\ \Updownarrow & \Downarrow \\ (a, b) \subseteq_\Gamma (c, d) & (a, b) \preceq_\Gamma (c, d) \\ \Updownarrow & \Updownarrow \\ S_a \cup S_b \subseteq S_c \cup S_d & |S_a \cup S_b| \leq |S_c \cup S_d| \end{array} \tag{3}$$

The $\Updownarrow$ on the upper-left is the definition of $\subseteq_\Gamma$. The $\Downarrow$ on the upper-right comes from the fact that $\preceq_\Gamma$ is a linearization (hence an extension) of $\leq_\Gamma$. The lower $\Updownarrow$-arrows follow from the representation lemma.

Let $\mathcal{M}$ be the model determined by the BT-family S. By (3), $\mathcal{M}$ satisfies the All- and $\mathsf{AtLeast}$-sentences in Γ. We would like to ensure as well that $\mathcal{M}$ does not satisfy φ. We have two cases:

- φ is $\mathsf{All}\ (a \cup b)\ (c \cup d)$. Since $\Gamma \nvdash \mathsf{All}\ (a \cup b)\ (c \cup d)$, by (3) we have $[\![a \cup b]\!] = S_a \cup S_b \not\subseteq S_c \cup S_d = [\![c \cup d]\!]$. So $\mathcal{M} \not\models \varphi$.
- φ is $\mathsf{AtLeast}\ (a \cup b)\ (c \cup d)$. Recall that when we applied Proposition 3.6, we took $x^\star$ to be (a, b). Now take y to be (c, d). Since $\Gamma \nvdash \mathsf{AtLeast}\ (a \cup b)\ (c \cup d)$, we have $(c, d) \not\leq_\Gamma (a, b)$ by definition. Thus, by Proposition 3.6, $(a, b) \prec_\Gamma (c, d)$. So by (3), $|[\![a \cup b]\!]| = |S_a \cup S_b| < |S_c \cup S_d| = |[\![c \cup d]\!]|$. So $\mathcal{M} \not\models \varphi$.

4 Outline of the Proof of the Representation Lemma

For the proof of completeness of $\mathcal{A}^{\cup}(\text{card})$, it remains to justify our representation lemma (Lemma 3.4). Given a suitable pair of relations $(\preceq, \Subset)$, we want to build a BT-family $S = (S_a)_{a \in BT}$ that preserves and reflects $\preceq$ and $\Subset$. Our plan is to start with a family that preserves and reflects $\Subset$. We then build our family of sets iteratively, ensuring that at each stage our family preserves and reflects $\Subset$ and at the final stage our family also preserves and reflects $\preceq$.

Proof of Lemma 3.4. Consider the preorder $\preceq$. We call a maximal set of $\preceq$-equivalent pairs a *size class*. We list the size classes in order, from $\prec$-largest to $\prec$-smallest. Let's say the size classes in this order are

$$C_1, C_2, \ldots, C_K$$

Since we are listing them from $\prec$-largest to $\prec$-smallest, we have the following fact: if $(a, b) \prec (c, d)$, and also $(a, b) \in C_i$, and finally $(c, d) \in C_j$, then $j < i$.

We will inductively construct a sequence of BT-families $S^0, S^1, \ldots, S^K$ such that at each step i we ensure that in S^i:

(1) The pairs preceding or in the size class C_i are correctly ordered according to $\preceq$, i.e. for all $(a, b), (c, d) \in \bigcup_{j \leq i} C_i$, $(a, b) \preceq (c, d)$ iff $s^i_{a,b} \leq s^i_{c,d}$.
(2) The sizes of all pairs strictly preceding C_i are larger than the sizes of all pairs in or following C_i. That is, for $j, k \in \{1, \ldots, K\}$ such that $j < i \leq k$, $q \in C_j$, and $p \in C_k$, $s^i_q > s^i_p$.
(3) S^i preserves and reflects $\Subset$.

If we do this for $i = 0, 1, \ldots, K$, then S^K will be a family of sets that preserves and reflects $\preceq$ and $\Subset$.

Constructing S^K. We begin by taking S^0 to be any family which preserves and reflects $\Subset$. One choice is to take, for each basic term a, $S^0_a = \{(c, d) \in \text{Pairs} : (a, a) \not\Subset (c, d)\}$. S^0 trivially satisfies assertions (1) and (2) from above and satisfies (3) by construction.

We now consider step $1 \leq i \leq K$. Suppose we have a family S^{i-1} that satisfies assertions (1)-(3) above for $i - 1$. In order to appropriately manipulate the sizes of sets S^i_a, we will apply the lemmas to follow. Both lemmas make use of a basic construction on families of sets, which we call *Clamp*.

Definition 4.1. Let S be a BT-family. Let $(c, d) \in$ Pairs, $r \in \omega$. We define a new family $Clamp(S, c, d, r)$ as follows: Let $R = \{*_1, \ldots, *_r\}$ be fresh points. For all $a \in BT$, let

$$Clamp(S, c, d, r)_a = \begin{cases} S_a \cup R & \text{if } (a, a) \not\Subset (c, d) \\ S_a & \text{otherwise} \end{cases}$$

In words, we add r new points simultaneously to all sets S_a, except when $\Subset$ "wants S_a to be a subset of $S_c \cup S_d$."

The following proposition summarizes the basic properties of the *Clamp* operation.

Proposition 4.2. *Let S be a BT-family, and fix $(c, d) \in$ Pairs and $r \in \omega$. Write T for $Clamp(S, c, d, r)$. Then:*

1. *For $(a,b) \Subset (c,d)$, $T_{a,b} = S_{a,b}$.*
2. *For (a,b) such that $(c,d) \prec (a,b)$, $T_{a,b} = S_{a,b} \cup \{*_1, \ldots, *_r\}$.*
3. *If S preserves and reflects $\Subset$, then so does T.*

Our first lemma allows us to equalize sizes of unions of pairs in the same size class. The second lemma ensures that the pairs in our size class C_i have greater size than all pairs in C_j for $j > i$.

Lemma 4.3. *Let S be a family which preserves and reflects $\Subset$. Let $k \geq 2$, and let $C = \{p_1, \ldots, p_k\}$ be a size class of $\preceq$. Then there is a BT-family T such that:*

1. *The unions corresponding to the pairs in C have equal size in T; i.e., for $1 \leq r, s \leq k$, $t_{p_r} = t_{p_s}$.*
2. *If (a,b) and (c,d) are any pairs which belong to larger size classes than C, then*
$$t_{a,b} \leq t_{c,d} \text{ if and only if } s_{a,b} \leq s_{c,d}.$$
3. *T preserves and reflects $\Subset$.*

Proof. Before we begin the construction of T, we have an observation. Let $\approx$ be the equivalence relation induced by $\Subset$. C, being a size class of $\approx$, splits into one or more $\approx$-classes. The observation is that if q_1 and q_2 are members of C which are in different $\Subset$ classes, then neither $q_1 \Subset q_2$ nor $q_2 \Subset q_1$. To see this, suppose towards a contradiction that $q_1 \Subset q_2$. Then since we also have $q_2 \preceq q_1$, we have $q_1 \Subset q_2$ by one of the properties of the suitable pair $(\preceq, \Subset)$. This gives a contradiction, since now $q_1 \approx q_2$.

Let us choose one pair in each $\approx$-class of C, and list the chosen pairs in size order according to S. That is, we have pairs $(a_1, b_1), \ldots, (a_k, b_k)$ so that every element of C is related by $\approx$ to exactly one pair on this list, and the order is chosen so that $s_{a_1,b_1} \leq s_{a_2,b_2} \leq \cdots \leq s_{a_k,b_k}$. Let

$$\begin{array}{rcl} T^1 & = & \mathit{Clamp}(S, a_2, b_2, s_{a_2,b_2} - s_{a_1,b_1}) \\ T^2 & = & \mathit{Clamp}(T^1, a_3, b_3, s_{a_3,b_3} - s_{a_2,b_2}) \\ & \vdots & \\ T^{k-1} & = & \mathit{Clamp}(T^{k-2}, a_k, b_k, s_{a_k,b_k} - s_{a_{k-1},b_{k-1}}) \end{array}$$

We take $T = T^{k-1}$. One may carefully verify that this T works. ☐

Lemma 4.4. *Let S be a family which preserves and reflects $\Subset$. Let $q_1, \ldots, q_\ell$ be a sequence of pairs in Pairs. Then there is a family T such that:*

1. *For $1 \leq i, j \leq k$, $s_{q_i} \leq s_{q_j}$ iff $t_{q_i} \leq t_{q_j}$.*
2. *For all pairs p which are $\prec$-below all q_j, we also have $t_p < t_{q_i}$ for all i.*
3. *T preserves and reflects $\Subset$.*

Proof. Let $m = \min_i s_i = \min_i s_{q_i}$. We call a pair p a *size competitor* if $p \prec q_j$ for all j, and yet $t_p \geq m$.

List the size competitors as $p_1, \ldots, p_k$. Note that for all size competitors p_i and all of the original points q_j, we have $q_j \not\Subset p_i$. For if we did have $q_j \Subset p_i$, then we would have $q_j \preceq p_i$; and the definition of a size competitor ensures that that $p_i \prec q_j$ for all i, j.

Let

$$\begin{array}{rcl} T^1 & = & \mathit{Clamp}(S, p_1, s_{p_1} - m + 1) \\ T^2 & = & \mathit{Clamp}(T^1, p_2, s_{p_2} - m + 1) \\ & \vdots & \\ T^k & = & \mathit{Clamp}(T^{k-1}, p_k, s_{p_k} - m + 1) \end{array}$$

We take $T = T^k$. Again, one may carefully verify that this T works. ☐

We may finally return to constructing the next family S^i. Consider our currently examined size class C_i. If C_i contains ≥ 2 pairs then we may apply Lemma 4.3 with size class C_i and family S^{i-1} in order to obtain a new family which we'll call T. (If C_i contains only 1 pair, let $T = S^{i-1}$). In T, all pairs in C_i have the same size (by part 1 of Lemma 4.3). By (1) for $i-1$ and by part 2 of Lemma 4.3, we have (1) for T. Lemma 4.3 also ensures that T preserves and reflects $\Subset$.

We list the pairs preceding or in size class C_i as $p_1, \ldots, p_k$. We apply Lemma 4.4 with these pairs $p_1, \ldots, p_k$ and family T. We let S^i be the resulting family. Lemma 4.4, part 1 ensures that (1) holds for S^i, since it held for T. And Lemma 4.4, part 2 ensures that (2) holds for S^i. Lemma 4.4 also ensures (3): that S^i preserves and reflects $\Subset$.

This completes the proof of our representation lemma and hence the completeness of $\mathcal{A}^{\cup}(\mathrm{card})$. ☐

5 Completeness of Logics with Intersection Terms

We obtain completeness of $\mathcal{A}^{\cap}(\mathrm{card})$ for free from the completeness of $\mathcal{A}^{\cup}(\mathrm{card})$. Let $\mathcal{L}^{\cup}$ be the language of $\mathcal{A}^{\cup}(\mathrm{card})$, and let $\mathcal{L}^{\cap}$ be the language of $\mathcal{A}^{\cap}(\mathrm{card})$. $\mathcal{L}^{\cup}$ and $\mathcal{L}^{\cap}$ share the same basic terms.

We translate $\mathcal{L}^{\cap}$ terms x and sentences ϕ to $\mathcal{L}^{\cup}$ as follows: For basic terms a, let $a^{\cup} = a$. For an intersection term, say $a \cap b$, let $(a \cap b)^{\cup} = \overline{a} \cup \overline{b}$. Let R be All or $\mathsf{AtLeast}$. Then for the $\mathcal{L}^{\cap}$-sentence $\psi = \mathsf{R}\ x\ y$, we let $\psi^{\cup}$ be $\mathsf{R}\ (y^{\cup})\ (x^{\cup})$. For a set Γ of $\mathcal{L}^{\cap}$ sentences, let $\Gamma^{\cup} = \{\psi^{\cup} : \psi \in \Gamma\}$.

Lemma 5.1. $\Gamma \vdash \phi$ *in* $\mathcal{A}^{\cap}(\mathrm{card})$ *iff* $\Gamma^{\cup} \vdash \phi^{\cup}$ *in* $\mathcal{A}^{\cup}(\mathrm{card})$.

We also need a semantic construction in the other direction. For a model $\mathcal{M}$ of $\mathcal{L}^{\cup}$, define a model $\mathcal{M}^{\cap}$ of $\mathcal{L}^{\cap}$ as follows: Let $\mathcal{M}^{\cap}$ use the same underlying universe M. For basic terms a, let $[\![a]\!]_{\mathcal{M}^{\cap}} = \overline{[\![a]\!]_{\mathcal{M}}}$. That is, each basic term's semantics in $\mathcal{M}^{\cap}$ is the complement of its semantics in $\mathcal{M}$.

Lemma 5.2. *For all models $\mathcal{M}$ of $\mathcal{L}^{\cup}$ and all $\mathcal{L}^{\cap}$-sentences ψ, $\mathcal{M}^{\cap} \models \psi$ iff $\mathcal{M} \models \psi^{\cup}$.*

From Lemmas 5.1 and 5.2, completeness of $\mathcal{A}^{\cap}(\mathrm{card})$ follows.

Theorem 5.3. *The logic $\mathcal{A}^{\cap}(\mathrm{card})$ is complete.*

6 Complexity of Our Logics

We now turn our attention towards the complexity of $\mathcal{A}^{\cup}(\mathrm{card})$ and $\mathcal{A}^{\cap}(\mathrm{card})$. As mentioned in Section 2, one of our primary motivations for restricting our language to binary terms was to argue that our logics are decidable in polynomial time. The proof is based on Theorem 1.5 of (Kruckman and Moss 2018), which is a variant of the proof of McAllester's Tractability Lemma (McAllester 1993).

Theorem 6.1. *The relation $\vdash$ is decidable in polynomial time.*

Furthermore, for $\mathcal{A}^{\cup}(\mathrm{card})$ and $\mathcal{A}^{\cap}(\mathrm{card})$ the following also holds:

Theorem 6.2. *If $\Gamma \not\vdash \varphi$ in either $\mathcal{A}^{\cup}(\mathrm{card})$ or $\mathcal{A}^{\cap}(\mathrm{card})$, then we can construct a countermodel $\mathcal{M}$ satisfying Γ but falsifying φ in polynomial time.*

The proof involves observing that the model-building procedure described in the proof of Theorem 3.5 can be performed in polynomial time (relative to the combined length of Γ and φ). We may first construct $\leq_\Gamma$ and $\subseteq_\Gamma$ over Pairs in polynomial time, since $\mathcal{A}^{\cup}(\mathrm{card})$ and $\mathcal{A}^{\cap}(\mathrm{card})$ are polynomial-time decidable (by Theorem 6.1). Of course, extending $\leq_\Gamma$ into the appropriate linear preorder $\preceq_\Gamma$ can be done in polynomial time. The rest of the work is to carefully check that the algorithm described in Section 4 can be done in polynomial time.

Remark 6.3. It follows from Theorem 6.1 that our logics with arbitrarily large finite union (or intersection) terms are also decidable in polynomial time. Given Γ, φ with arbitrary finite union terms (say), our decision procedure for $\Gamma \vdash_{\text{arb}} \varphi$ is simply to construct $\Gamma^\star$ and $\varphi^\star$ and then decide whether $\Gamma^\star \vdash \varphi^\star$ (see Section 2). Constructing $\Gamma^\star$ and $\varphi^\star$ takes polynomial steps in the size of Γ, φ. To verify that this is in fact a decision procedure for $\Gamma \vdash_{\text{arb}} \varphi$, we must check that $\Gamma \vdash_{\text{arb}} \varphi$ if and only if $\Gamma^\star \vdash \varphi^\star$. $\Gamma^\star \vdash \varphi^\star \implies \Gamma \vdash_{\text{arb}} \varphi$ is handled in Section 2. As for the converse, a proof tree $\mathcal{T}$ for $\Gamma \vdash_{\text{arb}} \varphi$ is transformed into a proof tree $\mathcal{T}^\star$ for $\Gamma^\star \vdash \varphi^\star$ by introducing the term substitutions t_i. One can verify that this is in fact a proof tree.

7 Completeness of Logics with More-Sentences

In this section, we consider the extension of our logic $\mathcal{A}^{\cup}(\mathrm{card})$ with More-sentences. We call the resulting logic $\mathcal{M}^{\cup}(\mathrm{card})$ (similarly, we call the corresponding logic with $\cap$-terms $\mathcal{M}^{\cap}(\mathrm{card})$). We may extend the argument in Section 3 to prove the completeness of $\mathcal{M}^{\cup}(\mathrm{card})$ (and hence, by Section 5, the completeness of $\mathcal{M}^{\cap}(\mathrm{card})$). Unfortunately, we cannot extend the complexity argument of Section 6 to obtain polynomial decidability for $\mathcal{M}^{\cup}(\mathrm{card})$. We discuss avenues for ameliorating this situation in Section 8.

We extend $\mathcal{A}^{\cup}(\mathrm{card})$ to $\mathcal{M}^{\cup}(\mathrm{card})$ by allowing, in addition to All- and AtLeast-sentences, the sentence More $x\ y$ (where x and y are terms). The semantics of More-sentences is similar to that for AtLeast-sentences:

$$\mathcal{M} \models \mathsf{More}\ x\ y \quad \text{iff} \quad |[\![x]\!]| > |[\![y]\!]|$$

$\mathcal{M}^{\cup}(\mathrm{card})$ employs the rules listed in Figure 2, in addition to those rules used in $\mathcal{A}^{\cup}(\mathrm{card})$ (similarly for $\mathcal{M}^{\cap}(\mathrm{card})$). The rules in Figure 2, with the exception of (RAA), we borrow from the logic $\mathcal{S}(\mathrm{card})$ described in (Moss 2016). Again, one may verify that each of the rules in Figure 2 is individually sound for our semantics.

Note in particular the rules (X) and (RAA). The (X) rule is *ex falso* (or *explosion*), fitted for our language of More- and AtLeast-sentences. Similarly, (RAA) is a special instance of *reductio ad absurdum*.

In contrast to $\mathcal{A}^{\cup}(\mathrm{card})$ and $\mathcal{A}^{\cap}(\mathrm{card})$, we must worry about expressing inconsistencies within $\mathcal{M}^{\cup}(\mathrm{card})$ and $\mathcal{M}^{\cap}(\mathrm{card})$. We say that a set Γ of sentences in $\mathcal{M}^{\cup}(\mathrm{card})$ (or $\mathcal{M}^{\cap}(\mathrm{card})$) is *inconsistent* whenever every sentence φ in the logic is provable from Γ (otherwise, we say that Γ is *consistent*). Note that for $\mathcal{M}^{\cup}(\mathrm{card})$, using rule (X), Γ is inconsistent if and only if there is a term z such that $\Gamma \vdash \mathsf{More}\ z\ z$.

Theorem 7.1. *The logic $\mathcal{M}^{\cup}(\mathrm{card})$ is complete.*

The proof is a straightforward extension of the proof of Theorem 3.5; observe that in order to build a model of Γ refuting More $x\ y$, it suffices to construct a model of $\Gamma \cup \{\mathsf{AtLeast}\ y\ x\}$. If $\Gamma \not\vdash \mathsf{More}\ x\ y$, then (RAA) ensures that $\Gamma \cup \{\mathsf{AtLeast}\ y\ x\}$ is consistent, which is necessary since we can only apply the modified version of the model construction to consistent Γ.

8 Discussion and Future Work

This paper has presented two complete, polynomial-time decidable logics for reasoning about the sizes of sets alongside the union or intersection of terms, respectively. These logics are the most basic for reasoning of this kind; $\mathcal{A}^{\cup}(\mathrm{card})$ and $\mathcal{A}^{\cap}(\mathrm{card})$ are both minimally expressive and decidable in polynomial time. Our logics may be viewed as more efficient fragments of BAPA and CardCompLogic, two more expressive NP-complete logics for reasoning about sizes with union and intersection. A direct corollary of our work is the completeness of the logic additionally permitting More-sentences.

Next Steps. Since decidability in both BAPA and CardCompLogic is NP-complete, it would be interesting to steadily build our fragment towards these logics and note at which point decidability is no longer decidable in polynomial time. The first step in this direction is to attempt to extend our polynomial decidability argument in Section 6 to the logic $\mathcal{M}^{\cup}(\mathrm{card})$. The main issue here is that $\mathcal{M}^{\cup}(\mathrm{card})$ makes use of the (RAA) rule, and we cannot put a bound on the height of an (RAA) application. One could remove the (RAA) rule from $\mathcal{M}^{\cup}(\mathrm{card})$ and attempt to replace it with simpler rules that may also ensure completeness. One such rule, not derivable from the rules of $\mathcal{M}^{\cup}(\mathrm{card})$ sans (RAA) is:

$$\frac{\mathsf{All}\ x\ a \quad \mathsf{All}\ x\ b \quad \mathsf{More}\ (a \cup b)\ b}{\mathsf{More}\ a\ x}\ (\text{DIAMOND})$$

(This rule is so-named because the terms x, a, b, and $a \cup b$ form a $\subseteq$-diamond.) Observe that this rule of inference is sound. If (DIAMOND) and rules like it ensure completeness without (RAA), then our argument for polynomial decidability follows without issue.

$$\frac{\textsf{More}\ x\ y \quad \textsf{AtLeast}\ y\ z}{\textsf{More}\ x\ z}\ (\textsc{more-l}) \qquad \frac{\textsf{AtLeast}\ x\ y \quad \textsf{More}\ y\ z}{\textsf{More}\ x\ z}\ (\textsc{more-r})$$

$$\frac{\textsf{More}\ x\ y}{\textsf{AtLeast}\ x\ y}\ (\textsc{more-atleast}) \qquad \frac{\textsf{More}\ z\ z}{\varphi}\ (\textsc{x}) \qquad \begin{array}{c}[\textsf{AtLeast}\ y\ x] \\ \vdots \\ \dfrac{\textsf{More}\ z\ z}{\textsf{More}\ x\ y}\end{array}\ (\textsc{raa})$$

Figure 2: The additional rules for the logics $\mathcal{M}^{\cup}(\mathrm{card})$ and $\mathcal{M}^{\cap}(\mathrm{card})$.

The next step would be to integrate union and intersection terms. But the further step of integrating union and intersection with term complement will likely result in an NP-complete logic, so this is where the road ends.

There is historical precedent in the syllogistic logic literature to allow Some sentences alongside All-sentences. As mentioned in Remark 2.2, the usual semantics for Some-sentences is that $\mathcal{M} \models \textsf{Some}\ x\ y$ whenever $[\![x]\!] \cap [\![y]\!] \neq \emptyset$. The main trouble with introducing Some-sentences is in addressing the following pesky rule:

$$\frac{\textsf{More}\ a\ b \quad \textsf{AtLeast}\ c\ d \quad \textsf{AtLeast}\ (b \cup d)\ (a \cup c)}{\textsf{Some}\ a\ c}$$

This rule simultaneously involves AtLeast-, More-, and Some-sentences with term union. Observe that this rule is sound as well, and that it is not provable from the rules of either $\mathcal{A}^{\cup}(\mathrm{card})$ or $\mathcal{S}^{\cup}$. One could hope to extend our model construction to model sets Γ which also include Some-sentences, but it is far from obvious how to integrate the above rule into the model-building process.

Finally, our two main logics can be integrated with SMT solvers in order to efficiently automate those inferences which just involve sizes and subset alongside union or intersection. A particularly appropriate SMT solver with which to test this is (Suter, Steiger, and Kuncak 2011), which extends the SMT solver Z3 with reasoning in quantifier-free BAPA.

Acknowledgements We thank the anonymous reviewers, as well as Vikraman Choudhury, Matthew Heimerdinger, and Chaitanya Koparkar, for their careful reviews and helpful comments.

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